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EMPLOYING WEAK $\psi - \varphi$ CONTRACTION ON FUZZY METRIC SPACES WITH APPLICATION

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ABSTRACT. We establish a common fixed point theorem satisfying a weak $\psi-\varphi$ contraction on partially ordered non-Archimedean fuzzy metric spaces. In the process, some multidimensional common fixed point results are derived from our main results. As an application, we study the existence of the solution to an integral equation and also give an example to show the usefulness of the obtained results. Our results generalize, extend and improve several well-known results of the existence literature.

1. INTRODUCTION

George and Veeramani [12] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [18] with the help of a continuous t-norm and defined the Hausdorff topology of fuzzy metric spaces. In [17], Istratescu introduced the concept of a non-Archimedean fuzzy metric space.

In [14], Guo and Lakshmikantham introduced the notion of a coupled fixed point for single-valued mappings. Using this notion, Gnana-Bhaskar and Lakshmikantham [2] established some coupled fixed point theorems by defining the mixed monotone property. After that, Lakshmikantham and Ciric [19] extended the notion of the mixed monotone property to the mixed $G-$monotone property and established coupled coincidence point results using a pair of commutative mappings, which generalized the results of Gnana-Bhaskar and Lakshmikantham [2]. For more details one can consult ([1], [4]- [11], [15], [16]).

On the other hand Gordji et al. [13] proved some fixed point theorems for (ψ, φ)-weak contractive mappings in a complete partially ordered metric space.

In this paper, we establish a common fixed point theorem satisfying a weak $\psi-\varphi$ contraction on partially ordered non-Archimedean fuzzy metric spaces. In the process, some multidimensional common fixed point results are derived from

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our main results. As an application, we study the existence of the solution to an integral equation and also give an example to show the fruitfulness of the obtained results. We generalize, extend, improve and fuzzify the results of Gnana-Bhaskar and Lakshmikantham [2], Lakshmikantham and Ciric [19] and several well-known results of the existing literature.

2. PRELIMINARIES

Definition 2.1. *[20]. A binary operation* $*: [0, 1] \times [0, 1] \to [0, 1]$ *is a continuous t-norm if it satisfies the following conditions:*

(1) $*$ *is commutative and associative,*
(2) $*$ *is continuous,*
(3) $a * 1 = a$ *for all* $a \in [0, 1]$,
(4) $a * b \le c * d$ *whenever* $a \le c$ *and* $b \le d$ *with* $a, b, c, d \in [0, 1]$.

A few examples of continuous t-norms are

$$a * b = ab,\ a * b = \min\{a,\ b\} \text{ and } a * b = \max\{a + b - 1,\ 0\}.$$

Definition 2.2. *[12]. The 3-tuple* $(X, M, *)$ *is called a fuzzy metric space if* X *is an arbitrary non-empty set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X^2 \times [0, \infty)$ *satisfying the following conditions: for each* $x, y, z \in X$ *and* $t, s > 0$,

$(FM-1)$ $M(x, y, t) > 0$,
$(FM-2)$ $M(x, y, t) = 1$ *iff* $x = y$,
$(FM-3)$ $M(x, y, t) = M(y, x, t)$,
$(FM-4)$ $M(x, z, t+s) \ge M(x, y, t) * M(y, z, s)$,
$(FM-5)$ $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ *is continuous.*

Remark 2.1. If in the above definition (FM-4) is replaced by

$$(NAFM-4)\ M(x,\ z,\ \max\{t,\ s\}) \ge M(x,\ y,\ t) * M(y,\ z,\ s),$$

or equivalently,

$$(NAFM-4)\ M(x,\ z,\ t) \ge M(x,\ y,\ t) * M(y,\ z,\ t),$$

then $(X,\ M,\ *)$ is called a non-Archimedean fuzzy metric space [17]. It is easy to check that (NAFM-4) implies (FM-4), that is, every non-Archimedean fuzzy metric space is itself a fuzzy metric space.

Example 2.1. *[12]. Let* (X, d) *be a metric space. Define t-norm by* $a * b = ab$ *and*

$$M(x,\ y,\ t) = \frac{t}{t + d(x,\ y)} \text{ for all } x,\ y \in X \text{ and } t > 0.$$

Then $(X, M, *)$ *is a fuzzy metric space. We call this fuzzy metric* M *induced by the metric* d *the standard fuzzy metric space.*

Remark 2.2. [12]. In the fuzzy metric space $(X, M, *)$, $M(x, y, \cdot)$ is non-decreasing for all $x, y \in X$.

Definition 2.3. *[12]. Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}_n$ in X is called Cauchy if for each $\varepsilon \in (0, 1)$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that*

$$M(x_n,\ x_m,\ t) > 1-\varepsilon \text{ whenever } n \geq m \geq n_0.$$

We say that $(X, M, *)$ is complete if every Cauchy sequence is convergent, that is, if there exists $y \in X$ such that $\lim\limits_{n\to\infty} M(x_n, y, t) = 1$, for all $t > 0$.

Definition 2.4. *[2]. Let $F : X^2 \to X$ be a given mapping. An element $(x, y) \in X^2$ is called a coupled fixed point of F if $F(x, y) = x$ and $F(y, x) = y$.*

Definition 2.5. *[2]. Let $(X, \preceq)$ be a partially ordered set and $F : X^2 \to X$ be a given mapping. We say that F has the mixed monotone property if for all $x, y \in X$, we have*

$$x_1,\ x_2 \in X,\ x_1 \preceq x_2 \implies F(x_1,\ y) \preceq F(x_2,\ y),$$
$$y_1,\ y_2 \in X,\ y_1 \preceq y_2 \implies F(x,\ y_1) \succeq F(x,\ y_2).$$

Definition 2.6. *[19]. Let $F : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x, y) \in X^2$ is called a coupled coincidence point of the mappings F and g if $F(x, y) = gx$ and $F(y, x) = gy$.*

Definition 2.7. *[19]. Let $F : X^2 \to X$ and $g : X \to X$ be given mappings. An element $(x, y) \in X^2$ is called a common coupled fixed point of the mappings F and g if $x = F(x, y) = gx$ and $y = F(y, x) = gy$.*

Definition 2.8. *[19]. The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be commutative if $gF(x, y) = F(gx, gy)$, for all $(x, y) \in X^2$.*

Definition 2.9. *[19]. Let $(X, \preceq)$ be a partially ordered set. Suppose $F : X^2 \to X$ and $g : X \to X$ are given mappings. We say that F has the mixed $g-$monotone property if for all $x, y \in X$, we have*

$$x_1,\ x_2 \in X,\ gx_1 \preceq gx_2 \implies F(x_1,\ y) \preceq F(x_2,\ y),$$
$$y_1,\ y_2 \in X,\ gy_1 \preceq gy_2 \implies F(x,\ y_1) \succeq F(x,\ y_2).$$

If g is the identity mapping on X, then F satisfies the mixed monotone property.

Definition 2.10. *([2], [11]). A partially ordered metric space $(X,\ d,\ \preceq)$ is a metric space $(X,\ d)$ provided with a partial order $\preceq$. A partially ordered metric space $(X, d, \preceq)$ is said to be non-decreasing-regular (respectively, non-increasing-regular) if for every sequence $(x_n) \subseteq X$ such that $x_n \to x$ and $x_n \preceq x_{n+1}$ (respectively, $x_n \succeq x_{n+1}$) for all $n \geq 0$, we have $x_n \preceq x$ (respectively, $x_n \succeq x$) for all $n \geq 0$. $(X, d, \preceq)$ is said to be regular if it is both non-decreasing-regular and non-increasing-regular.*

Definition 2.11. *[11]. Let $(X, \preceq)$ be a partially ordered set and $F, G : X \to X$ be two mappings. We say that F is $(G, \preceq)-$non-decreasing if $Fx \preceq Fy$ for all*

$x, y \in X$ such that $Gx \preceq Gy$. If G is the identity mapping on X, we say that F is $\preceq -$non-decreasing. If F is $(G, \preceq)-$non-decreasing and $Gx = Gy$, then $Fx = Fy$.

