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THE EXISTENCE OF LI-YORKE CHAOS IN CERTAIN PREDATOR-PREY SYSTEM OF DIFFERENCE EQUATIONS

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Dedicated to the 70th birthday of the dear Professor Mustafa R. S. Kulenović, our teacher and mentor

ABSTRACT. This paper investigates an autonomous predator-prey system of difference equations with three equilibrium points and exhibits chaos in the sense of Li-Yorke in the positive equilibrium point. Numerical simulations are presented to illustrate our results.

1. INTRODUCTION AND PRELIMINARIES

A simple criterion for chaos in one dimensional discrete dynamical systems, "period three implies chaos" was provided by Li and Yorke [12]. This definition is the first description of chaos. F. R. Marotto mentioned that the essential properties of chaos are the following: there exist an infinite number of periodic solutions of various periods; there exists an uncountably infinite set of points which exhibit random behavior; and there is a high sensitivity to initial conditions [13–15]. Marotto extended Li-Yorke chaos in one-dimension to multi-dimension through introducing the notion of snap-back repeller by his famous theorem in 1978, a few years after Li and Yorke defined for chaos. Due to a technical flaw, Marotto redefined a snap-back repeller in 2005 [15]. We will give Marotto's definition for a "snap-back repeller" and then his theorem, which are quoted from [13] and [15].

Definition 1.1. *[15] Suppose $\overline{z}$ is a fixed point of a map T with all eigenvalues of* $\det J_T(\overline{z})$ *exceeding* 1 *in magnitude, and suppose there exist a point $z_0 \neq \overline{z}$ in a repelling neighbourhood $B_r(\overline{z})$ of $\overline{z}$ and an integer $M > 1$, such that $z_M = \overline{z}$ and* $\det J_T(z_k) \neq 0$ *for $1 \leq k \leq M$, where $z_k = T^k(z_0)$. Then $\overline{z}$ is called a snap-back repeller of T.*

Remark 1.1. [15] It is easy to see that Definition 1.1 still implies that the sequence $\{z_k\}_{k=-\infty}^{M}$, where $z_{k+1} = T(z_k)$ for all $k < M$, satisfies $z_M = \overline{z}$ and $z_k \to \overline{z}$ as $k \to -\infty$, making this set of points a homoclinic orbit. Also, since all z_k for $k \leq 0$ lie within

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the local unstable manifold of the map T at the fixed point $\bar{z}$, where T is $1-1$, and since $\det J_T(z_k) \neq 0$ for $1 \leq k \leq M$, then this homoclinic orbit is transverzal in the sense that T is $1-1$ in a neighborhood of each z_k for all $k \leq M$.

Theorem 1.1. *[13] If a map T possesses a snap-back repeller, then T is chaotic in the sense of Li-Yorke. That is, there exist*

1. *a positive integer N, such that T has a point of period p, for each integer $p \geq N$,*
2. *a "scrambled set" of T, i.e., an uncountable set S containing no periodic points of T, such that*
   a) $T(S) \subset S$
   b) $\limsup_{n\to\infty} \|T^n(x) - T^n(y)\| > 0$ *for all $x, y \in S$, with $x \neq y$,*
   c) $\limsup_{n\to\infty} \|T^n(x) - T^n(y)\| > 0$ *for all $x \in S$, with $x \neq y$ and periodic point y of T,*
3. *an uncountable subset S_0 of S such that* $\liminf_{n\to\infty} \|T^n(x) - T^n(y)\| = 0$, *for every $x, y \in S_0$.*

In this paper, we investigate Li-Yorke chaos in a positive equilibrium point of an autonomous system of difference equations of the predator-prey type

$$\begin{aligned} x_{n+1} &= ax_n(1-x_n) - bx_ny_n \\ y_{n+1} &= -cy_n + dx_ny_n, \end{aligned} \tag{1.1}$$

where x_n and y_n represent population density of a prey and predator, respectively, and a, b, c and d are positive parameters, considered in [20]. See also [10], Problem 21, p.185 for a similar system that appears in the fishery. This is the well known Lotka-Volterra predator-prey model, which is one of the most important population models. Here a represents the natural growth rate of the prey in the absence of predators, b represents the effect of predation on the prey, c represents the natural death rate of the predator in the absence of prey, and d represents the efficiency and propagation rate of the predator in the presence of prey. In recent years, the study of the complex dynamics of the predator-prey models, including aspects such as stability, periodic solutions, bifurcations, and chaotic behavior, has drawn the attention of many excellent researchers.

System (1.1) was investigated in [20]. Authors first showed that by using appropriate change of variables, parameter b can be eliminated from System (1.1), then they gave necessary and sufficient conditions for the existence and local stability of the equilibrium points. They proved that there are flip and Neimark-Sacker bifurcations. Our the goal is to show the existence of a chaotic phenomenon in the sense of Li-Yorke. Similar dynamics have been proven in [2], where the authors investigated a discrete-time predator-prey system with the Allee effect, and they proved that there are flip and Hopf bifurcations, and there exists a chaotic the phenomenon in the sense of Li-Yorke.

In [20] it was shown that System (1.1), for $b = 1$, i.e.

$$\begin{aligned} x_{n+1} &= ax_n(1-x_n) - x_ny_n \\ y_{n+1} &= -cy_n + dx_ny_n, \end{aligned} \tag{1.2}$$

has a unique extinction equilibrium point $E_1(0,0)$, a unique exclusion equilibrium point $E_2\left(\frac{a-1}{a},0\right)$ for $a>1$, and a unique coexistence equilibrium point $E_3(\bar{x},\bar{y}) = \left(\frac{1+c}{d}, \frac{d(a-1)-a(1+c)}{d}\right)$ for $d > \frac{a(1+c)}{a-1}$ and $a>1$. Therefore, $\bar{z} = E_3(\bar{x},\bar{y})$ is the unique positive equilibrium point of System (1.2), which is locally asymptotically stable if and only if one of the following conditions holds:

(a) $1 < a \leq 3$, $c > 0$ and $\frac{a(1+c)}{a-1} < d < \frac{a(2+c)}{a-1}$;
(b) $3 < a \leq 5$, $c > 0$ and $\frac{a(1+c)(3+c)}{3+a-c+ac} < d < \frac{a(2+c)}{a-1}$;
(c) $5 < a \leq 9$, $0 < c < \frac{9-a}{a-5}$ and $\frac{a(1+c)(3+c)}{3+a-c+ac} < d < \frac{a(2+c)}{a-1}$.

