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# FIXED POINTS OF HYPERBOLIC CONTRACTION MAPPINGS ON HYPERBOLIC VALUED METRIC SPACES

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ABSTRACT. In this paper, we give some elementary topological concepts and results on hyperbolic valued metric space and then, we introduce two fixed point theorems for hyperbolic valued metric spaces by defining hyperbolic contraction mapping. We also give an example which support the main result.

## 1. PRELIMINARIES, BACKGROUND AND NOTATION

As we have known, fixed point theory has been an important issue of modern analysis and applied mathematics, particularly, whose importance comes from finding roots of algebraic equation and numerical analysis. The main purpose of researchers is to obtain new results in different metric spaces (see [2, 4–10, 16–18, 20]).

The most comprehensive study of analysis in the bicomplex setting is certainly the book of Price [19]. Alpay et al. [3] developed a general theory of functional analysis with bicomplex scalars. After this studies, many articles have been published in this area and many important results have been gained (see [1, 12–15]).

The definition of hyperbolic valued metric space was presented by Kumar and Saini [11]. In this study, we develop a fixed point theory by defining some topological structures related to hyperbolic valued metric spaces. We also support the main result with an example.

Now, we give basic properties of bicomplex numbers and hyperbolic numbers which will be used in our subsequent discussion. For further details on the following definitions and results, we refer the reader to [3, 15, 19].

Let $i$ and $j$ be independent imaginary units such that $i^2 = j^2 = -1$, $ij = ji$ and $\mathbb{C}(i)$ be the set of complex numbers with the imaginary unit $i$.The set of bicomplex numbers $\mathbb{BC}$ is defined by

$$\mathbb{BC} = \{z = z_1 + jz_2 : z_1, z_2 \in \mathbb{C}(i)\}.$$

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The set $\mathbb{BC}$ forms a ring with respect to the addition and multiplication defined as

$$z+w=(z_1+jz_2)+(w_1+jw_2)=(z_1+w_1)+j(z_2+w_2),$$
$$z.w=(z_1+jz_2).(w_1+jw_2)=(z_1w_1-z_2w_2)+j(z_1w_2+z_2w_1).$$

The set of hyperbolic numbers $\mathbb{D}$ is defined by

$$\mathbb{D}=\{x+ky:x,y\in\mathbb{R}\},$$

where $k^2=1$ and $k=i.j$.

The set $\mathbb{D}$ is a subring of the set $\mathbb{BC}$, and also $\mathbb{D}$ is a ring and a module over itself.

There are three types of conjugates in $\mathbb{BC}$ :

$$z^{\dagger_1}=\overline{z_1}+j\overline{z_2},$$
$$z^{\dagger_2}=z_1-jz_2,$$
$$z^{\dagger_3}=\overline{z_1}-j\overline{z_2},$$

where $\overline{z_1},\overline{z_2}$ are the complex conjugates of $z_1,z_2\in\mathbb{C}(i)$. Also, we know three types moduli for any $z\in\mathbb{BC}$:

$$|z|_i^2=z.z^{\dagger_2}=z_1^2+z_2^2\in\mathbb{C}(i),$$
$$|z|_j^2=z.z^{\dagger_1}=\left(|z_1|^2-|z_2|^2\right)+j\left(2\Re\left(z_1.\overline{z_2}\right)\right)\in\mathbb{C}(j),$$
$$|z|_k^2=z.z^{\dagger_3}=\left(|z_1|^2+|z_2|^2\right)+k\left(-\Im\left(z_1.\overline{z_2}\right)\right)\in\mathbb{D}.$$

Let $z=z_1+jz_2$ be any bicomplex number in $\mathbb{BC}$. We say that $z$ is invertible if $|z|_i\neq 0$, that is, $z_1^2+z_2^2\neq 0$ and its inverse is given by $z^{-1}=\frac{z^{\dagger_2}}{|z|_i^2}$. If, on the other hand, $z\neq 0$ but $|z|_i=0$, then $z$ is a zero divisor.