Definition 2.12. *[3]. Let $(X, M, *)$ be an ordered fuzzy metric space. Two mappings $F, G : X \to X$ are said to be compatible if*

$$\lim_{n\to\infty} M(GFx_n,\ FGx_n,\ t) = 1,$$

provided that (x_n) is a sequence in X such that

$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n = x \in X.$$

Definition 2.13. *[15]. Let $F : X^2 \to X$ and $g : X \to X$ be two mappings. We say that the pair (F, g) is compatible if*

$$\lim_{n\to\infty} M(F(gx_n,\ gy_n),\ g(F(x_n,\ y_n),\ F(y_n,\ x_n)),\ t) = 1,$$
$$\lim_{n\to\infty} M(F(gy_n,\ gx_n),\ g(F(y_n,\ x_n),\ F(x_n,\ y_n)),\ t) = 1,$$

whenever (x_n) and (y_n) are sequences in X such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} F(x_n,\ y_n) = x \in X,$$
$$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} F(y_n,\ x_n) = y \in X.$$

Definition 2.14. *[15]. Let X be a non-empty set. The mappings $F : X^2 \to X$ and $g : X \to X$ are called weakly compatible if $F(x, y) = gx$ and $F(y, x) = gy$ implies that $g(F(x, y), F(y, x)) = F(gx, gy)$ and $g(F(y, x), F(x, y)) = F(gy, gx)$, for all $x, y \in X$.*

3. Fixed point results

In the sequel, X is a non-empty set and $\beta : X \to X$ is a mapping. For simplicity, we denote $\beta(x)$ by βx where $x \in X$.

Let Ψ denote the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying:
(i_ψ) ψ is continuous and non-decreasing,
(ii_ψ) $\psi(t) = 0 \Leftrightarrow t = 0$.
Let Φ denote the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying:
(i_φ) φ is lower semi-continuous and non-decreasing,
(ii_φ) $\varphi(t) = 0 \Leftrightarrow t = 0$.
Let Θ denote the set of all functions $\theta : [0, +\infty) \to [0, +\infty)$ satisfying:
(i_θ) θ is continuous,
(ii_θ) $\theta(t) = 0 \Leftrightarrow t = 0$.

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and $(X, M, *)$ be a complete non-Archimedean fuzzy metric space. Suppose $\alpha, \beta : X \to X$ are two mappings such that α is $(\beta, \preceq)-$non-decreasing and $\alpha(X) \subseteq \beta(X)$ for which there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that*

$$\psi\left(\frac{1}{M(\alpha x,\ \alpha y,\ t)}-1\right) \tag{3.1}$$
$$\leq \psi\left(\frac{1}{A(x,\ y)}-1\right)-\varphi\left(\psi\left(\frac{1}{A(x,\ y)}-1\right)\right)+\theta\left(\frac{1}{B(x,\ y)}-1\right),$$

where

$$A(x,\ y)=\min\{M(\beta x,\ \beta y,\ t),\ M(\beta x,\ \alpha x,\ t),\ M(\beta y,\ \alpha y,\ t)\}, \tag{3.2}$$

and

$$B(x,\ y)=\max\{M(\beta y,\ \alpha x,\ t),\ M(\beta y,\ \alpha y,\ t)\}, \tag{3.3}$$

for all $x, y\in X$ such that $\beta x\preceq\beta y$. There exists $x_0\in X$ such that $\beta x_0\preceq\alpha x_0$. Also assume that, at least, one of the following conditions holds:

(a) (X, M) *is complete, α and β are continuous and the pair (α, β) is compatible,*
(b) $(\beta(X), M)$ *is complete and $(X, M, \preceq)$ is non-decreasing-regular,*
(c) $(X,\ M)$ *is complete, β is continuous and monotone-non-decreasing, the pair (α, β) is compatible and $(X, M, \preceq)$ is non-decreasing-regular.*

Then α and β have a coincidence point. Furthermore, if the following condition holds:

(d) *Suppose that for every $x, y\in X$ there exists $u\in X$ such that αu is comparable to αx and αy and also the pair (α, β) is weakly compatible.*

Then α and β have a unique common fixed point.

Proof. Let $x_0\in X$ be arbitrary and since $\alpha x_0\in\alpha(X)\subseteq\beta(X)$, there exists $x_1\in X$ such that $\alpha x_0=\beta x_1$. Then $\beta x_0\preceq\alpha x_0=\beta x_1$. Since α is $(\beta, \preceq)-$non-decreasing, $\alpha x_0\preceq\alpha x_1$. Now $\alpha x_1\in\alpha(X)\subseteq\beta(X)$, so there exists $x_2\in X$ such that $\alpha x_1=\beta x_2$. Then $\beta x_1=\alpha x_0\preceq\alpha x_1=\beta x_2$. Since α is $(\beta, \preceq)-$non-decreasing, $\alpha x_1\preceq\alpha x_2$. Repeating this argument, there exists a sequence $(x_n)_{n\geq 0}$ such that (βx_n) is $\preceq-$non-decreasing, $\beta x_{n+1}=\alpha x_n\preceq\alpha x_{n+1}=\beta x_{n+2}$ and

$$\beta x_{n+1}=\alpha x_n \text{ for all } n\geq 0. \tag{3.4}$$

Suppose that for each $n\geq 0$, $M(\beta x_n,\ \beta x_{n+1},\ t)<1$. It is clear that $B(x_n,\ x_{n+1})=1$ for all $n\geq 0$. Now, by the contractive condition (3.1), (ii_θ) and by the monotonicity of ψ, we have

$$\psi\left(\frac{1}{M(\beta x_{n+1},\ \beta x_{n+2},\ t)}-1\right)$$
$$=\psi\left(\frac{1}{M(\alpha x_n,\ \alpha x_{n+1},\ t)}-1\right)$$
$$\leq\psi\left(\frac{1}{A(x_n,\ x_{n+1})}-1\right)-\varphi\left(\psi\left(\frac{1}{A(x_n,\ x_{n+1})}-1\right)\right).$$

Thus
$$\psi\left(\frac{1}{M(\beta x_{n+1},\ \beta x_{n+2},\ t)}-1\right) \tag{3.5}$$
$$\leq \psi\left(\frac{1}{A(x_n,\ x_{n+1})}-1\right)-\varphi\left(\psi\left(\frac{1}{A(x_n,\ x_{n+1})}-1\right)\right),$$
which since $\varphi \geq 0$ implies that
$$\psi\left(\frac{1}{M(\beta x_{n+1},\ \beta x_{n+2},\ t)}-1\right) \leq \psi\left(\frac{1}{A(x_n,\ x_{n+1})}-1\right).$$
Since ψ is non-decreasing, therefore we obtain
$$\frac{1}{M(\beta x_{n+1},\ \beta x_{n+2},\ t)}-1 \leq \frac{1}{A(x_n,\ x_{n+1})}-1. \tag{3.6}$$
Again
$$\begin{aligned} &A(x_n,\ x_{n+1}) \\ &= \min\{M(\beta x_n,\ \beta x_{n+1},\ t),\ M(\beta x_n,\ \alpha x_n,\ t),\ M(\beta x_{n+1},\ \alpha x_{n+1},\ t)\} \\ &= \min\{M(\beta x_n,\ \beta x_{n+1},\ t),\ M(\beta x_n,\ \beta x_{n+1},\ t),\ M(\beta x_{n+1},\ \beta x_{n+2},\ t)\} \\ &= \min\{M(\beta x_n,\ \beta x_{n+1},\ t),\ M(\beta x_{n+1},\ \beta x_{n+2},\ t)\}. \end{aligned}$$
If $M(\beta x_{n+1}, \beta x_{n+2}, t) \leq M(\beta x_n, \beta x_{n+1}, t)$, then
$$A(x_n,\ x_{n+1}) = M(\beta x_{n+1},\ \beta x_{n+2},\ t). \tag{3.7}$$
Thus, by (3.5) and (3.7), we have
$$\psi\left(\frac{1}{M(\beta x_{n+1},\ \beta x_{n+2},\ t)}-1\right)$$
$$\leq \psi\left(\frac{1}{M(\beta x_{n+1},\ \beta x_{n+2},\ t)}-1\right)-\varphi\left(\psi\left(\frac{1}{M(\beta x_{n+1},\ \beta x_{n+2},\ t)}-1\right)\right),$$
which is only possible when $M(\beta x_{n+1},\ \beta x_{n+2},\ t) = 1$, a contradiction. Hence, $M(\beta x_n, \beta x_{n+1}, t) \leq M(\beta x_{n+1}, \beta x_{n+2}, t)$. Then
$$A(x_n,\ x_{n+1}) = M(\beta x_n,\ \beta x_{n+1},\ t). \tag{3.8}$$
Thus, by (3.6), we get
$$\frac{1}{M(\beta x_{n+1},\ \beta x_{n+2},\ t)}-1 \leq \frac{1}{M(\beta x_n,\ \beta x_{n+1},\ t)}-1. \tag{3.9}$$
This shows that the sequence $(\delta_n)_{n\geq 0}$ defined by
$$\delta_n = \frac{1}{M(\beta x_n,\ \beta x_{n+1},\ t)}-1, \tag{3.10}$$
is a non-increasing sequence. Thus there exists $\delta \geq 0$ such that
$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty}\left(\frac{1}{M(\beta x_n,\ \beta x_{n+1},\ t)}-1\right) = \delta. \tag{3.11}$$