Also, in [20] it was shown that System (1.2) undergoes a flip bifurcation and a Neimark-Sacker bifurcation. Some numerical simulations were presented to exhibit the complex dynamical behaviors, such as the period $6,16,18,20,21,24,27$, and 37 orbits, attracting invariant cycles, quasi-periodic orbits, chaotic behaviors, which appear and disappear suddenly, coexisting chaotic attractors, etc.

2. LI-YORKE CHAOS

In this section, we prove that System (1.2) exhibits chaos in the sense of Li-Yorke in a positive equilibrium point and present the conditions for the existence of chaotic behavior.

First, we give the conditions under which the positive equilibrium point $\bar{z} = \left(\frac{1+c}{d}, \frac{d(a-1)-a(1+c)}{d}\right)$ of System (1.2) is a snap-back repeller. The corresponding map associated with System (1.2) is given with

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax(1-x)-xy \\ -cy+dxy \end{pmatrix}. \tag{2.1}$$

The Jacobian matrix is of the form

$$J_T(x,y) = \begin{pmatrix} a-2ax-y & -x \\ dy & -c+dx \end{pmatrix},$$

which implies

$$\mathrm{tr} J_T(x,y) = (d-2a)x - y + a - c$$

and

$$\det J_T(x,y) = -2adx^2 + a(2c+d)x + c(y-a). \tag{2.2}$$

The corresponding characteristic equation is

$$\lambda^2 - \mathrm{tr}(J_T(x,y))\lambda + \det J_T(x,y) = 0, \tag{2.3}$$

which in a positive equilibrium $\bar{z}$ becomes

$$\lambda^2 + \frac{a+ac-2d}{d}\lambda - \frac{a(c+1)(2+c-d)+cd}{d} = 0. \tag{2.4}$$

In order to prove the existence of Li-Yorke chaos, we will consider the corresponding eigenvalues which are complex-conjugate with modulus greater then one. It is equivalent with the following conditions

$$\left.\begin{array}{r}\left(\frac{a+ac-2d}{d}\right)^2+4\left(\frac{a(c+1)(2+c-d)+cd}{d}\right)<0\\ -\frac{a(c+1)(2+c-d)+cd}{d}>1\end{array}\right\},$$

see [10]. It implies

$$d>d_{\max},$$

where

$$d_{\max}=\begin{cases}\frac{a(2+c)}{a-1} & \iff \quad 1<a\leq\frac{5c+9}{c+1},\\ \frac{a\left(c+1+\sqrt{(c+1)(a+c)}\right)}{2(a-1)} & \iff \quad a>\frac{5c+9}{c+1}.\end{cases} \tag{2.5}$$

Our next step is to determine a neighborhood $U_{\bar{z}}$ of $\bar{z}=(\bar{x},\bar{y})$ in which the norms of eigenvalues exceed 1 for all $(x,y)\in U_{\bar{z}}$. That means that we need to solve the following system of inequalities

$$\left.\begin{array}{r}Z_1(x,y)=(-\mathrm{tr}J_T(x,y))^2-4\det J_T(x,y)<0,\\ Z_2(x,y)=\det J_T(x,y)-1>0,\end{array}\right\}$$

where

$$\begin{aligned}Z_1(x,y)&=(2a+d)^2x^2-2(2a+d)(a+c)x+y^2\\&\quad+2(2a-d)xy-2(a+c)y+(a+c)^2,\\ Z_2(x,y)&=-2adx^2+a(2c+d)x+cy-ac-1.\end{aligned}$$

The quadratic curves $Z_1(x,y)=0$ and $Z_2(x,y)=0$ represent an ellipse and a parabola, respectively. For equation $Z_1(x,y)=0$ we have

$$y'=-\frac{\frac{\partial Z_1(x,y)}{\partial x}}{\frac{\partial Z_1(x,y)}{\partial y}}=-\frac{(2a+d)^2x+(2a-d)y-(2a+d)(a+c)}{(2a-d)x+y-c-a}=0$$

if

$$\begin{gathered}(2a+d)^2x+(2a-d)y-(2a+d)(a+c)=0,\\ (2a-d)x+y-c-a\neq 0.\end{gathered}$$

The last equations combining with equation $Z_1(x,y)=0$ provides that the minimum value of y along the ellipse $Z_1(x,y)=0$ is

$$y_{min}=y\left(\frac{a+c}{2a+d}\right)=0,$$

and the maximum value of y along the ellipse $Z_1(x,y)=0$ is

$$y_{max}=y\left(\frac{(a+c)d}{2a(2a+d)}\right)=\frac{(a+c)(2a+d)}{2a}.$$

The ellipse $Z_1(x,y)=0$ intersects axes O_x and O_y in the following points

$$E_x=6\left(\frac{a+c}{2a+d},0\right),\quad E_y=(0,a+c).$$

Let first find a region $I_1 \subset \mathbb{R}^2$ where $Z_1(x,y) < 0$ for all $(x,y) \in I_1$. If D_1 is the corresponding discriminant, then we have

$$D_1 = 16dy\left(-2ay + (2a+d)(a+c)\right) > 0 \Longleftrightarrow 0 < y < \frac{(a+c)(2a+d)}{2a}.$$