The ring $\mathbb{BC}$ is not a division ring, since one can see that if $e_1=\frac{1+ij}{2}$ and $e_2=\frac{1-ij}{2}$, then $e_1$ and $e_2$ are zero divisors. The numbers $e_1$ and $e_2$ form idempotent basis of bicomplex numbers and hence any bicomplex number $z=z_1+jz_2$ is uniquely written as

$$z=e_1\beta_1+e_2\beta_2 \tag{1.1}$$

where $\beta_1=z_1-iz_2,\beta_2=z_1+iz_2\in\mathbb{C}(i)$. Formula $(1.1)$ is called the idempotent representation of $z$.

The sum and product of bicomplex numbers is also stated by using idempotent representation $(1.1)$. Specifically, if $z=\beta_1e_1+\beta_2e_2$, $w=\gamma_1e_1+\gamma_2e_2\in\mathbb{BC}$, then

$$z+w=(\beta_1+\gamma_1)e_1+(\beta_2+\gamma_2)e_2,$$
$$z.w=\beta_1\gamma_1e_1+\beta_2\gamma_2e_2,$$
$$z^n=\beta_1^ne_1+\beta_2^ne_2.$$

Let $\alpha=x+ky$ be any hyperbolic number. Then, we have the equality

$$\alpha=e_1\alpha_1+e_2\alpha_2,$$

where $\alpha_1 = x+y, \alpha_2 = x-y \in \mathbb{R}$. If $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$, then $\alpha$ is called a positive hyperbolic number. Therefore, the set of positive hyperbolic numbers $\mathbb{D}^+$ is denoted by

$$\mathbb{D}^+ = \{\alpha = e_1\alpha_1 + e_2\alpha_2 : \alpha_1 \geq 0, \alpha_2 \geq 0\}.$$

For two hyperbolic numbers $\alpha$ and $\beta$; if their difference $\beta - \alpha \in \mathbb{D}^+$ ( or $\beta - \alpha \in \mathbb{D}^+ - \{0\}$), then we write $\alpha \precsim \beta$ ( or $\alpha \precnsim \beta$ ). For $\alpha = e_1\alpha_1 + e_2\alpha_2$, $\beta = \beta_1 e_1 + \beta_2 e_2 \in \mathbb{D}$ with real numbers $\alpha_1, \alpha_2, \beta_1$ and $\beta_2$, we have that

$$\alpha \precsim \beta \text{ if and only if } \alpha_1 \leq \beta_1 \text{ and } \alpha_2 \leq \beta_2,$$
$$\alpha \precnsim \beta \text{ if and only if } \alpha \neq \beta \text{ and } \alpha_1 \leq \beta_1 \text{ and } \alpha_2 \leq \beta_2,$$
$$\alpha \prec \beta \text{ if and only if } \alpha_1 < \beta_1 \text{ and } \alpha_2 < \beta_2.$$

This relation $\precsim$ is reflexive, anti - symmetric, transitive and so defines a partial order on $\mathbb{D}$.

We know that the hyperbolic valued module $|z|_k$ of a bicomplex number $z = e_1\beta_1 + e_2\beta_2$ is also given as

$$|z|_k = e_1 |\beta_1| + e_2 |\beta_2|.$$

One can easily see that

$$|z.w|_k = |z|_k . |w|_k$$

for any $z, w \in \mathbb{BC}$.

The following statements are true for $\alpha, \beta, \gamma \in \mathbb{D}$ :

(i) If $\alpha \precsim \beta$ then $\alpha + \gamma \precsim \beta + \gamma$.
(ii) If $\alpha \prec \beta$ then $\alpha + \gamma \prec \beta + \gamma$.
(iii) If $\alpha \precsim \beta$ and $\beta \precnsim \gamma$, then $\alpha \precnsim \gamma$.
(iv) If $\alpha \prec \beta$ and $0 \prec \gamma$, then $\alpha\gamma \prec \beta\gamma$.
(v) If $\alpha \prec \beta$ and $\gamma \prec 0$, then $\beta\gamma \prec \alpha\gamma$.
(vi) If $\alpha \precsim \beta$ and $0 \precnsim \gamma$, then $\alpha\gamma \precsim \beta\gamma$.
(vii) If $\alpha \precsim \beta$ and $\gamma \precnsim 0$, then $\beta\gamma \precsim \alpha\gamma$.
(viii) If $\alpha \precsim \beta$ and $\gamma \precsim \delta$, then $\alpha + \gamma \precsim \beta + \delta$.
(ix) If $\alpha \precnsim \beta$ and $\gamma \precnsim \delta$, then $\alpha + \gamma \precnsim \beta + \delta$.
(x) If $\alpha, \beta \in \mathbb{D}^+$, then $\alpha \precsim \beta$ (or $\alpha \precnsim \beta$ or $\alpha \prec \beta$) implies that $|\alpha| \leq |\beta|$ (or $|\alpha| < |\beta|$) where $|.|$ shows Euclidean norm in $\mathbb{BC}$ (see [3, 15]).
(xi) If $\alpha \in \mathbb{D}^+$, then $|\alpha|_k = \alpha$.