Then
$$\lim_{n\to\infty}\left(\frac{1}{A(x_n,\, x_{n+1})}-1\right)=\delta. \tag{3.12}$$
We shall prove that $\delta=0$. Assume to the contrary that $\delta>0$. Now, by contractive condition (3.1), (ii_θ) and by the monotonicity of ψ, we have
$$\begin{aligned}&\psi\left(\frac{1}{M(\beta x_{n+1},\, \beta x_{n+2},\, t)}-1\right)\\&=\psi\left(\frac{1}{M(\alpha x_n,\, \alpha x_{n+1},\, t)}-1\right)\\&\le\psi\left(\frac{1}{A(x_n,\, x_{n+1})}-1\right)-\varphi\left(\psi\left(\frac{1}{A(x_n,\, x_{n+1})}-1\right)\right).\end{aligned}$$
Letting $n\to\infty$ in the above inequality, by using (i_ψ), (i_φ), (3.11) and (3.12), we get
$$\psi(\delta)\le\psi(\delta)-\varphi(\psi(\delta)),$$
which is only possible when $\delta=0$. Thus
$$\delta=\lim_{n\to\infty}\delta_n=\lim_{n\to\infty}\left(\frac{1}{M(\beta x_n,\, \beta x_{n+1},\, t)}-1\right)=0. \tag{3.13}$$
or
$$\lim_{n\to\infty}M(\beta x_n,\, \beta x_{n+1},\, t)=1. \tag{3.14}$$
Now we claim that $(\beta x_n)_{n\ge0}$ is a Cauchy sequence in X. Suppose that $(\beta x_n)_{n\ge0}$ is not a Cauchy sequence. Then there exists an $\varepsilon>0$ for which we can find two sequences of positive integers $(m(k))$ and $(n(k))$ such that for all positive integers k,
$$M(\beta x_{n(k)},\, \beta x_{m(k)},\, t)\le 1-\varepsilon \text{ for } n(k)>m(k)>k. \tag{3.15}$$
Assuming that $n(k)$ is the smallest such positive integer, we get
$$M(\beta x_{n(k)-1},\, \beta x_{m(k)},\, t)>1-\varepsilon. \tag{3.16}$$
Now, by (3.15), (3.16) and (NAFM-4), we have
$$\begin{aligned}1-\varepsilon\ge r_k&=M(\beta x_{n(k)},\, \beta x_{m(k)},\, t)\\&\ge M(\beta x_{n(k)},\, \beta x_{n(k)-1},\, t)*M(\beta x_{n(k)-1},\, \beta x_{m(k)},\, t)\\&>M(\beta x_{n(k)},\, \beta x_{n(k)-1},\, t)*(1-\varepsilon).\end{aligned}$$
Letting $k\to\infty$ in the above inequality, by using (3.14), we have
$$\lim_{k\to\infty}r_k=\lim_{k\to\infty}M(\beta x_{n(k)},\, \beta x_{m(k)},\, t)=1-\varepsilon. \tag{3.17}$$
By (NAFM-4), we have
$$\begin{aligned}&M(\beta x_{n(k)+1},\, \beta x_{m(k)+1},\, t)\\&\ge M(\beta x_{n(k)+1},\, \beta x_{n(k)},\, t)*M(\beta x_{n(k)},\, \beta x_{m(k)},\, t)*M(\beta x_{m(k)},\, \beta x_{m(k)+1},\, t).\end{aligned}$$

Letting $k \to \infty$ in the above inequalities, using (3.14) and (3.17), we have

$$\lim_{k\to\infty} M(\beta x_{n(k)+1},\ \beta x_{m(k)+1},\ t) = 1-\varepsilon. \tag{3.18}$$

As $n(k) > m(k)$, $\beta x_{n(k)} \succeq \beta x_{m(k)}$, by using contractive condition (3.1), (ii_θ) and by the monotonicity of ψ, we have

$$\begin{aligned}
&\psi\left(\frac{1}{M(\beta x_{n(k)+1},\ \beta x_{m(k)+1},\ t)}-1\right)\\
&=\psi\left(\frac{1}{M(\alpha x_{n(k)},\ \alpha x_{m(k)},\ t)}-1\right)\\
&\leq \psi\left(\frac{1}{A(x_{n(k)},\ x_{m(k)})}-1\right)-\varphi\left(\psi\left(\frac{1}{A(x_{n(k)},\ x_{m(k)})}-1\right)\right)\\
&\leq \psi\left(\frac{1}{M(\beta x_{n(k)},\ \beta x_{m(k)},\ t)}-1\right)-\varphi\left(\psi\left(\frac{1}{M(\beta x_{n(k)},\ \beta x_{m(k)},\ t)}-1\right)\right).
\end{aligned}$$

Letting $k \to \infty$ in the above inequality, by using the property of ψ, θ, φ and (3.17), (3.18), we have

$$\psi\left(\frac{\varepsilon}{1-\varepsilon}\right) \leq \psi\left(\frac{\varepsilon}{1-\varepsilon}\right)-\varphi\left(\psi\left(\frac{\varepsilon}{1-\varepsilon}\right)\right) < \psi\left(\frac{\varepsilon}{1-\varepsilon}\right),$$

which is a contradiction due to $\varepsilon > 0$. It means that $(\beta x_n)_{n\geq 0}$ is a Cauchy sequence in X.

We claim that α and β have a coincidence point distinguishing between cases $(a)-(c)$.

Suppose now that (a) holds, that is, (X, M) is complete, α and β are continuous and the pair $(\alpha,\ \beta)$ is compatible. Since $(X,\ M)$ is complete, there exists $z \in X$ such that $(\beta x_n) \to z$. By (3.4), we also have that $(\alpha x_n) \to z$. As α and β are continuous, then $(\alpha\beta x_n) \to \alpha z$ and $(\beta\beta x_n) \to \beta z$. By using the fact that the pair $(\alpha,\ \beta)$ is compatible, we deduce that

$$\lim_{n\to\infty} M(\beta\alpha x_n,\ \alpha\beta x_n,\ t) = 1.$$

In such a case, we conclude that

$$M(\beta z,\ \alpha z,\ t) = \lim_{n\to\infty} M(\beta\beta x_{n+1},\ \alpha\beta x_n,\ t) = \lim_{n\to\infty} M(\beta\alpha x_n,\ \alpha\beta x_n,\ t) = 1,$$

that is, z is a coincidence point of α and β.

