Cover of the Monograph

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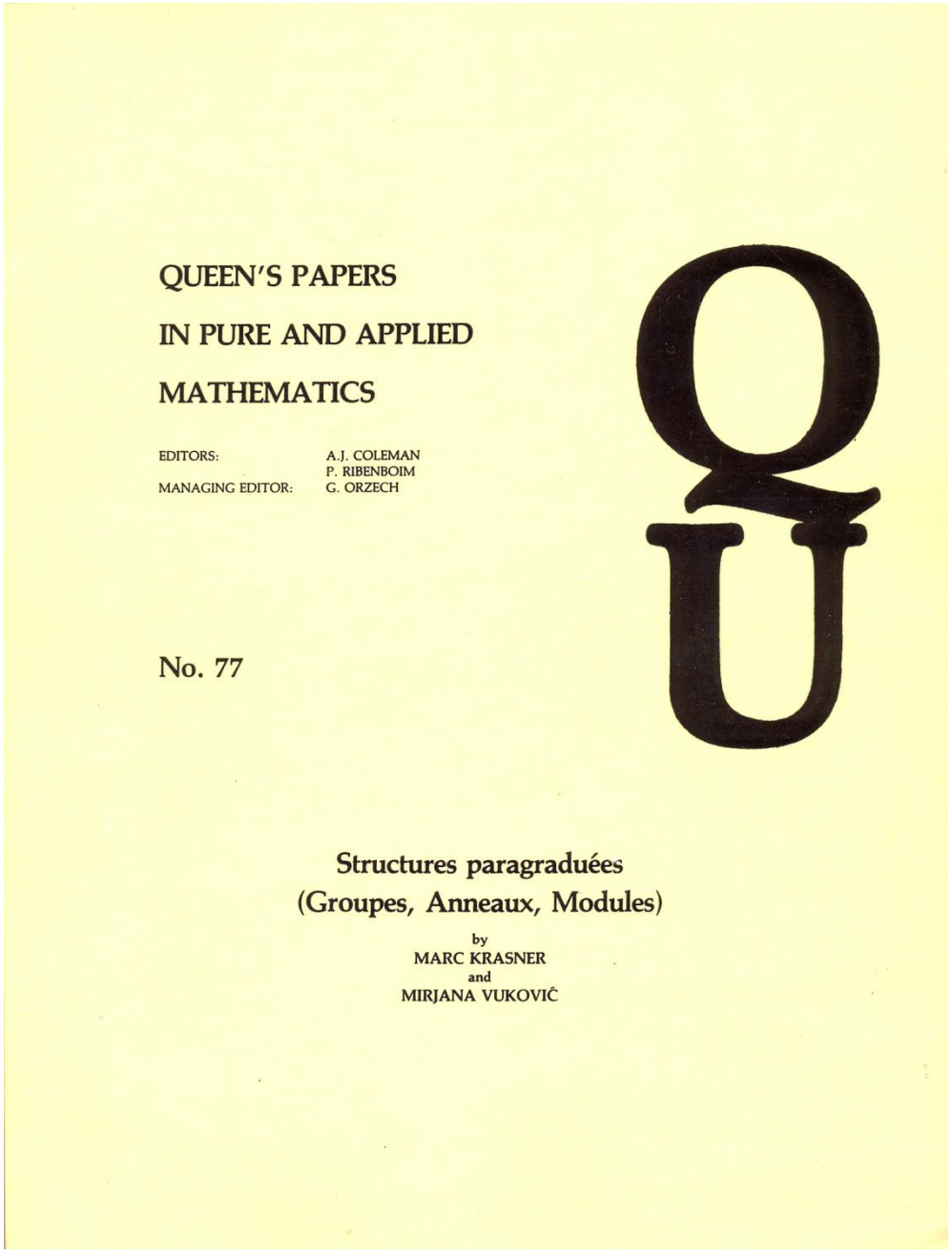
***STRUTURES PARAGRADUEES* (*groupes, anneaux, modules*)**
***PARAGRADED STRUCTURES* (*Groups, Rings, Modules*)**
Queen's Papers in pure and applied Mathematics, No. 77, Queen's University, Kingston, ON (1987), pp. VIII + 163.