Now,

$$Z_1(x,y) < 0 \Longleftrightarrow (x,y) \in I_1,$$

where

$$I_1 = \left\{(x,y) : x \in (\widehat{x}_-, \widehat{x}_+),\ y \in \left(0, \tfrac{(a+c)(2a+d)}{2a}\right)\right\},$$

and

$$\widehat{x}_\pm = \tfrac{(-2a+d)y+(a+c)(2a+d)\pm 2\sqrt{d((a+c)(2a+d)-2ay)y}}{(2a+d)^2}.$$

Let us now find a region $I_2 \subset \mathbb{R}^2$ such that $Z_2(x,y) > 0$ for all $(x,y) \in I_2$. For the corresponding discriminant $D_2 = a\left(a(2c+d)^2 + 8d(-ac+cy-1)\right)$, we have that

$$D_2 > 0 \Longleftrightarrow y > \frac{-ad^2 + 4(ac+2)d - 4ac^2}{8cd}.$$

Hence,

$$Z_2(x,y) > 0 \Longleftrightarrow (x,y) \in I_2,$$

where

$$I_2 = \left\{(x,y) : x \in (\widetilde{x}_-, \widetilde{x}_+),\ y \in \left(\tfrac{-ad^2+4(ac+2)d-4ac^2}{8cd}, +\infty\right)\right\}$$

and

$$\widetilde{x}_\pm = \tfrac{a(2c+d)\pm\sqrt{a(ad^2-4(ac-2cy+2)d+4ac^2)}}{4ad}.$$

Therefore, the region $U_{\bar{z}} = I_1 \cap I_2$, where $Z_1(x,y) < 0$ and $Z_2(x,y) > 0$, is of the form

$$U_{\bar{z}} = \left\{(x,y) : x \in (\widetilde{x}_-, \widetilde{x}_+) \cap (\widehat{x}_-, \widehat{x}_+),\ y \in \left(\widehat{y}_-, \tfrac{(a+c)(2a+d)}{2a}\right)\right\} \tag{2.6}$$

with

$$\widehat{y}_- = \max\left\{0, \frac{-ad^2 + 4(ac+2)d - 4ac^2}{8cd}\right\}.$$

It is easy to check that

$$\frac{-ad^2 + 4(ac+2)d - 4ac^2}{8cd} < \frac{(a+c)(2a+d)}{2a}$$

holds if $d > \frac{a(2+c)}{a-1}$.

Also, notice that the area $U_{\bar{z}}$ is not an empty set since $\bar{z} \in U_{\bar{z}}$ for $d > d_{\max}$.

Thus, we have the following result.

Lemma 2.1. *Let $a > 1$, $c > 0$ and $d > d_{\max}$, where $d_{\max}$ is given by* (2.5)*. Then $U_{\bar{z}}$ defined by (2.6) is a repelling area of the equilibrium point $\bar{z}$.*

In order to prove that equilibrium point $\bar{z} = (\bar{x}, \bar{y}) = \left(\frac{1+c}{d}, \frac{d(a-1)-a(1+c)}{d}\right)$ is a snap-back repeller, our next step is to find one point $z_0 = (x_0, y_0) \in U_{\bar{z}}$ such that $z_0 \neq \bar{z}$, $T^M(z_0) = \bar{z}$ for some positive integer M and $\det J_T(z_k) \neq 0$ for $1 \leq k \leq M$, where map T is defined by (2.1). Let it be $M=2$. Then, we need to find a point $z_0=(x_0,y_0)\in U_{\bar{z}}$ and point $z_1=(x_1,y_1)\notin U_{\bar{z}}$ such that

$$z_1=T(z_0),\ z_2=T(z_1)=T^2(z_0)=\bar{z}\ \text{ and } \det J_T(z_1)\neq 0,$$

since

$$\det J_T(z_2)=\det J_T(\bar{z})=-\frac{a(c+1)(2+c-d)+cd}{d}>1.$$

By calculating the inverse iterations of the fixed point $\bar{z}$ twice, we are looking for the point $z_0=(x_0,y_0)$, x_0, $y_0>0$, as the solution of the system

$$\left.\begin{array}{r} ax(1-x)-xy=x_1 \\ -cy+dxy=y_1 \end{array}\right\} \tag{2.7}$$

for $z_1=(x_1,y_1)$ which satisfies the system

$$\left.\begin{array}{r} ax_1(1-x_1)-x_1y_1=\bar{x} \\ -cy_1+dx_1y_1=\bar{y} \end{array}\right\}. \tag{2.8}$$

From second equation in (2.8) we have $y_1=\frac{\bar{y}}{-c+dx_1}$ for $x_1\neq\frac{c}{d}$ and it implies

$$\begin{aligned} y_1<0 \quad &\text{if} \quad x_1<\frac{c}{d}, \\ y_1>0 \quad &\text{if} \quad x_1>\frac{c}{d}. \end{aligned}$$

Further, we get

$$ax_1(1-x_1)-x_1\left(\tfrac{\bar{y}}{-c+dx_1}\right)-\bar{x}=0,$$

from which

$$\frac{(c+1-dx_1)\left(adx_1^2+a(1-d)x_1+c\right)}{d(c-dx_1)}=0.$$

It is obvious that the equilibrium point $\bar{z}$ is one solution (for $x_1=\frac{c+1}{d}$). So, we can have at most two real solutions different from equilibrium point which we get by solving equation

$$adx_1^2+a(1-d)x_1+c=0. \tag{2.9}$$

A discriminant of Equation (2.9) is always positive for $a>1$ and $d>d_{\max}$. Indeed,

$$\begin{aligned} ad^2-2(a+2c)d+a &= (ad-2(a+2c))\,d+a \\ &> \left(a\left(\tfrac{a(2+c)}{a-1}\right)-2(a+2c)\right)d+a \\ &= \frac{(a-2)^2cd+2ad+a(a-1)}{a-1}>0 \end{aligned}$$

and Equation (2.9) has two positive (positivity is easy to prove) solutions

$$(x_1)_\pm = \frac{a(d-1) \pm \sqrt{a(ad^2 - 2(a+2c)d + a)}}{2ad},$$

so we have

$$(y_1)_\pm = \frac{\bar{y}}{-c + d(x_1)_\pm} = \frac{d(a-1) - a(c+1)}{d\left(-c + d(x_1)_\pm\right)}.$$