Suppose now that (b) holds, that is, $(\beta(X), M)$ is complete and $(X, M, \preceq)$ is non-decreasing-regular. As (βx_n) is a Cauchy sequence in the complete space $(\beta(X),\ M)$, there exist $y \in \beta(X)$ such that $(\beta x_n) \to y$. Let $z \in X$ be any point such that $y = \beta z$. In this case $(\beta x_n) \to \beta z$. Indeed, as $(X,\ M,\ \preceq)$ is non-decreasing-regular and (βx_n) is $\preceq -$non-decreasing and converging to βz, we deduce that $\beta x_n \preceq \beta z$ for all $n \geq 0$. Applying the contractive condition (3.1) and by the monotonicity of ψ, we get

$$\begin{aligned}
&\psi\left(\frac{1}{M(\beta x_{n+1},\ \alpha z,\ t)}-1\right)\\
&=\psi\left(\frac{1}{M(\alpha x_n,\ \alpha z,\ t)}-1\right)\\
&\le \psi\left(\frac{1}{A(x_n,\ z)}-1\right)-\varphi\left(\psi\left(\frac{1}{A(x_n,\ z)}-1\right)\right)+\theta\left(\frac{1}{B(x_n,\ z)}-1\right),
\end{aligned}\tag{3.19}$$

where

$$\begin{aligned}
A(x_n,\ z)&=\min\{M(\beta x_n,\ \beta z,\ t),\ M(\beta x_n,\ \alpha x_n,\ t),\ M(\beta z,\ \alpha z,\ t)\}\\
&=\min\{M(\beta x_n,\ \beta z,\ t),\ M(\beta x_n,\ \beta x_{n+1},\ t),\ M(\beta z,\ \alpha z,\ t)\},
\end{aligned}$$

and

$$\begin{aligned}
B(x_n,\ z)&=\max\{M(\beta z,\ \alpha x_n,\ t),\ M(\beta z,\ \alpha z,\ t)\}\\
&=\max\{M(\beta z,\ \beta x_{n+1},\ t),\ M(\beta z,\ \alpha z,\ t)\}.
\end{aligned}$$

Letting $n\to\infty$ in (3.19), by using (i_ψ), (i_φ) and (ii_θ), we get

$$\psi\left(\frac{1}{M(\beta z,\ \alpha z,\ t)}-1\right)\le\psi\left(\frac{1}{M(\beta z,\ \alpha z,\ t)}-1\right)-\varphi\left(\psi\left(\frac{1}{M(\beta z,\ \alpha z,\ t)}-1\right)\right),$$

which is possible only when $M(\beta z, \alpha z, t)=1$, that is, z is a coincidence point of α and β.

Suppose now that (c) holds, that is, (X, M) is complete, β is continuous and monotone non-decreasing, the pair (α, β) is compatible and $(X, M, \preceq)$ is non-decreasing-regular. As (X, M) is complete, there exists $z\in X$ such that $(\beta x_n)\to z$. By (3.4), we also have that $(\alpha x_n)\to z$. As β is continuous, then $(\beta\beta x_n)\to\beta z$. Furthermore, since the pair (α, β) is compatible, we have

$$\lim_{n\to\infty}M(\beta\beta x_{n+1},\ \alpha\beta x_n,\ t)=\lim_{n\to\infty}M(\beta\alpha x_n,\ \alpha\beta x_n,\ t)=1.$$

As $(\beta\beta x_n)\to\beta z$ the previous property means that $(\alpha\beta x_n)\to\beta z$.

Indeed, as $(X, M, \preceq)$ is non-decreasing-regular and (βx_n) is $\preceq$-non-decreasing and converging to z, we deduce that $\beta x_n\preceq z$. It follows, by the monotonicity of β, that $\beta\beta x_n\preceq\beta z$. Applying the contractive condition (3.1) and by the monotonicity of ψ, we get

$$\begin{aligned}
&\psi\left(\frac{1}{M(\alpha\beta x_n,\ \alpha z,\ t)}-1\right)\\
&\le\psi\left(\frac{1}{A(\beta x_n,\ z)}-1\right)-\varphi\left(\psi\left(\frac{1}{A(\beta x_n,\ z)}-1\right)\right)+\theta\left(\frac{1}{B(\beta x_n,\ z)}-1\right),
\end{aligned}\tag{3.20}$$

where

$$A(\beta x_n,\ z)=\min\{M(\beta\beta x_n,\ \beta z,\ t),\ M(\beta\beta x_n,\ \alpha\beta x_n,\ t),\ M(\beta z,\ \alpha z,\ t)\},$$

and

$$B(\beta x_n,\ z)=\max\{M(\beta z,\ \alpha\beta x_n,\ t),\ M(\beta z,\ \alpha z,\ t)\}.$$

Letting $n \to \infty$ in (3.20), by using (i_ψ), (i_φ) and (ii_θ), we get

$$\psi\left(\frac{1}{M(\beta z,\ \alpha z,\ t)} - 1\right) \leq \psi\left(\frac{1}{M(\beta z,\ \alpha z,\ t} - 1\right) - \varphi\left(\psi\left(\frac{1}{M(\beta z,\ \alpha z,\ t)} - 1\right)\right),$$

which is possible when $M(\beta z, \alpha z, \alpha) = 1$, that is, z is a coincidence point of α and β.

Hence, the set of coincidence points of β and α is non-empty. Therefore, suppose that x and y are coincidence points of α and β, that is, $\alpha x = \beta x$ and $\alpha y = \beta y$. Now, we show that $\beta x = \beta y$. By the assumption, there exists $u \in X$ such that αu is comparable with αx and αy. Put $u_0 = u$ and choose $u_1 \in X$ so that $\beta u_0 = \alpha u_1$. Then, we can inductively define sequences (βu_n) where $\beta u_{n+1} = \alpha u_n$ for all $n \geq 0$. Hence $\alpha x = \beta x$ and $\alpha u = \alpha u_0 = \beta u_1$ are comparable. Suppose that $\beta u_1 \preceq \beta x$ (the proof is similar to that in the other case). We claim that $\beta u_n \preceq \beta x$ for each $n \in \mathbb{N}$. In fact, we will use mathematical induction. Since $\beta u_1 \preceq \beta x$, our claim is true for $n = 1$.

We presume that $\beta u_n \preceq \beta x$ holds for some $n > 1$. Since α is $\beta-$non-decreasing with respect to $\preceq$, we get $\beta u_{n+1} = \alpha u_n \preceq \alpha x = \beta x$, and this proves our claim. Since $\beta u_n \preceq \beta x$ and so by using contractive condition (3.1), (ii_θ) and by the monotonicity of ψ, we have

$$\begin{aligned}\psi\left(\frac{1}{M(\beta u_{n+1},\ \beta x,\ t)} - 1\right) &= \psi\left(\frac{1}{M(\alpha u_n,\ \alpha x,\ t)} - 1\right)\\ &\leq \psi\left(\frac{1}{A(u_n,\ x)} - 1\right) - \varphi\left(\psi\left(\frac{1}{A(u_n,\ x)} - 1\right)\right),\end{aligned}$$

where

$$\begin{aligned}A(u_n,\ x) &= \min\{M(\beta u_n,\ \beta x,\ t),\ M(\beta u_n,\ \alpha u_n,\ t),\ M(\beta x,\ \alpha x,\ t)\}\\ &= \min\{M(\beta u_n,\ \beta x,\ t),\ M(\beta u_n,\ \beta u_{n+1},\ t)\}.\end{aligned}$$

Letting $n \to \infty$ in the above inequality and by using (ii_ψ) and (ii_φ), we get

$$\lim_{n\to\infty} M(\beta u_n,\ \beta x,\ t) = 1. \tag{3.21}$$

Similarly, one can prove that

$$\lim_{n\to\infty} M(\beta u_n,\ \beta y,\ t) = 0. \tag{3.22}$$

Hence, by (3.21) and (3.22), we get

$$\beta x = \beta y. \tag{3.23}$$

Since $\beta x = \alpha x$, by weak compatibility of α and β, we have $\beta\beta x = \beta\alpha x = \alpha\beta x$. Let $z = \beta x$. Then $\beta z = \alpha z$. Thus z is a coincidence point of β and α. Then from (3.23) with $y = z$, it follows that $\beta x = \beta z$, that is, $z = \beta z = \alpha z$. Therefore, z is a common fixed point of α and β. To prove the uniqueness, assume that w is another common fixed point of α and β. Then by (3.23) we have $w = \beta w = \beta z = z$. Hence the common fixed point of α and β is unique. □