A sequence in $\mathbb{BC}$ is a function defined by $z : \mathbb{N} \to \mathbb{BC}, n \to z_n$. This sequence converges to a point $z^* \in \mathbb{BC}$ if and only if to each $\varepsilon > 0$ there corresponds an $n_0(\varepsilon)$ such that $|z_n - z^*| < \varepsilon$ for all $n \geq n_0(\varepsilon)$. It is denoted by $z_n \to z^*$ as $n \to +\infty$. The sequence $z = (z_n)$ is a Cauchy sequence in $\mathbb{BC}$ if and only if to each $\varepsilon > 0$ there corresponds an $n_0(\varepsilon)$ such that $|z_n - z_m| < \varepsilon$ for all $n, m \geq n_0(\varepsilon)$. Also, $z = (z_n)$ converges to a point in $\mathbb{BC}$ if and only if it is a Cauchy sequence in $\mathbb{BC}$. On the other hand, for any sequence $(z_n)$ in $\mathbb{BC}$ such that $z : \mathbb{N} \to \mathbb{BC}$, $z_n = \beta_{1n}e_1 + \beta_{2n}e_2$ and for any $z^* = \beta_1^* e_1 + \beta_2^* e_2 \in \mathbb{BC}$, we have that $z_n \to z^*$ as $n \to +\infty$ if and only if $\beta_{1n} \to \beta_1^*$ and $\beta_{2n} \to \beta_2^*$ as $n \to +\infty$.

The following definition and lemma are recently introduced by Kumar and Saini [11].

**Definition 1.1.** Let $X$ be a nonempty set and $d_{\mathbb{D}} : X \times X \to \mathbb{D}$ be a function such that for any $x, y, z \in X$, the following properties hold :
(i) $0_{\mathbb{D}} \precsim d_{\mathbb{D}}(x,y)$ *and* $d_{\mathbb{D}}(x,y) = 0$ *if and only if* $x = y$,
(ii) $d_{\mathbb{D}}(x,y) = d_{\mathbb{D}}(y,x)$,
(iii) $d_{\mathbb{D}}(x,z) \precsim d_{\mathbb{D}}(x,y) + d_{\mathbb{D}}(y,z)$.
*Then* $d_{\mathbb{D}}$ *is called a hyperbolic valued or* $\mathbb{D}$ *- valued metric on* $X$ *and the pair* $(X, d_{\mathbb{D}})$ *is called a hyperbolic valued or* $\mathbb{D}$ *- valued metric space* [11].

**Example 1.2.** 1) Let $X = \mathbb{R}$ and a mapping $d_{\mathbb{D}} : \mathbb{R} \times \mathbb{R} \to \mathbb{D}$ be defined by

$$d_{\mathbb{D}}(x,y) := (1 + ij)|x - y|$$

*for any* $x, y \in X$ *where* $|.|$ *is the usual real modulus. Then,* $(\mathbb{R}, d_{\mathbb{D}})$ *is a hyperbolic valued metric space.*
2) Let $X = \mathbb{BC}$, *the mapping* $d_{\mathbb{D}} : \mathbb{BC} \times \mathbb{BC} \to \mathbb{D}$ *defined by* $d_{\mathbb{D}}(x,y) := |x - y|_k$ *for any* $x, y \in \mathbb{BC}$ *is a hyperbolic valued metric.*

**Lemma 1.3.** *Every hyperbolic valued metric space is first countable [11].*

## 2. Some topological concepts on hyperbolic valued metric spaces

In this part, we define some topological structures related to hyperbolic valued metric spaces and we discuss some of their properties which will be required in the subsequent section.