We get $\det J_T\left((z_1)_\pm\right) \neq 0$, where $(z_1)_\pm = \left((x_1)_\pm, (y_1)_\pm\right)$. Namely,

$$\det J_T\left((z_1)_+\right) = -\frac{\left(-a(3+2c-d) + \sqrt{a(ad^2-2(a+2c)d+a)}\right)\sqrt{a(ad^2-2(a+2c)d+a)}}{2ad} \neq 0,$$

if $ad^2 - 2(a+2c)d + a \neq a(3+2c-d)^2$, which is true for $d \neq \frac{a(c+2)(c+1)}{a+(a-1)c}$. The last condition is always satisfied because $\frac{a(c+2)(c+1)}{a+(a-1)c} < \frac{a(2+c)}{a-1} < d$. Also, it easy to see that $\det J_T\left((z_1)_+\right) < 0 < 1$, which means $\left((z_1)_+\right) \notin U_{\bar{z}}$. Similarly, we conclude $\left((z_1)_-\right) \notin U_{\bar{z}}$.

Hence, we have two points

$$(z_1)_\pm = \left(\frac{a(d-1) \pm \sqrt{a(ad^2-2(a+2c)d+a)}}{2ad}, \frac{a(2c-d+1) \pm \sqrt{a(ad^2-2(a+2c)d+a)}}{2cd}\right).$$

In the following analysis we will solve System (2.7) for $z_1 = (z_1)_+$, i.e. we will restrict our consideration to the case when both coordinates are positive. From the second equation in System (2.7) we get

$$y = \frac{y_1}{-c + dx}$$

and since $y_1 > 0$, it implies $x > \frac{c}{d}$. After substituting y in the first equation of System (2.7), we obtain

$$\left.\begin{array}{l} ax(1-x) - \frac{xy_1}{dx-c} - x_1 = 0 \\ y = \frac{y_1}{dx-c} \end{array}\right\}$$

assuming that $x > \frac{c}{d}$. Let

$$h(d,x) = ax(1-x) - \frac{xy_1}{dx-c} - x_1.$$

Considering the fact that $\bar{x} = \frac{c+1}{d}$, it holds

$$h(d,\bar{x}) = 0 \Longleftrightarrow \frac{a\Upsilon(a,c,d) + (a(c+1)+cd)\sqrt{a(ad^2-2(a+2c)d+a)}}{2acd^2} = 0 \tag{2.10}$$

where

$$\Upsilon(a,c,d) = cd^2 - (a + c + (2c+3)ac)d + a(c+1)\left(2c^2 + 4c + 1\right).$$

Equation (2.10) possesses a solution only if

$$\Upsilon(a,c,d) < 0, \tag{2.11}$$

and in that case, Equation (2.10) is equivalent to

$$\Upsilon_1(a,c,d) \cdot \Upsilon_2(a,c,d) = 0,$$

where

$$\Upsilon_1(a,c,d) = a(c+2)(c+1) - d(a+(a-1)c),$$
$$\Upsilon_2(a,c,d) = (a+(a+1)c)d^2 - a(c+1)(a+(a+1)c)d + a^2(c+1)^3.$$

Now, we have the following two cases.
a) Notice $\Upsilon_1(a,c,d) = 0$ if $d = \frac{a(c+1)(c+2)}{(a+(a-1)c)}$. Since

$$\frac{a(c+2)(c+1)}{(a+(a-1)c)} - \frac{a(c+2)}{a-1} = -\frac{a(c+2)}{(a-1)(a+(a-1)c)} < 0$$

it implies

$$d < d_{\max},$$

so d is not appropriate.
b) $\Upsilon_2(a,c,d) = 0$ implies

$$d_{\pm} = \frac{a(c+1)\left((a+c+ac) \pm \sqrt{(a(c+1)-3c-4)(a+c+ac)}\right)}{2(a+c+ac)}$$

for $a \geq \frac{3c+4}{c+1}$. It has to be $\Upsilon(a,c,d_+) < 0$, or equivalently

$$-\frac{\Upsilon_3(a,c) + \left(a(c+1)^2 + c\right)\sqrt{(a(c+1)-3c-4)(a+c+ac)}}{2(a(c+1)+c)} < 0,$$

where

$$\Upsilon_3(a,c) = a^2(c+1)^3 - a(c+1)\left(c^2+4c+2\right) - c\left(4c^2+7c+2\right).$$

Since $\Upsilon_3(0,c) < 0$ and $\Upsilon_3\left(\frac{3c+4}{c+1}, c\right) = 2(c+1)(c+2)^2 > 0$, it is obviously $\Upsilon_3(a,c) > 0$ and $\Upsilon(a,c,d_+) < 0$ are satisfied.
Let us now check when $d_+ > d_{\max}$.
1.) We see that $d_{\max} = \frac{a(2+c)}{a-1}$ if $\frac{3c+4}{c+1} \leq a \leq \frac{5c+9}{c+1}$, so $d_+ > d_{\max}$ if

$$\frac{a(c+1)\left((a+c+ac) + \sqrt{(a(c+1)-3c-4)(a+c+ac)}\right)}{2(a+c+ac)} > \frac{a(c+2)}{a-1},$$

i.e.