If we put $\theta(t)=0$ in the Theorem 3.1, we get the following result:

Corollary 3.1. *Let $(X,\preceq)$ be a partially ordered set and $(X,M,*)$ be a complete non-Archimedean fuzzy metric space. Suppose $\alpha,\beta: X\to X$ are two mappings such that α is $(\beta,\preceq)-$non-decreasing and $\alpha(X)\subseteq\beta(X)$ for which there exist $\varphi\in\Phi$ and $\psi\in\Psi$ such that*

$$\psi\left(\frac{1}{M(\alpha x,\ \alpha y,\ t)}-1\right)\leq\psi\left(\frac{1}{A(x,\ y)}-1\right)-\varphi\left(\psi\left(\frac{1}{A(x,\ y)}-1\right)\right),$$

where $A(x,y)$ is defined in (3.2), for all $x,y\in X$ such that $\beta x\preceq\beta y$. There exists $x_0\in X$ such that $\beta x_0\preceq\alpha x_0$. Furthermore, suppose one of the conditions $(a)-(c)$ of Theorem 3.1 holds. Then α and β have a coincidence point. Moreover, if condition (d) of Theorem 3.1 holds. Then α and β have a unique common fixed point.

If we put $\varphi(t)=t-t\varphi_1(t)$ for all $t\geq 0$ in Corollary 3.1, then we get the following result:

Corollary 3.2. *Let $(X,\preceq)$ be a partially ordered set and $(X,M,*)$ be a complete non-Archimedean fuzzy metric space. Suppose $\alpha,\beta: X\to X$ are two mappings such that α is $(\beta,\preceq)-$non-decreasing and $\alpha(X)\subseteq\beta(X)$ for which there exist $\varphi_1\in\Phi$ and $\psi\in\Psi$ such that*

$$\psi\left(\frac{1}{M(\alpha x,\ \alpha y,\ t)}-1\right)\leq\varphi_1\left(\psi\left(\frac{1}{A(x,\ y)}-1\right)\right)\psi\left(\frac{1}{A(x,\ y)}-1\right),$$

where $A(x,y)$ is defined in (3.2), for all $x,y\in X$ such that $\beta x\preceq\beta y$. There exists $x_0\in X$ such that $\beta x_0\preceq\alpha x_0$. Futhermore, suppose one of the conditions $(a)-(c)$ of Theorem 3.1 holds. Then α and β have a coincidence point. Moreover, if condition (d) of Theoem 3.1 holds. Then α and β have a unique common fixed point.

If we put $\psi(t)=2t$ for all $t\geq 0$ in Corollary 3.2, then we get the following result:

Corollary 3.3. *Let $(X,\preceq)$ be a partially ordered set and $(X,M,*)$ be a complete non-Archimedean fuzzy metric space. Suppose $\alpha,\beta: X\to X$ are two mappings such that α is $(\beta,\preceq)-$non-decreasing and $\alpha(X)\subseteq\beta(X)$ for which there exists $\varphi_1\in\Phi$ such that*

$$\frac{1}{M(\alpha x,\ \alpha y,\ t)}-1\leq\varphi_1\left(2\left(\frac{1}{A(x,\ y)}-1\right)\right)\left(\frac{1}{A(x,\ y)}-1\right),$$

where $A(x,y)$ is defined in (3.2), for all $x,y\in X$ such that $\beta x\preceq\beta y$. There exists $x_0\in X$ such that $\beta x_0\preceq\alpha x_0$. Furthermore, suppose one of the conditions $(a)-(c)$ of Theorem 3.1 holds. Then α and β have a coincidence point. Moreover, if condition (d) of Theoem 3.1 holds. Then α and β have a unique common fixed point.

If we put $\varphi_1(t)=k$ where $0<k<1$, for all $t\geq 0$ in Corollary 3.3, then we get the following result:

Corollary 3.4. *Let $(X, \preceq)$ be a partially ordered set and $(X, M, *)$ be a complete non-Archimedean fuzzy metric space. Suppose $\alpha, \beta : X \to X$ are two mappings such that α is $(\beta, \preceq)-$non-decreasing, $\alpha(X) \subseteq \beta(X)$ and*

$$\frac{1}{M(\alpha x,\ \alpha y,\ t)} - 1 \leq k\left(\frac{1}{A(x,\ y)} - 1\right),$$

where $A(x, y)$ is defined in (3.2), for all $x, y \in X$ such that $\beta x \preceq \beta y$ and $k < 1$. There exists $x_0 \in X$ such that $\beta x_0 \preceq \alpha x_0$. Furthermore, suppose one of the conditions $(a)-(c)$ of Theorem 3.1 holds. Then α and β have a coincidence point. Moreover, if condition (d) of Theorem 3.1 holds. Then α and β have a unique common fixed point.

If $\beta = I$ (the identity mapping) in the Corollary 3.4, we get the following:

Corollary 3.5. *Let $(X, \preceq)$ be a partially ordered set and $(X, M, *)$ be a complete non-Archimedean fuzzy metric space. Suppose $\alpha : X \to X$ is a non-decreasing mapping satisfying*

$$\frac{1}{M(\alpha x,\ \alpha y,\ t)} - 1 \leq k\left(\frac{1}{A_1(x,\ y)} - 1\right),$$

where

$$A_1(x,\ y) = \min\{M(x,\ y,\ t),\ M(x,\ \alpha x,\ t),\ M(y,\ \alpha y,\ t)\},$$

for all $x, y \in X$ such that $x \preceq y$ and $k < 1$. If there exists $x_0 \in X$ such that $x_0 \preceq \alpha x_0$, then α has a fixed point.

Example 3.1. *Suppose that $X = [0, 1]$, equipped with the usual metric $d : X \times X \to [0, +\infty)$ with the natural ordering of real numbers $\leq$ and $*$ is defined by $a * b = ab$, for all $a, b \in [0, 1]$. Define*

$$M(x,\ y,\ t) = \frac{t}{t + d(x,\ y)},\ \textit{for all } x,\ y \in X \textit{ and } t > 0.$$

Clearly $(X, M, *)$ is a complete non-Archimedean fuzzy metric space. Let $\alpha, \beta : X \to X$ be defined as

$$\alpha x = \frac{x^2}{3} \textit{ and } \beta x = x^2 \textit{ for all } x \in X.$$

Define $\psi : [0, +\infty) \to [0, +\infty)$ by

$$\psi(t) = t \textit{ for all } t \geq 0,$$

define $\varphi : [0, +\infty) \to [0, +\infty)$ by

$$\varphi(t) = \frac{2t}{3} \textit{ for all } t \geq 0,$$

define $\theta : [0, +\infty) \to [0, +\infty)$ by

$$\theta(t) = \frac{t}{2} \textit{ for all } t \geq 0.$$

Now, for all $x, y \in X$ with $\beta x \preceq \beta y$, we have

$$\begin{aligned}
&\psi\left(\frac{1}{M(\alpha x,\ \alpha y,\ t)}-1\right)\\
&=\frac{1}{M(\alpha x,\ \alpha y,\ t)}-1\\
&=\frac{1}{3}\left(\frac{1}{M(\beta x,\ \beta y,\ t)}-1\right)\\
&\leq\frac{1}{3}\left(\frac{1}{A(x,\ y)}-1\right)\\
&\leq\psi\left(\frac{1}{A(x,\ y)}-1\right)-\varphi\left(\psi\left(\frac{1}{A(x,\ y)}-1\right)\right)+\theta\left(\frac{1}{B(x,\ y)}-1\right).
\end{aligned}$$

Thus the contractive condition of Theorem 3.1 is satisfied for all $x, y \in X$. In addition all the other conditions of Theorem 3.1 are satisfied and $z = 0$ is a unique common fixed point of α and β.