**Definition 2.1.** Let $(X, d_{\mathbb{D}})$ be a hyperbolic valued metric space, $x \in X$, and $0_{\mathbb{D}} \precnsim r \in \mathbb{D}$, we define a set $B_{\mathbb{D}}(x,r) = \{y \in X : d_{\mathbb{D}}(x,y) \precnsim r\}$ which is called a hyperbolic open ball of hyperbolic radius $r$ with center $x$. Similarly, a hyperbolic closed ball of hyperbolic radius $r$ with center $x$ is defined by $\overline{B_{\mathbb{D}}}(x,r) = \{y \in X : d_{\mathbb{D}}(x,y) \precsim r\}$.

**Definition 2.2.** Let $(X, d_{\mathbb{D}})$ be a hyperbolic valued metric space and $A \subset X$. A point $x \in X$ is called a interior point of $A$ if there exists $0 \precnsim r \in \mathbb{D}$ such that $B_{\mathbb{D}}(x,r) \subset A$. A point $x \in X$ is called a limit point of $A$ if $(B_{\mathbb{D}}(x,r) - \{x\}) \cap A \neq \varnothing$ for every $0 \precnsim r \in \mathbb{D}$. The set of interior points of $A$ is denoted by $A^{\circ_{\mathbb{D}}}$ and the set of limit points of $A$ is denoted by $A'^{\mathbb{D}}$. We say that the subset $A$ is a hyperbolic open set if each element of $A$ belong to $A^{\circ_{\mathbb{D}}}$. We say that the subset $A$ is a hyperbolic closed set if each limit point of $A$ belong to $A$.

**Lemma 2.3.** *Let* $(X, d_{\mathbb{D}})$ *be a hyperbolic valued metric space. Then, each hyperbolic open ball of* $X$ *is a hyperbolic open set.*

*Proof.* Let $x \in X$, $0 \precnsim r \in \mathbb{D}$ and $B_{\mathbb{D}}(x,r)$ be a hyperbolic open ball. We show that $B_{\mathbb{D}}(x,r)$ is a hyperbolic open set. Suppose that $\delta = r - d_{\mathbb{D}}(x,y)$ and $z \in B_{\mathbb{D}}(y,\delta)$ for each $y \in B_{\mathbb{D}}(x,r)$ .Thus, we can write $d_{\mathbb{D}}(y,z) \precnsim \delta = r - d_{\mathbb{D}}(x,y)$. Then, it follows that $d_{\mathbb{D}}(x,z) \precsim d_{\mathbb{D}}(x,y) + d_{\mathbb{D}}(y,z) \precnsim r$. This means that $z \in B_{\mathbb{D}}(x,r)$. Therefore, $B_{\mathbb{D}}(y,\delta) \subset B_{\mathbb{D}}(x,r)$ which shows that $B_{\mathbb{D}}(x,r)$ is a hyperbolic open set. This completes the proof. □

**Lemma 2.4.** *Let $(X,d_{\mathbb{D}})$ be a hyperbolic valued metric space. Then, the following statements hold :*

(i) *The sets $X$ and $\varnothing$ are hyperbolic open sets.*
(ii) *The intersection of any finite family of hyperbolic open sets is also a hyperbolic open set.*
(iii) *The union of any countable family of hyperbolic open sets is also a hyperbolic open set.*
(iv) *The union of any uncountable family of hyperbolic open sets is also a hyperbolic open set.*

*Proof.* The first one is clear. Let us prove (ii). Let $A_1, A_2, \ldots, A_m$ be hyperbolic open sets and $y$ be any point of $A_1 \cup A_2 \cup \ldots \cup A_m$. Then, there is a natural number $m_0 \in \{1,2,\ldots,m\}$ such that $y \in A_{m_0}$. Since $A_{m_0}$ is a hyperbolic open set, there exists $0 \precnsim r \in \mathbb{D}$ such that $B_{\mathbb{D}}(y,r) \subset A_{m_0}$. Therefore, we conclude that $B_{\mathbb{D}}(y,r) \subset A_1 \cup A_2 \cup \ldots \cup A_m$. This means that the union of any finite family of hyperbolic open sets is a hyperbolic open set.