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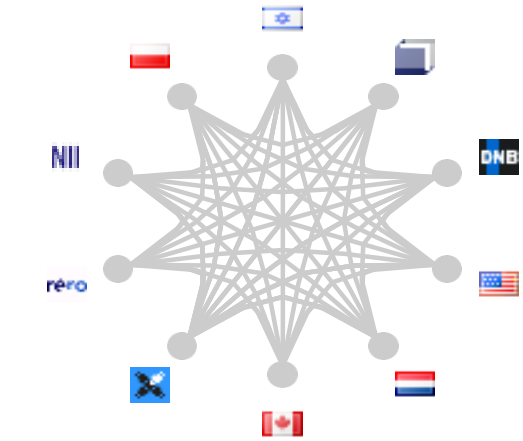

REVIEW OF THE MONOGRAPH

***Paragraded Structures* (*Groups, Rings, Modules*), Queen' s Papers in pure and applied Mathematics, No. 77, Queen's University, Kingston, ON (1987), pp. VIII + 163.**

"In this monograph Marc Krasner and Mirjana Vuković developed a general theory of algebraic graded structures (groups, rings, modules), which are at the same time a generalization of the classical graduation, as defined by Bourbaki, as well as an extension of the work ***"Anneax graduées généraux"*** (*Colloque d' Algèbre*, Rennes (1980), pp. 209-308), done by M. Krasner in which is included also the notion of a graduation introduced by M. Krasner in 1940's in his paper ***"Une généralisation de la notion de corps-corpoide. Un corpoide remarquable de la theory des corps values"*** (*Comptes Rendus, T.* 219 (1944), 345-347). One of the most important properties of the paragraded structures (groups, rings, modules) is that the homogeneous part of the direct product and the direct sum is the direct product and the direct sum of the corresponding homogeneous parts, respectively. This work extends many classical results and opens a completely new branch of the research." (*V. Perić*)

The monograph *"**Structures paragraduées* (*groupes, anneaux, modules*)**"* consists of detailed **Introduction** (**pp. 1-54**), where the historical developement of the graded structures (groups, rings, modules) are presented and a motivation for the introduction of more general paragraded and extragraded structures, and sketch of table of contents is given. The main part of this monograph is divided into the following three chapters:

The **Chapter I** (**pp. 55-98**), called ***Extragraded and paragraded groups,*** the **Chapter II** (**pp. 99-127**), where we observe ***structures derived from paragraded groups***, and the **Chapter III** (**pp. 128-162**) dedicated to ***paragraded rings*** and ***modules*** which are defined from paragraduation of the additive groups similarly as in the case of the graduation.

The ***graduation of a multiplicative group*** G is defined as a mapping $\gamma : \Delta \to Sg(G)$ which joins an element $\delta \in \Delta$ to some subgroup $G_\delta \in Sg(G)$ of the group G, such that the group G is the direct sum of the groups G_δ ($\delta \in \Delta$). The elements of a set Δ are called grades (degrees) and they are essential if $G_\delta \neq \{1\}$, and empty if $G_\delta = \{1\}$. The set $H = \cup_{\delta \in \Delta} G_\delta$ is the ***homogeneous part*** of the graded group G, and elements of that set are ***homogeneous elements***. If there are empty grades, then we usually distinguish one of them and mark it as 0, that is grade of 1.

The theory of graduation by Marc Krasner has three aspects:
For the homogeneous element $x \neq 1$, there is exactly $\delta = \delta(x) \in \Delta$ for which $x \in G_\delta$ and we call it the ***grade of the homogeneous element*** x.

1. ***non-homogeneous***, where graded group G is observed as the group with certain graduation;
2. ***semi-homogeneous***, where the group (G, H) is observed as pair of the group G and the so-called ***homogroupoid*** $H \subseteq G$, so possiblly graduation of the group G can be reconstructed (determined up to the equivalence) using the homogeneous part H;
3. ***homogeneous***, where we observe the homogroupoid (H; $\cdot$) and possibly we can construct a so-called ***linearization*** (H; $\cdot$) of this homogroupoid, which is in fact the graded group for which (H; $\cdot$) is the homogeneous part.

The homogroupoids are certain partial structures (given by axioms **1°° - 4°°**, **pp. 5-6**).