4. Coupled fixed point results

Next, we deduce the two dimensional version of Theorem 3.1. Given $n \in \mathbb{N}$ where $n \geq 2$, let X^n be the n^{th} Cartesian product $X \times X \times \ldots \times X$ (n times). For the ordered fuzzy metric space $(X, M, \preceq)$, let us consider the ordered fuzzy metric space $(X^2, M_\delta, \sqsubseteq)$, where $M_\delta : X^2 \times X^2 \times [0, \infty) \to [0, 1]$ defined by

$$M_\delta(Y,\ V,\ t) = \min\{M(x,\ u,\ t),\ M(y,\ v,\ t)\},\ \forall Y = (x,\ y),\ V = (u,\ v) \in X^2,$$

and $\sqsubseteq$ was introduced by

$$(u,\ v) \sqsubseteq (x,\ y) \Leftrightarrow x \succeq u \text{ and } y \preceq v,\ \text{ for all } (u,\ v),\ (x,\ y) \in X^2.$$

It is easy to check that M_δ is a non-Archimedean fuzzy metric on X^2. Moreover $(X, M, *)$ is complete if and only if $(X^2, M_\delta, *)$ is complete. We define the mapping $\Omega_F, \Omega_G : X^2 \to X^2$, for all $(x, y) \in X^2$, by

$$\Omega_F(x,\ y) = (F(x,\ y),\ F(y,\ x)) \text{ and } \Omega_g(x,\ y) = (gx,\ gy).$$

Lemma 4.1. *Let $(X, \preceq)$ be a partially ordered set and $(X, M, *)$ be a complete non-Archimedean fuzzy metric space. Let $F : X^2 \to X$ and $g : X \to X$ be two mappings. Then*

(1) *(X, M) is complete if and only if (X^2, M_δ) is complete.*
(2) *If $(X, M, \preceq)$ is regular, then $(X^2, M_\delta, \sqsubseteq)$ is also regular.*
(3) *If F is $M-$continuous, then Ω_F is M_δ-continuous.*
(4) *F has the mixed monotone property with respect to $\preceq$ if and only if Ω_F is $\sqsubseteq$-non-decreasing.*
(5) *F has the mixed $g-$monotone property with respect to $\preceq$ if and only if then Ω_F is $(\Omega_g, \sqsubseteq)$-non-decreasing.*

(6) *If there exist two elements* x_0, $y_0 \in X$ *with* $gx_0 \preceq F(x_0, y_0)$ *and* $gy_0 \succeq F(y_0, x_0)$, *then there exists a point* $(x_0, y_0) \in X^2$ *such that* $\Omega_g(x_0, y_0) \sqsubseteq \Omega_F(x_0, y_0)$.

(7) *If* $F(X^2) \subseteq g(X)$, *then* $\Omega_F(X^2) \subseteq \Omega_g(X^2)$.

(8) *If* F *and* g *are commuting in* $(X, M, \preceq)$, *then* Ω_F *and* Ω_g *are also commuting in* $(X^2, M_\delta, \sqsubseteq)$.

(9) *If* F *and* g *are compatible in* $(X, M, \preceq)$, *then* Ω_F *and* Ω_g *are also compatible in* $(X^2, M_\delta, \sqsubseteq)$.

(10) *If* F *and* g *are weak compatible in* $(X, M, \preceq)$, *then* Ω_F *and* Ω_g *are also weak compatible in* $(X^2, M_\delta, \sqsubseteq)$.

(11) *) A point* $(x, y) \in X^2$ *is a coupled coincidence point of* F *and* g *if and only if it is a coincidence point of* Ω_F *and* Ω_g.

(12) $(x,y) \in X^2$ *is a coupled fixed point of* F *if and only if it is a fixed point of* Ω_F.

Proof. Items (1), (2), (3), (6), (7), (11) and (12) are obvious.

(4) Let F has the mixed monotone property with respect to $\preceq$. Then we have to show that Ω_F is $\sqsubseteq$-non-decreasing.

Let (x, y), $(u, v) \in X^2$ be such that $(x, y) \sqsubseteq (u, v)$. Then $x \preceq u$ and $y \succeq v$. As F has the mixed monotone property with respect to $\preceq$ and so $F(x, y) \preceq F(u, v)$ and $F(y, x) \succeq F(v, u)$. Thus $\Omega_F(x, y) \sqsubseteq \Omega_F(u, v)$.

Thus, if F has the mixed monotone property with respect to $\preceq$, then Ω_F is $\sqsubseteq$-non-decreasing and vice-versa.

(5) Let F has the mixed $g-$monotone property with respect to $\preceq$. Then we have to show that Ω_F is $(\Omega_g, \sqsubseteq)$-non-decreasing.

Let (x, y), $(u, v) \in X^2$ be such that $\Omega_g(x, y) \sqsubseteq \Omega_g(u, v)$. Then $gx \preceq gu$ and $gy \succeq gv$. As F has the mixed $g-$monotone property with respect to $\preceq$ and so $F(x, y) \preceq F(u, v)$ and $F(y, x) \succeq F(v, u)$. Thus $\Omega_F(x, y) \sqsubseteq \Omega_F(u, v)$.

Thus, if F has the mixed $g-$monotone property with respect to $\preceq$, then Ω_F is $(\Omega_g, \sqsubseteq)$-non-decreasing and vice-versa.

(8) Let $(x, y) \in X^2$. Since g and F are commutative, by the definition of Ω_g and Ω_F, we have $\Omega_g\Omega_F(x, y) = \Omega_g(F(x, y), F(y, x)) = (gF(x, y), gF(y, x)) = (F(gx, gy), F(gy, gx)) = \Omega_F(gx, gy) = \Omega_F\Omega_g(x, y)$, which shows that Ω_g and Ω_F are commutative.

(9) Let $((x_n, y_n)) \subseteq X^2$ be any sequence such that $\Omega_F(x_n, y_n) \overset{M_\delta}{\to} (x, y)$ and $\Omega_g(x_n, y_n) \overset{M_\delta}{\to} (x, y)$. Then,

$$(F(x_n, y_n), F(y_n, x_n)) \overset{M_\delta}{\to} (x, y) \Rightarrow F(x_n, y_n) \overset{M}{\to} x \text{ and } F(y_n, x_n) \overset{M}{\to} y,$$

and

$$(gx_n, gy_n) \overset{M_\delta}{\to} (x, y) \Rightarrow gx_n \overset{M}{\to} x \text{ and } gy_n \overset{M}{\to} y.$$

Therefore

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x \in X,$$
$$\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y \in X.$$

Since the pair (F, g) is compatible, we have

$$\lim_{n\to\infty} M(F(gx_n, gy_n), gF(x_n, y_n), t) = 1,$$
$$\lim_{n\to\infty} M(F(gy_n, gx_n), gF(y_n, x_n), t) = 1.$$

In particular,

$$\begin{aligned}
&\lim_{n\to\infty} M_\delta(\Omega_g\Omega_F(x_n, y_n), \Omega_F\Omega_g(x_n, y_n), t)\\
&= \lim_{n\to\infty} M_\delta(\Omega_g(F(x_n, y_n), F(y_n, x_n)), \Omega_F(gx_n, gy_n), t)\\
&= \lim_{n\to\infty} M_\delta((gF(x_n, y_n), gF(y_n, x_n)), (F(gx_n, gy_n), F(gy_n, gx_n)), t)\\
&= \lim_{n\to\infty} \min\left\{ \begin{array}{c} M(gF(x_n, y_n), F(gx_n, gy_n), t), \\ M(gF(y_n, x_n), F(gy_n, gx_n), t) \end{array} \right\}\\
&= 1.
\end{aligned}$$

Hence, the mappings Ω_F and Ω_g are compatible in $(X^2, M_\delta, \sqsubseteq)$.