$$\frac{a(c+1)\left(\sqrt{(a(c+1)-3c-4)(a+c+ac)}\right)}{2(a+c+ac)} > -\frac{a(a(c+1)-3c-5)}{2(a-1)}$$

which is satisfied by assumption $\frac{3c+5}{c+1} \leq a \leq \frac{5c+9}{c+1}$.
For $\frac{3c+4}{c+1} \leq a < \frac{3c+5}{c+1}$ inequality $d_+ > d_{\max}$ holds if

$$\frac{(c+1)^2((a(c+1)-3c-4)(a+c+ac))}{(a+c+ac)^2} > \left(-\frac{a-3c+ac-5}{a-1}\right)^2,$$

i.e.

$$\frac{a^2(c+1)^2 + a(c+1)\left(c^2-c-4\right) - \left(9c+10c^2+3c^3+1\right)}{(a-1)^2(a+c+ac)} > 0,$$

from which

$$a > \frac{-\left(c^2-c-4\right)+(c+2)\sqrt{(c+5)(c+1)}}{2(c+1)}.$$

So $d_+ > d_{\max}$ if

$$\frac{-\left(c^2-c-4\right)+(c+2)\sqrt{(c+5)(c+1)}}{2(c+1)} < a \leq \frac{5c+9}{c+1}.$$

2.) If $a > \frac{5c+9}{c+1}$, then $d_{\max} = \frac{a\left(c+1+\sqrt{(c+1)(a+c)}\right)}{2(a-1)}$. So $d_+ > d_{\max}$ if

$$\frac{a(c+1)\left((a(c+1)+c)+\sqrt{(a(c+1)-3c-4)(a(c+1)+c)}\right)}{2(a(c+1)+c)} > \frac{a\left(c+1+\sqrt{(c+1)(a+c)}\right)}{2(a-1)},$$

i.e.

$$\frac{(c+1)\sqrt{(a-3c+ac-4)(a(c+1)+c)}}{(a(c+1)+c)} > \frac{\left((c+1)(2-a)+\sqrt{(c+1)(a+c)}\right)(a(c+1)+c)}{(a-1)(a(c+1)+c)}.$$

For $a > \frac{5c+9}{c+1}$ the right side of the previous inequality is negative. Indeed,

$$\begin{aligned}\sqrt{(c+1)(a+c)} &\leq (c+1)(a-2)\\ &\Longleftrightarrow (c+1)(a+c) \leq (c+1)^2(a-2)^2\\ &\Longleftrightarrow (a+c) \leq (c+1)(a-2)^2\\ &\Longleftrightarrow a \geq \frac{3c+4}{c+1}.\end{aligned}$$

Now, we have that $d_+ > d_{\max}$ for $a > \frac{5c+9}{c+1}$.
Hence,

$$d^* = \frac{a(c+1)\left(a+c+ac+\sqrt{(a(c+1)-3c-4)(a+c+ac)}\right)}{2(a+c+ac)} \tag{2.12}$$

for

$$a > \frac{-\left(c^2-c-4\right)+(c+2)\sqrt{(c+5)(c+1)}}{2(c+1)}, \tag{2.13}$$

where $d^* = d_+$.

Remark 2.1. Similarly we can prove that d_- does not satisfy condition $d > d_{\max}$.

Function $h(d,x) = ax(1-x) - \frac{xy_1}{dx-c} - x_1$, is continuous under conditions $x > \frac{c}{d}$ and $d^* > d_{\max}$. Now, we have

$$\frac{\partial h(d,\bar{x})}{\partial x} = 0 \Longleftrightarrow \frac{a(d-2c-3)+\sqrt{a(ad-2(a+2c)d+a)}}{2d} = 0.$$

Let us show $\frac{\partial h(d^*,\bar{x})}{\partial x} \neq 0$. Indeed, $\frac{\partial h(d,\bar{x})}{\partial x} = 0$ for $d < 2c+3$ and

$$a^2(2c+3-d)^2 = a\left(ad^2-2(a+2c)d+a\right). \tag{2.14}$$

From (2.14) we get

$$d=\frac{a\,(c+1)\,(c+2)}{a+(a-1)\,c}$$

and we see that $d<2c+3$ holds. Also,

$$\frac{a\,(c+2)\,(c+1)}{(a+(a-1)\,c)}-\frac{a\,(c+2)}{a-1}=-\frac{a\,(c+2)}{(a-1)\,(a+(a-1)\,c)}<0.$$

is satisfied. So,

$$d=\frac{a\,(c+1)\,(c+2)}{a+(a-1)\,c}<\frac{a\,(c+2)}{a-1}<d^{*}.$$

Therefore, under certain conditions on the parameters, we have that

1° $h\,(d^{*},\overline{x})=0$,
2° $h\,(d,x)$ is continuous for $d>d_{\max}$ and $x>\frac{c}{d}$,
3° $\frac{\partial h(d^{*},\overline{x})}{\partial x}\neq 0$.

By Implicit Function Theorem there exist a unique function $x=x_0(d)$ and $\eta>0$ such that

(i) $x_0(d^{*})=\overline{x}$,
(ii) $h\,(d,x_0\,(d))=0$ for $d\in(d^{*}-\eta,d^{*}+\eta)$,
(iii) $x=x_0\,(d)$ is continuous in $d\in(d^{*}-\eta,d^{*}+\eta)$.

Let $M=2$, $z_0=(x_0,y_0)=\left(x_0,\frac{y_1}{-c+dx_0}\right)$. Then, z_0 belongs to $U_{\overline{z}}$ for $d-d^{*}$ small enough.
Finally, let

$$U^{*}=\left\{(x,y):\frac{(x-\overline{x})^{2}}{r_{x_0}^{2}}+\frac{(y-\overline{y})^{2}}{r_{y_0}^{2}}\leq 1\right\},$$

be a repelling neighborhood, where

$$r_{x_0}=|\overline{x}-x_0|+\delta_1,\quad r_{y_0}=\rho\,(\overline{z},z_0)+\delta_2,$$

$$\rho\,(\overline{z},z_0)=\sqrt{(\overline{x}-x_0)^{2}+(\overline{y}-y_0)^{2}},$$

and δ_1 and δ_2 are some positive small enough constants, such that $U^{*}\subset U_{\overline{z}}$. Then equilibrium point $\overline{z}=\left(\frac{1+c}{d},\frac{d(a-1)-a(1+c)}{d}\right)$ is a snap-back repeller in U^{*}.
So, we proved the following result.