The proofs of (iii) and (iv) are similar to the proof of (ii). □

**Corollary 2.5.** *Every hyperbolic valued metric space is a topological space based on the set of all hyperbolic open sets.*

*Proof.* The proof of Corollary 2.1 depends on properties of hyperbolic open sets which are in Lemma 2.2. □

**Proposition 2.6.** *Let $(X,d_{\mathbb{D}})$ be a hyperbolic valued metric space and $A \subset X$. Then, the set $A$ is hyperbolic closed if and only if $X - A$ is hyperbolic open.*

*Proof.* The proof of this proposition is direct applications of definitions. □

**Definition 2.7.** Let $(X,d_{\mathbb{D}})$ be a hyperbolic valued metric space, $(x_n)$ be any sequence in $X$ and $x \in X$. If for every $0 \precnsim \varepsilon \in \mathbb{D}$ there exists $n_0 \in \mathbb{N}$ depending on $\varepsilon$ such that for all $n \geq n_0$

$$d_{\mathbb{D}}(x_n,x) \precnsim \varepsilon,$$

*then we say that $(x_n)$ is convergent with respect to the metric $d_{\mathbb{D}}$. We denote this by $\lim_{\mathbb{D}\, n\to+\infty} x_n = x$ or $x_n \overset{d_{\mathbb{D}}}{\to} x$ as $n \to +\infty$.*

*If for every $0 \precnsim \varepsilon \in \mathbb{D}$ there exists $n_0 \in \mathbb{N}$ depending on $\varepsilon$ such that for all $n,m \geq n_0$*

$$d_{\mathbb{D}}(x_n,x_m) \precnsim \varepsilon,$$

*then we say that $(x_n)$ is a Cauchy sequence with respect to the metric $d_{\mathbb{D}}$.*

*If every Cauchy sequence with respect to the metric $d_{\mathbb{D}}$ is convergent with respect to the metric $d_{\mathbb{D}}$ in $(X,d_{\mathbb{D}})$, then we say that $(X,d_{\mathbb{D}})$ is a complete hyperbolic valued metric space.*

**Theorem 2.8.** *Let $(X,d_{\mathbb{D}})$ be a hyperbolic valued metric space, $x\in X$ and $A\subset X$. Then, $x\in A'^{\mathbb{D}}$ if and only if there exists a sequence $(x_n)$ contained in A with $x_n\neq x$ for all $n\in\mathbb{N}$ and $x_n\overset{d_{\mathbb{D}}}{\to}x$ as $n\to+\infty$.*

*Proof.* Let $x\in A'^{\mathbb{D}}$.Then, $(B_{\mathbb{D}}(x,\varepsilon)-\{x\})\cap A\neq\varnothing$ for every $0\precnsim\varepsilon\in\mathbb{D}$. Since the hyperbolic valued metric space $(X,d_{\mathbb{D}})$ is first countable, each point $x$ in $X$ has a countable neighbourhood basis. For each $x\in X$ we can take the neighbourhood basis of $x$, $B^x_{\mathbb{D}}$ to be the set of all hyperbolic open balls centered at $x$ with radius $\frac{1}{n}e_1+\frac{1}{n}e_2$, $B^x_{\mathbb{D}}=\left\{B_{\mathbb{D}}\left(x,\frac{1}{n}e_1+\frac{1}{n}e_2\right):n\in\mathbb{N}\right\}$. In that case, $\left\{B_{\mathbb{D}}\left(x,\frac{1}{n}e_1+\frac{1}{n}e_2\right)-\{x\}\right\}\cap A\neq\varnothing$ for all $n\in\mathbb{N}$. Hence, we can find a point $x_n\in X$ different from $x$ with $x_n\in B_{\mathbb{D}}\left(x,\frac{1}{n}e_1+\frac{1}{n}e_2\right)$ and $x_n\in A$ for each $n\in\mathbb{N}$. Thus, we obtain that there exists a sequence $(x_n)$ contained in $A$ with $x_n\neq x$ for all $n\in\mathbb{N}$ and $x_n\overset{d_{\mathbb{D}}}{\to}x$ as $n\to+\infty$.