(10) Let $(x, y) \in X^2$ be a coincidence point of Ω_g and Ω_F. Then $\Omega_g(x, y) = \Omega_F(x, y)$, that is, $(gx, gy) = (F(x, y), F(y, x))$, that is, $gx = F(x, y)$ and $gy = F(y, x)$. Since g and F are weak compatible, by the definition of Ω_g and Ω_F, we have $\Omega_g\Omega_F(x, y) = \Omega_g(F(x, y), F(y, x)) = (gF(x, y), gF(y, x)) = (F(gx, gy), F(gy, gx)) = \Omega_F(gx, gy) = \Omega_F\Omega_g(x,y)$, which shows that Ω_g and Ω_F commute at their coincidence point, that is, Ω_g and Ω_F are weak compatible. $\square$

Theorem 4.1. *Let $(X, \preceq)$ be a partially ordered set and $(X, M, *)$ be a complete non-Archimedean fuzzy metric space. Assume $F : X^2 \to X$ and $g : X \to X$ are two mappings such that F has the mixed $g-$monotone property with respect to $\preceq$ on X for which there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ satisfying*

$$\begin{aligned}
&\psi\left(\frac{1}{M(F(x, y), F(u, v), t)} - 1\right)\\
&\leq \psi\left(\frac{1}{A_M^g(x, y, u, v)} - 1\right) - \varphi\left(\psi\left(\frac{1}{A_M^g(x, y, u, v)} - 1\right)\right)\\
&+ \theta\left(\frac{1}{B_M^g(x, y, u, v)} - 1\right),
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
&A_M^g(x, y, u, v)\\
&= \min\left\{ \begin{array}{c} M(gx, gu, t), M(gx, F(x, y), t), M(gu, F(u, v), t), \\ M(gy, gv, t), M(gy, F(y, x), t), M(gv, F(v, u), t) \end{array} \right\},
\end{aligned} \tag{4.2}$$

and

$$B_M^g(x, y, u, v) = \max\left\{ \begin{array}{c} M(gu, F(x, y), t), M(gu, F(u, v), t), \\ M(gv, F(y, x), t), M(gv, F(v, u), t) \end{array} \right\}, \tag{4.3}$$

for all $x, y, u, v \in X$, where $gx \preceq gu$ and $gy \succeq gv$. Suppose that $F(X^2) \subseteq g(X)$, g is continuous, monotone non-decreasing and the pair (F, g) is compatible. Also suppose that either

(a) F is continuous or

(b) $(X, d, \preceq)$ is regular.

If there exist two elements $x_0, y_0 \in X$ with

$$gx_0 \preceq F(x_0, y_0) \text{ and } gy_0 \succeq F(y_0, x_0),$$

then F and g have a coupled coincidence point. In addition, suppose that for every $(x, y), (x^*, y^*) \in X^2$, there exists a point $(u, v) \in X^2$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$, and also the pair (F, g) is weakly compatible. Then F and g have a unique common coupled fixed point.

Proof. Let $x, y, u, v \in X$, with $gx \preceq gu$ and $gy \succeq gv$. Then, by using (4.1), we have

$$\begin{aligned}
&\psi\left(\frac{1}{M(F(x, y), F(u, v), t)} - 1\right) \\
&\leq \psi\left(\frac{1}{A_M^g(x, y, u, v)} - 1\right) - \varphi\left(\psi\left(\frac{1}{A_M^g(x, y, u, v)} - 1\right)\right) \\
&+ \theta\left(\frac{1}{B_M^g(x, y, u, v)} - 1\right),
\end{aligned}$$

Furthermore taking into account that $gy \succeq gv$ and $gx \preceq gu$, the contractive condition (4.1) also guarantees that

$$\begin{aligned}
&\psi\left(\frac{1}{M(F(y, x), F(v, u), t)} - 1\right) \\
&\leq \psi\left(\frac{1}{A_M^g(x, y, u, v)} - 1\right) - \varphi\left(\psi\left(\frac{1}{A_M^g(x, y, u, v)} - 1\right)\right) \\
&+ \theta\left(\frac{1}{B_M^g(x, y, u, v)} - 1\right).
\end{aligned}$$

Combining them, we get

$$\begin{aligned}
&\max\left\{\begin{array}{l} \psi\left(\dfrac{1}{M(F(x, y), F(u, v), t)} - 1\right), \\ \psi\left(\dfrac{1}{M(F(y, x), F(v, u), t)} - 1\right) \end{array}\right\} \\
&\leq \psi\left(\frac{1}{A_M^g(x, y, u, v)} - 1\right) - \varphi\left(\psi\left(\frac{1}{A_M^g(x, y, u, v)} - 1\right)\right) \\
&+ \theta\left(\frac{1}{B_M^g(x, y, u, v)} - 1\right).
\end{aligned}$$

Since ψ is non-decreasing,

$$\psi\left(\max\left\{\begin{array}{c}\left(\frac{1}{M(F(x,y),F(u,v),t)}-1\right),\\ \left(\frac{1}{M(F(y,x),F(v,u),t)}-1\right)\end{array}\right\}\right)$$
$$\leq\psi\left(\frac{1}{A_M^g(x,y,u,v)}-1\right)-\varphi\left(\psi\left(\frac{1}{A_M^g(x,y,u,v)}-1\right)\right)$$
$$+\theta\left(\frac{1}{B_M^g(x,y,u,v)}-1\right). \tag{4.4}$$

Thus, it follows from (4.4) that

$$\psi\left(\frac{1}{M_\delta(\Omega_F(x,y),\Omega_F(u,v),t)}-1\right)$$
$$=\psi\left(\frac{1}{M_\delta((F(x,y),F(y,x)),(F(u,v),F(v,u)),t)}-1\right)$$
$$=\psi\left(\frac{1}{\min\{M(F(x,y),F(u,v),t),M(F(y,x),F(v,u),t)\}}-1\right)$$
$$=\psi\left(\max\left\{\begin{array}{c}\left(\frac{1}{M(F(x,y),F(u,v),t)}-1\right),\\ \left(\frac{1}{M(F(y,x),F(v,u),t)}-1\right)\end{array}\right\}\right)$$
$$\leq\psi\left(\frac{1}{A_M^g(x,y,u,v)}-1\right)-\varphi\left(\psi\left(\frac{1}{A_M^g(x,y,u,v)}-1\right)\right)$$
$$+\theta\left(\frac{1}{B_M^g(x,y,u,v)}-1\right)$$
$$\leq\psi\left(\frac{1}{A_{M_\delta}((x,y),(u,v))}-1\right)-\varphi\left(\psi\left(\frac{1}{A_{M_\delta}((x,y),(u,v))}-1\right)\right)$$
$$+\theta\left(\frac{1}{B_{M_\delta}((x,y),(u,v))}-1\right),$$

where

$$A_{M_\delta}((x,y),(u,v))=\min\left\{\begin{array}{c}M_\delta(\Omega_g(x,y),\Omega_g(u,v),t),\\ M_\delta(\Omega_g(x,y),\Omega_F(x,y),t),\\ M_\delta(\Omega_g(u,v),\Omega_F(u,v),t)\end{array}\right\},$$

and

$$B_{M_\delta}((x,y),(u,v))=\max\left\{\begin{array}{c}M_\delta(\Omega_g(u,v),\Omega_F(x,y),t)\\ M_\delta(\Omega_g(u,v),\Omega_F(u,v),t)\end{array}\right\}.$$

It is only necessary to apply Theorem 3.1 to the mappings $\alpha=\Omega_F$ and $\beta=\Omega_g$ in the ordered metric space $(X^2, M_\delta, \sqsubseteq)$ taking into account all items of Lemma 4.1. □

Corollary 4.1. *Let $(X, \preceq)$ be a partially ordered set and $(X, M, *)$ be a complete non-Archimedean fuzzy metric space. Assume $F : X^2 \to X$ and $g : X \to X$ are two mappings such that F has the mixed $g-$monotone property with respect to $\preceq$ on X for which there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ satisfying (4.1), where $A_M^g(x, y, u, v)$ and $B_M^g(x, y, u, v)$ are defined in (4.2) and (4.3) respectively, for all $x, y, u, v \in X$, where $gx \preceq gu$ and $gy \succeq gv$. Suppose that $F(X^2) \subseteq g(X)$, g is continuous, monotone non-decreasing and the pair (F, g) is commuting. Also suppose that either*

(a) *F is continuous or*
(b) *$(X, d, \preceq)$ is regular.*

If there exist two elements $x_0, y_0 \in X$ with

$$gx_0 \preceq F(x_0, y_0) \text{ and } gy_0 \succeq F(y_0, x_0),$$

then F and g have a coupled coincidence point. In addition, suppose that for every $(x, y), (x^*, y^*) \in X^2$, there exists a point $(u, v) \in X^2$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$, and also the pair (F, G) is weakly compatible. Then F and g have a unique common coupled fixed point.