Theorem 2.1. *Assume that the conditions in Lemma 2.1 hold. If the conditions (2.12) and (2.13) are satisfied, then there exists d near d^{*} such that $\overline{z}=(\overline{x},\overline{y})$ is a snap-back repeller of System* (1.2) *and consequently, System* (1.2) *is chaotic in the sense of Li-Yorke.*

3. The Lyapunov dimension of the attractor and numerical simulations

In many articles, the appearance of chaos is confirmed by positive Lyapunov coefficients (e.g., [7, 16]). But, in the previous section, we proved the existence of chaos, and in this section, we will make several appropriate numerical simulations.

By Kaplan and Yorke [8], and by Alligood, Sauer and Yorke [1] the Lyapunov dimension of the attractor for n-dimensional map T is defined as follows.

Definition 3.1. *[1] Let T be a map on $\mathbb{R}^n$. Consider an orbit with Lyapunov exponents $L_1 \geq L_2 \geq \cdots \geq L_n$, and let j denote the largest integer such that*

$$\sum_{k=1}^{j} L_k \geq 0.$$

Define the Lyapunov dimension d_L of the orbit by

$$d_L = D_{KY} = \begin{cases} 0 & \text{if} \quad \text{no such } j \text{ exists,} \\ j + \frac{1}{|L_{j+1}|} \sum\limits_{k=1}^{j} L_k & \text{if} \quad j < n, \\ n & \text{if} \quad j = n. \end{cases}$$

Specially, for two-dimensional map we have

$$d_L = 1 + \frac{L_1}{|L_2|}, \tag{3.1}$$

where

$$L_1 \geq 0 > L_2 \text{ and } |L_2| > L_1. \tag{3.2}$$

In this section, we will calculate of the Lyapunov dimension of the attractor.

Also, we illustrate phase portraits (the snap-back repeller with the corresponding area and chaotic attractor), bifurcation diagrams, maximum Lyapunov exponents corresponding to bifurcation diagrams for System (1.2) to demonstrate the above theoretical analysis and show the new attractive, complex dynamical behaviors by using numerical simulations.

Example 3.1. *For $a = 4.2, c = 0.1$ and $d = 3.0$ we have*

$$\bar{z} = (0.36667, 1.66), \quad z_0 = (0.37156, 0.87785), \quad z_1 = (0.65454, 0.89074),$$
$$\delta_1 = 0.026733, \quad \delta_2 = 0.21783$$

and

$$\rho(\bar{z}, z_0) = \sqrt{(0.36667 - 0.37156)^2 + (1.66 - 0.87785)^2} \approx 0.78217,$$

$$r_{x_0}^2 = (|0.36667 - 0.37156| + 0.026733)^2 = 0.001,$$
$$r_{y_0}^2 = (\rho(\bar{z}, z_0) + \delta_2)^2 = (0.78217 + 0.21783)^2 = 1,$$

$$U^{*}=\left\{(x,y):\frac{(x-0.36667)^{2}}{0.001}+\frac{(y-1.66)^{2}}{1}\leq 1\right\}\subset U_{\bar{z}}.$$

Figure 1 represents phase portrait with 10 *iterations with repelling area* $U_{\bar{z}}$ *and neighborhood* U^* *of the snap-back repeller* $\bar{z}$*. Figure 2 represents phase portrait with* 10050 *iterations.*

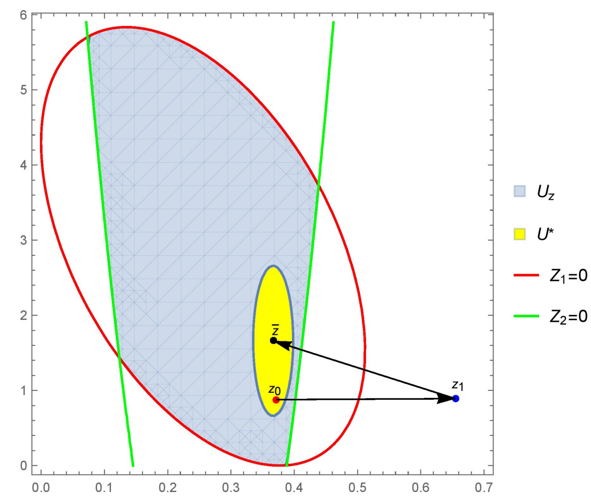

FIGURE 1. Snap-back repeller $\bar{z}=(0.36667,1.66)$

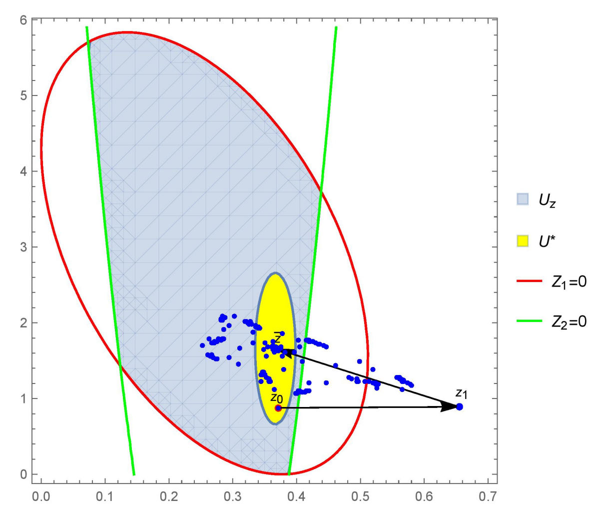

FIGURE 2. Chaotic attractor

In Figure 3 and 4 we plotted the bifurcation diagrams generated by code Bif2D [18] and maximum Lyapunov exponents generated by package lce [17].