The graduation of a ring (A,+, $\cdot$) is a γ: $\Delta \to Sg$(A) of its additive group (A, +), such that for every pair of grades $\xi, \eta \in \Delta$ there exists $\varsigma \in \Delta$ with $A_\xi A_\eta \subseteq A_\varsigma$.

The homogeneous part (H,+, $\cdot$) of a graded ring (A,+, $\cdot$), as the partial structure of the ring induced by the ring structure can be characterized with a certain system of axioms (**1°- 6°, pp. 15**). Such partial structures are called ***aneids***. The unitary graded ring is the graded ring with the unit element which is homogeneous. If also $H^* = H\backslash\{0\}$ is the multiplicative group, then the graded ring (A,+, $\cdot$) is called a ***graded corp*** and the ***aneid*** (H,+, $\cdot$) is called a ***corpoid***.

Every aneid (H,+, $\cdot$) determines a ***graded ring*** with homogeneous part H up to the H-isomorphism which is called a ***linearization*** of the aneid H and is denoted by (H,+, $\cdot$).

The ***graded module*** is the graded Abelian group (M, +) which is a module over the graded ring (A,+, $\cdot$), such that for every grade $\delta \in \Delta$ of the ring and for every grade $d \in D$ of the group there exists $t \in D$ of the group for which we have $A_\delta M_d \subseteq M_t$. In the homoneous part N of the group M we have apart from partial operation + and externel partial operation on the aneid (H,+, $\cdot$) which determine a partial structure (H, N) of so-called ***H-moduloid.***

The graded groups (rings, modules) are objects of a certain category whose ***morphisms*** are the so-called ***homomorphisms.*** The category is not closed with respect to the direct product in which the homogeneous part would be the direct product of the homogeneous parts of the factors. Similar holds for the ***homogroupoids*** (***aneids, moduloids***).

The chapter I is divided into three paragraphs. In the first one the theory of paragraded and extagraded groups is presented from the non-homogeneous aspect.

First, an extragraduation of the group G is defined as a generalization of the graduation. It is a certain mapping E: $\Delta \to Sg(G)$ of a partially ordered set $(\Delta , <)$, which is from bellow complete semi-lattice and from beyond inductively ordered set $Sg(G)$ of the subgroups of the group G, which fulfill a certain system of axioms (**I - VI, pp. 56- 60**). One of them (**axiom IV**) requests that H has to generate the group G. Four axioms result with certain commutative relations in H, which together with a relation of the form $xy = z$ in H, create a set of relations in H. If the mapping E fulfills only first five axioms, and G is generated with H and the system of the relations $\mathbb{R}$, then E is called a paragraduation of the group G. The main result of this paragraph is the **Theorem 1 (p. 66)** which states that each extragraduation is at the same time a paragraduation. The homogeneous part is, of course, the set $H = \cup_{\delta \in \Delta} G_\delta$, while the grade of the homogeneous element x is defined as $inf \{ \delta \in \Delta : x \in G_\delta \}$.

Each graduation is a special paragraduation. Namely, it is sufficient that the set of empty grades is well ordered, taking the empty grade 0 as its first element, and after all of them to put mutually incomparable essential grades of the given graduation.

In ***the second paragraph*** of the first chapter we observe extragraduation and paragraduation from the semi-homogeneous aspect. There, the proof of the **Theorem 4** (**pp. 87-90**) is given apart the other things, which gives sufficient and necessary conditions for the certain subset H of the group G to be the homogeneous part of that group for the ***paragraduation***, resp. ***extragraduation.***

The third paragraph has a homogeneous aspect. Some of the conditions (**1º, 4º**, and partially of **condition 5º**) from the **Theorem 4** are translated to the language of the partial operation in H (**conditions II* - IV***) and to these we add condition **I*,** so H has a unit element 1. The partial structure $(H, \cdot)$ which fulfills those conditions is called a ***quasigroupoid.*** The so-called ***morphisant function*** $u(x, y)$ determines some commutative relations which together with the relation of the form $xy = z$ in H, in the paragroupoid H, creates a set of the relations $\mathbb{R}$.