Corollary 4.2. *Let $(X, \preceq)$ be a partially ordered set and $(X, M, *)$ be a complete non-Archimedean fuzzy metric space. Assume $F : X^2 \to X$ is a mapping such that F has the mixed monotone property with respect to $\preceq$ on X and there exist $\varphi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that*

$$\begin{aligned}
&\psi\left(\frac{1}{M(F(x, y), F(u, v), t)} - 1\right) \\
&\leq \psi\left(\frac{1}{A_M(x, y, u, v)} - 1\right) - \varphi\left(\psi\left(\frac{1}{A_M(x, y, u, v)} - 1\right)\right) \\
&+ \theta\left(\frac{1}{B_M(x, y, u, v)} - 1\right),
\end{aligned}$$

where

$$\begin{aligned}
&A_M(x, y, u, v) \\
&= \min\left\{ \begin{array}{c} M(x, u, t), M(x, F(x, y), t), M(u, F(u, v), t), \\ M(y, v, t), M(y, F(y, x), t), M(v, F(v, u), t) \end{array} \right\},
\end{aligned}$$

and

$$B_M(x, y, u, v) = \max\left\{ \begin{array}{c} M(u, F(x, y), t), M(u, F(u, v), t), \\ M(v, F(y, x), t), M(v, F(v, u), t) \end{array} \right\},$$

for all $x, y, u, v \in X$, where $x \preceq u$ and $y \succeq v$. Also suppose that either

(a) *F is continuous or*
(b) *$(X, d, \preceq)$ is regular.*

If there exist two elements $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0,\, y_0) \text{ and } y_0 \succeq F(y_0,\, x_0),$$

then F has a coupled fixed point.

In a similar way, we may state the results analogous to Corollary 3.1, Corollary 3.2, Corollary 3.3, Corollary 3.4 and Corollary 3.5 for Theorem 4.1, Corollary 4.1 and Corollary 4.2.

5. Application

In this section, we give an application to our results. Consider the integral equation

$$u(t) = \int_0^T K(t,\, s,\, u(s))ds + h(t),\; t \in [0,\, T], \tag{5.1}$$

where $T > 0$. We introduce the following space:

$$C[0,\, T] = \{u : [0,\, T] \to \mathbb{R} : u \text{ is continuous on } [0,\, T]\},$$

equipped with the metric

$$d(x,\, y) = \sup_{t\in[0,\, T]} |x(t) - y(t)|\,, \text{ for each } x,\, y \in C[0,\, T].$$

It is clear that $(C[0,T],d)$ is a regular complete metric space. It is easy to check that $(C[0,T],M,*)$ is a complete non-Archimedean fuzzy metric space with respect to the fuzzy metric

$$M(x,\, y,\, t) = \frac{t}{t+d(x,\, y)}, \text{ for all } x,\, y \in X \text{ and } t > 0,$$

where $*$ is defined by $a*b = ab$, for all $a,\, b \in I$. Furthermore, $C[0,\, T]$ can be equipped with the partial order $\preceq$ as follows: for $x, y \in C[0,\, T]$,

$$x \preceq y \Longleftrightarrow x(t) \le y(t), \text{ for each } t \in [0,\, T].$$

Now, we state the main result of this section.

Theorem 5.1. *We assume that the following hypotheses hold:*

(i) *$K : [0,\, T] \times [0,\, T] \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are continuous.*

(ii) *For all $s, t, u, v \in C[0,\, T]$ with $v \preceq u$, we have*

$$K(t,\, s,\, v(s)) \le K(t,\, s,\, u(s)).$$

(iii) *There exists a continuous function $G : [0,\, T] \times [0,\, T] \to [0,\, +\infty)$ such that*

$$|K(t,\, s,\, x) - K(t,\, s,\, y)| \le G(t,\, s)\cdot\frac{|x-y|}{2},$$

for all $s,\, t \in C[0,\, T]$ and $x, y \in \mathbb{R}$ with $x \succeq y$.

(iv)
$$\sup_{t\in[0,\, T]} \int_0^T G(t,\, s)^2 ds \le \frac{1}{T}.$$

Then the integral (5.1) has a solution $u^* \in C[0,\, T]$.

Proof. Define $F : C[0,\, T] \to C[0,\, T]$ by

$$Fu(t) = \int_0^T K(t,\ s,\ u(s))ds + h(t),\ \text{for all } t \in [0,\ T] \text{ and } u \in C[0,\ T].$$

We first prove that F is non-decreasing. Assume that $v \preceq u$. From (ii), for all s, $t \in [0, T]$, we have $K(t, s, v(s)) \leq K(t, s, u(s))$. Thus, we get,

$$Fv(t) = \int_0^T K(t,\ s,\ v(s))ds + h(t) \leq \int_0^T K(t,\ s,\ u(s))ds + h(t) = Fu(t).$$

Now, for all u, $v \in C[0, T]$ with $v \preceq u$, due to (iii) and by using Cauchy-Schwarz inequality, we get

$$\begin{aligned}
|Fu(t) - Fv(t)| &\leq \int_0^T |K(t,\ s,\ u(s)) - K(t,\ s,\ v(s))|\, ds \\
&\leq \int_0^T G(t,\ s) \cdot \frac{|u(s) - v(s)|}{2} ds \\
&\leq \left(\int_0^T G(t,\ s)^2 ds \right)^{\frac{1}{2}} \left(\int_0^T \left(\frac{|u(s) - v(s)|}{2} \right)^2 ds \right)^{\frac{1}{2}}.
\end{aligned}$$

Thus

$$|Fu(t) - Fv(t)| \leq \left(\int_0^T G(t,\ s)^2 ds \right)^{\frac{1}{2}} \left(\int_0^T \left(\frac{|u(s) - v(s)|}{2} \right)^2 ds \right)^{\frac{1}{2}}. \tag{5.2}$$

Taking (iv) into account, we estimate the first integral in (5.2) as follows:

$$\left(\int_0^T G(t,\ s)^2 ds \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{T}}. \tag{5.3}$$

For the second integral in (5.2) we proceed in the following way:

$$\left(\int_0^T \left(\frac{|u(s) - v(s)|}{2} \right)^2 ds \right)^{\frac{1}{2}} \leq \sqrt{T} \cdot \frac{d(u,\ v)}{2}. \tag{5.4}$$

Combining (5.2), (5.3) and (5.4), we conclude that

$$d(Fu,\ Fv) \leq \frac{1}{2} d(u,\ v).$$

It yields

$$\begin{aligned}
\frac{1}{M(Fu,\ Fv,\ t)} - 1 &= \frac{1}{2} \left(\frac{1}{M(u,\ v,\ t)} - 1 \right) \\
&\leq \frac{1}{2} \left(\frac{1}{A_1(u,\ v)} - 1 \right),
\end{aligned}$$

for all u, $v \in C[0, T]$ with $v \preceq u$. Thus the contractive condition of Corollary 3.5 is satisfied for $k = 1/2 \in (0, 1)$. Hence, all hypotheses of Corollary 3.5 are satisfied. Thus, F has a fixed point $u^* \in C[0, T]$ which is a solution of (5.1). □

6. Conclusion

Using the same techniques, we can obtain triple, quadruple and in general, multidimensional coincidence point theorems from Theorem 3.1.

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