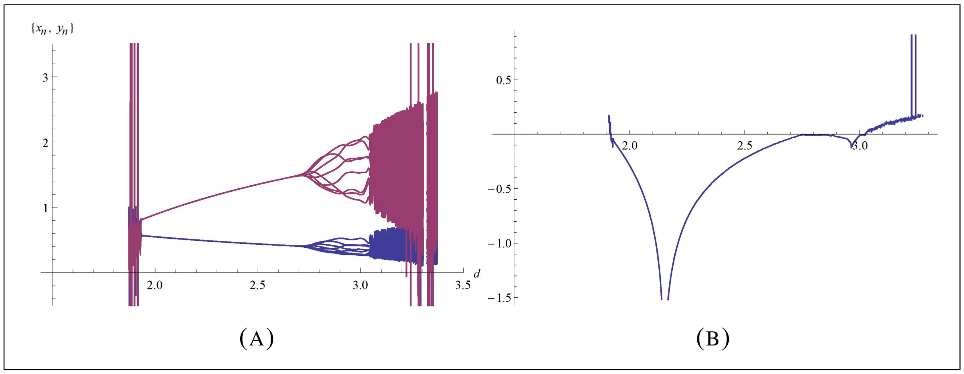

FIGURE 3. For $a = 4.2$, $c = 0.1$ and initial point $(0.3, 1.6)$:
(A) Bifurcation diagram for $d \in (1.44381, 3.5)$;
(B) The maximum Lyapunov exponents for $d \in (1.44381, 3.3)$

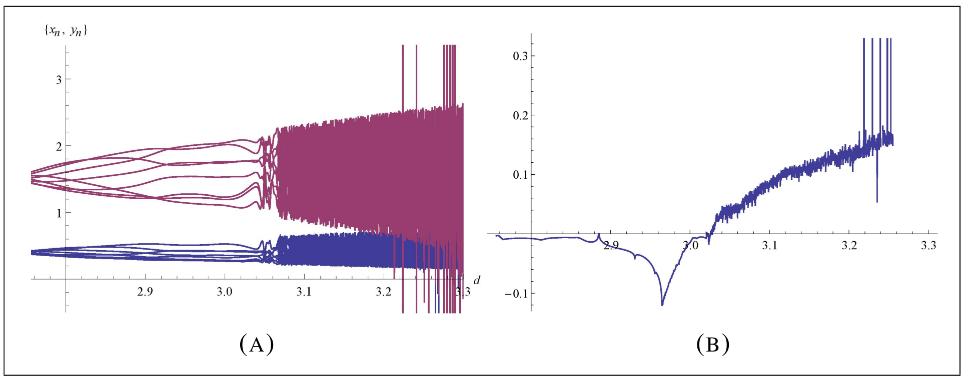

FIGURE 4. For $a = 4.2$, $c = 0.1$, $d \in (2.7563, 3.3)$:
(A) Bifurcation diagram; (B) The maximum Lyapunov exponents

Example 3.2. *For the values of parameters $a = 4.2$ and $c = 0.1$ we compute the Lyapunov dimension in the cases where conditions* (3.2) *are satisfied. In Figure 5 we see that conditions (3.2) are satisfied when d belongs to interval* $(3.02, 3.04)$. *By* (3.1)*, Lyapunov dimension for $d = 3.021$ is*

$$d_L = 1 + \frac{L_1}{|L_2|} = 1 + \frac{0.0009727049991202961}{0.006455582474271305} = 1.1507.$$

Other data for different values of parameter d are given in Table 1.

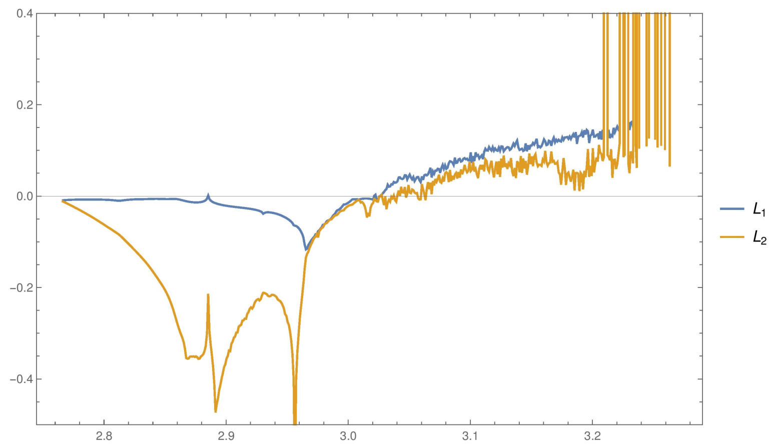

FIGURE 5. Lyapunov exponents L_1 and L_2 for $a = 4.2$, $c = 0.1$

d	For 500 iterations we have $\{L_1, L_2\}$ given with:	d_L
3.020	$\{-0.00771621443415771, -0.016385081323215037\}$	-
3.021	$\{\mathbf{0.0009727049991202961}, -\mathbf{0.006455582474271305}\}$	**1.1507**
3.022	$\{\mathbf{0.002805663684499562}, -\mathbf{0.007879936996153374}\}$	**1.3561**
3.023	$\{-0.0008108421924574601, -0.008248321634339448\}$	-
3.024	$\{-0.016420925424534723, -0.018735853081581006\}$	-
3.025	$\{-0.005298020860912034, -0.009123929620639026\}$	-
3.026	$\mathbf{0.0010739514748464175}, -\mathbf{0.0022726373249094198}\}$	**1.4726**
3.027	$\{0.004722314055858313, 0.004276530818779462\}$	-
3.028	$\{\mathbf{0.0007449252704960492}, -\mathbf{0.024522106324883518}\}$	**1.0304**
3.029	$\{0.004589591217875484, 0.0014255288558806855\}$	-
3.030	$\{0.011466987871733437, -0.006282177838434302\}$	-
3.031	$\{\mathbf{0.019241955983153588}, -\mathbf{0.02519276681624431}\}$	**1.7638**
3.070	$\{0.056465817134740845, 0.03238431775602044\}$	-
3.100	$\{0.08402039436980291, 0.053151159101234934\}$	-

TABLE 1. The calculations are conducted for values of parameters $a = 4.2$, $c = 0.1$ and initial condition $z_0 = (0.3, 1.6)$

Notice that if $d \in \{3.021, 3.022, 3.026, 3.028, 3.031\}$ conditions (3.2) hold.
Let us now check how the number of iterations affects the Lyapunov dimension for $d = 3.028$. The data are given in Table 2.