If $\boldsymbol{F}$ is a free group generated by the set H, and N_u is its normal subgroup which corresponds to the set of the relations $\mathbb{R}$, then the mapping

$$u^*: x \to xN_u \ (x \in H)$$

is a homomorphism in the sense that for every $x, y \in H$ with $xy \in H$,

$$u^*(xy) = u^*(x)\, u^*(y).$$

In order that the paragraded group G exists with H as its homogeneous part, it is necessary and sufficient that the following conditions are fulfilled:

1) $x N_u \cap H = \{ x \}$, $(x \in H)$ and
2) $xyN_u \cap H \neq \emptyset \Rightarrow xy \in H$ (**p. 94**).

In that case H is called an ***u-paragroupoid***, and the corresponding paragraded group $(H, \cdot, u)$ is called its ***linearization,*** while the function u is called the ***linearizant function***. In the case that $(H, \cdot, u)$ is an extragraded group, the partial structure $(H, \cdot, u)$ is called an ***u-extragroupoid***. According to the **Theorem 4,** it is the case when the transcription **V*** of the **conditions 5º** holds (**pp. 95-96**).

In Chapter II we introduce structures derived from the paragraded groups. We start with homogeneous groups and prove that from the paragraduation (extragraduation) of the group G we get the corresponding graduation of its homogeneous subgroup g **(Lemma 1.),** show how the ***homogeneous part*** h of the homogeneous group g can be characterised (**Lemma 3**.), in other words a ***subgroupoid h*** of the ***extragroupoid*** H **(Lemma 4.).**

Further, we define a factor structure H/h and for that structure we proved ***The first*** and ***The second theorem of isomorphism***. With the help of the ***quasihomogeneous isomorphism*** Φ of the paragraded group $(G,\cdot, E)$ to the paragraded group $(G',\cdot, E')$, $(G,\cdot, E^{(\Phi)})$ – the so-called ***aglutinate*** of $(G,\cdot, E)$ is defined by Φ, which is a paragraded group if $(G',\cdot, E')$ is.

For the family of paragraded groups the direct product, and the direct sum is defined, in the natural way. Those are, again paragraded groups, and the direct sum is a homogeneous subgroup of the direct product. The direct product (direct sum) of the paragraded groups is an extragraded group, if and only if all of the factors are extragraded groups. However, the direct product of the graded groups is only an extragraded group, except if at most one of the factors is with non-trivial graduation, i.e. $\Delta_\alpha \neq \{ 0 \}$.

Corresponding construction of the direct product of the ***quasi-groupoids*** gives a so-called ***multigroupoid***. In the set of $Hom(H, H')$ of the quasi-homomorphisms of the quasigroupoid $(H, +)$ to the quasigroupoid $(H',+)$, the additive operation is defined which makes it into a commutative quasigroupoid.

If $(H', +)$ is ***commutatively linearizable***, that is if there exists **0 – *linearization*** $(H',+,0)$ such that the direct product $(H',+,0)^H$ is one commutative linearization of the direct product $(H',+)^H$. In this case **(*Hom*(H, H'), +)** is a paragroupoid.

Chapter III is dedicated to **p*aragraded rings*** and ***modules,*** which are obtained from the paragraduation of correspondings additive groups similarly as in the case of the graduation. We will not be discussing in details the contents of this chapter, also very rich with the results.

„From what has been here said, it can be seen that large work *"Paragraded structures"* contains many important results. They refer to first time introduced here, paragraded structures (groups, rings, modules) that generalize earlier known graded structures, with good justifications for that generalization.

In this work M.Krasner and M.Vuković developed a general theory of algebraic graded structures which is at the same time a generalization of classical graduation, as defined by Bourbaki, as well as an extension of the work done by M. Krasner (*Anneax gradués généraux, Colloque d'algèbre, Rennes* (1980), pp. 209 – 308). So, this work extends many classical results and opens a completely new branch of research.“ (*Academician Veselin Perić*)

It is safe to say that, in addition to the connection with the ***Kurosh's famous Russian school of algebra*** I am double theards to one of the oldest and most famous School of Mathematics, both through:
- ***academician Manojlo Maravić*** and ***the well-known Belgrade School of analysis of Mihailo Petrović Alas*** and

- algebra and Marc Krasner.