Conversely, suppose that there exists a sequence $(x_n)$ contained in $A$ with $x_n\neq x$ for all $n\in\mathbb{N}$ and $x_n\overset{d_{\mathbb{D}}}{\to}x$ as $n\to+\infty$. In this case, for every $0\precnsim\varepsilon\in\mathbb{D}$ there exists $n_0\in\mathbb{N}$ depending on $\varepsilon$ such that $0\precnsim d_{\mathbb{D}}(x_n,x)\precnsim\varepsilon$ for all $n\geq n_0$. Thus, we can write $x_n\in B_{\mathbb{D}}(x,\varepsilon)-\{x\}$ for every $0\precnsim\varepsilon\in\mathbb{D}$ and for all $n\geq n_0$. This implies that $(B_{\mathbb{D}}(x,\varepsilon)-\{x\})\cap A\neq\varnothing$ for every $0\precnsim\varepsilon\in D$, that is, $x\in A'^{\mathbb{D}}$. The proof is completed. □

**Theorem 2.9.** *Let $(X,d_{\mathbb{D}})$ be a hyperbolic valued metric space and $A\subset X$. Then, the inclusion $(A'^{\mathbb{D}})'^{\mathbb{D}}\subset A'^{\mathbb{D}}$ holds.*

*Proof.* Let $x$ be any limit point of $A'^{\mathbb{D}}$. Then, for every $0\precnsim\varepsilon\in\mathbb{D}$, $(B_{\mathbb{D}}(x,\varepsilon)-\{x\})\cap A'^{\mathbb{D}}\neq\varnothing$. In this case, $\left(B_{\mathbb{D}}\left(x,\frac{\varepsilon}{2}\right)-\{x\}\right)\cap A'^{\mathbb{D}}\neq\varnothing$. This implies that there exists $y\in A'^{\mathbb{D}}$ different from $x$ with $y\in B_{\mathbb{D}}\left(x,\frac{\varepsilon}{2}\right)$. On the other hand, since $y\in A'^{\mathbb{D}}$, for every $0\precnsim\varepsilon\in\mathbb{D}$, $(B_{\mathbb{D}}(y,\varepsilon)-\{y\})\cap A\neq\varnothing$. In this case, $\left(B_{\mathbb{D}}\left(y,\frac{\varepsilon}{2}\right)-\{y\}\right)\cap A\neq\varnothing$.This implies that there exists $z\in A$ different from $y$ with $z\in B_{\mathbb{D}}\left(y,\frac{\varepsilon}{2}\right)$.

We assume that $x\notin A'^{\mathbb{D}}$. Then, we can find a hyperbolic number $0\precnsim\varepsilon_0\in\mathbb{D}$ such that $(B_{\mathbb{D}}(x,\varepsilon_0)-\{x\})\cap A=\varnothing$. We claim that $z\neq x$. In fact, if $z=x$, it would be obtained that $x\in(B_{\mathbb{D}}(y,\varepsilon)-\{y\})\cap A$ for every $0\precnsim\varepsilon\in\mathbb{D}$, that is, $x\in A'^{\mathbb{D}}$ which contradicts our assumption. Therefore, it is seen that $z\neq x$. Also, this means that $z\in(B_{\mathbb{D}}(x,\varepsilon)-\{x\})\cap A$ for every $0\precnsim\varepsilon\in\mathbb{D}$, but this is a contradiction. Then, our assumption is wrong, $x\in A'^{\mathbb{D}}$ and so the inclusion $(A'^{\mathbb{D}})'^{\mathbb{D}}\subset A'^{\mathbb{D}}$ holds. This completes the proof. □

**Proposition 2.10.** *Let $(X,d_{\mathbb{D}})$ be a hyperbolic valued metric space and $A\subset X$. Then, $A\cup A'^{\mathbb{D}}$ is a hyperbolic closed set. This set is called a hyperbolic closure of the set $A$, which is denoted by $\overline{A}^{\mathbb{D}}$.*