Number of iteration	$\{L_1, L_2\}$	d_L
500	$\{0.0007449252704960492, -0.024522106324883518\}$	1.0304
1000	$\{0.004264708217339782, -0.01818074631866523\}$	1.2346
10000	$\{0.01014786024923329, -0.013084603762557025\}$	1.7756
100000	$\{0.010594945814845724, -0.013379956641930453\}$	1.7919

TABLE 2. The calculations are conducted for values of parameters $a = 4.2$, $c = 0.1$, $d = 3.028$ and initial condition $z_0 = (0.3, 1.6)$

Example 3.3. *For* $a = 4.2,\ c = 0.1,\ d = 3.028$ *we have nine-coexisting chaotic sets. See Figure 6 (B).*
In Figure 6 (A): $z_0 = (0.3, 1.6)$*, 100000 orbits,* $d_L = 1.7919$*.*
In Figure 6 (B): $z_0 = (0.2666, 1.542)$*, 100000 orbits,* $d_L = 1.7793$*.*
Then the equilibrium point is $\bar{z} = (0.3632760898282695, 1.6742404227212682)$*.*

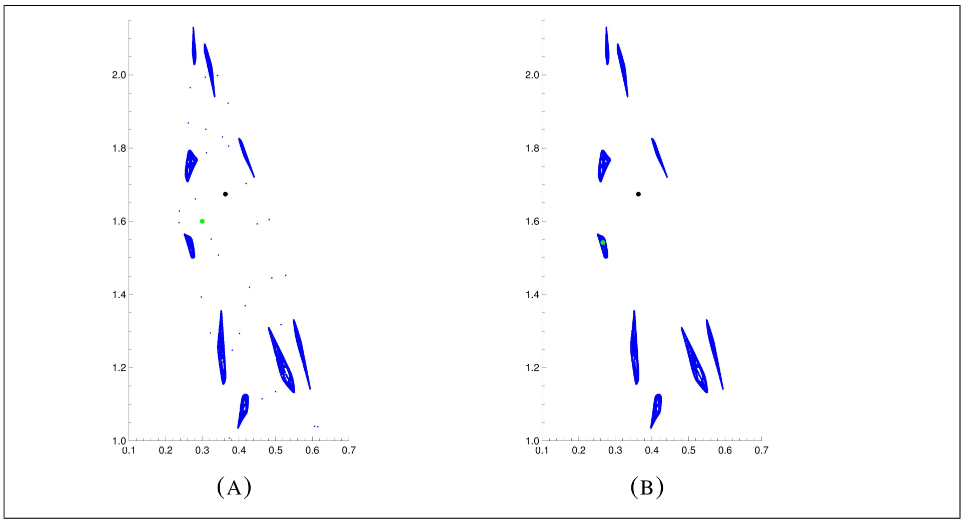

FIGURE 6. Phase portraits for $a = 4.2$, $c = 0.1$, $d = 3.028$ of System (1.2), initial point z_0 (green) and eqilibrium point $\bar{z}$ (black)

Example 3.4. *For* $a = 4.2,\ c = 0.1,\ d = 3.07$ *and* $a = 4.2,\ c = 0.1,\ d = 3.1$ *we have a chaotic attractor. See Figure 7.*
In Figure 7 (A): $d = 3.07$*, 100000 orbits and*
$(L_1, L_2) = (0.056561824648223076, 0.031407648031095456)$.
In Figure 7 (B): $d = 3.1$*, 100000 orbits and*
$(L_1, L_2) = (0.08714779637266407, 0.05465502376283619)$.

Equilibrium points are $\bar{z} = (0.3583061889250814, 1.695114006514658)$ *and* $\bar{z} = (0.3548387096774194, 1.709677419354839)$.

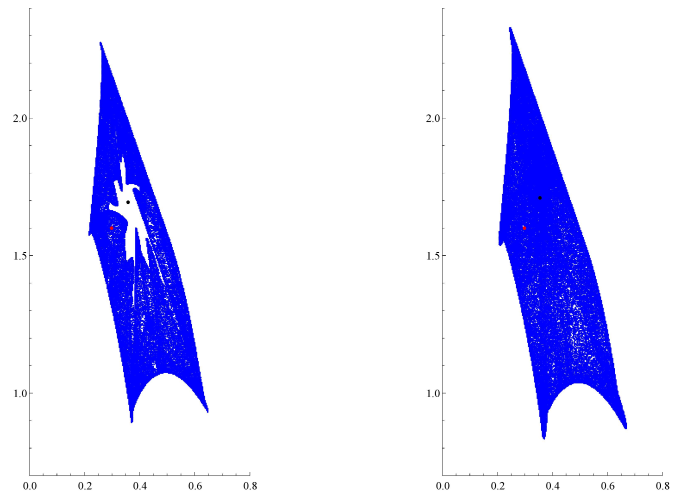

FIGURE 7. Phase portraits for $a = 4.2$, $c = 0.1$ of System (1.2), initial point $z_0 = (0.2666, 1.542)$ (red) and equilibrium point $\bar{z}$ (black)

AUTHOR'S CONTRIBUTIONS

All five authors contributed to each part of this study equally and read and approved the final version of the manuscript.

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