*Proof.* The proof depends on definition of limit point, Proposition 2.1 and Theorem 2.2. □

**Proposition 2.11.** *Let $(X,d_{\mathbb{D}})$ be a hyperbolic valued metric space, $x\in X$ and $A\subset X$. Then,*

(i) $\overline{A}^{\mathbb{D}} = A$ *if and only if* $A$ *is hyperbolic closed.*

(ii) $x \in \overline{A}^{\mathbb{D}}$ *if and only if* $B_{\mathbb{D}}(x,\varepsilon) \cap A \neq \varnothing$ *for every* $0 \precnsim \varepsilon \in \mathbb{D}$.

(iii) $x \in \overline{A}^{\mathbb{D}}$ *if and only if there exists a sequence* $(x_n)$ *contained in* $A$ *such that* $x_n \overset{d_{\mathbb{D}}}{\to} x$ *as* $n \to +\infty$.

*Proof.* The proof of all parts of Proposition 2.3 is clear from Proposition 2.2. □

**Proposition 2.12.** *Let* $(X, d_{\mathbb{D}})$ *be a hyperbolic valued metric space,* $(x_n)$ *be a sequence in* $X$, $x \in X$ *and* $x_n \overset{d_{\mathbb{D}}}{\to} x$ *as* $n \to +\infty$. *Then,*

(i) *The limit point of the sequence* $(x_n)$ *is unique.*

(ii) *The sequence* $(x_n)$ *is a Cauchy sequence with respect to the metric* $d_{\mathbb{D}}$.

(iii) *All subsequences of the sequence* $(x_n)$ *converges to* $x$ *with respect to the metric* $d_{\mathbb{D}}$.

*Proof.* (i) The proof is clear with a routine verification.

(ii) Since $x_n \overset{d_{\mathbb{D}}}{\to} x$ as $n \to +\infty$, for every $0_{\mathbb{D}} \precnsim \varepsilon \in \mathbb{D}$ there exists $n_0 \in \mathbb{N}$ depending on $\varepsilon$ such that $d_{\mathbb{D}}(x_n, x) \precnsim \frac{\varepsilon}{2}$ for all $n \geq n_0$. In this case,

$$d_{\mathbb{D}}(x_n, x_m) \precsim d_{\mathbb{D}}(x_n, x) + d_{\mathbb{D}}(x_m, x) \precnsim \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for every $0 \precnsim \varepsilon \in \mathbb{D}$ and for all $n, m \geq n_0$.

This means that $(x_n)$ is a Cauchy sequence with respect to the metric $d_{\mathbb{D}}$.

(iii) Let $(y_n)$ be any subsequence of $(x_n)$. Since $x_n \overset{d_{\mathbb{D}}}{\to} x$ as $n \to +\infty$, for every $0 \precnsim \varepsilon \in \mathbb{D}$ there exists $n_1 \in \mathbb{N}$ depending on $\varepsilon$ such that for all $n \geq n_1$, $d_{\mathbb{D}}(x_n, x) \precnsim \frac{\varepsilon}{2}$ and also, $(x_n)$ is a Cauchy sequence with respect to the metric $d_{\mathbb{D}}$ by (ii), that is, there exists $n_2 \in \mathbb{N}$ depending on $\varepsilon$ such that for all $n, m \geq n_2$, $d_{\mathbb{D}}(x_n, x_m) \precnsim \frac{\varepsilon}{2}$. Since $d_{\mathbb{D}}(y_n, x) \precsim d_{\mathbb{D}}(y_n, x_n) + d_{\mathbb{D}}(x_n, x)$ for all $n \in \mathbb{N}$, if we take $n_0 = \max\{n_1, n_2\}$, then

$$d_{\mathbb{D}}(y_n, x) \precnsim \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \geq n_0$ which implies that $y_n \overset{d_{\mathbb{D}}}{\to} x$ as $n \to +\infty$.

Then, the proof of all parts of Proposition 2.4 is complete. □

**Lemma 2.13.** *Let* $(X, d_{\mathbb{D}})$ *be a hyperbolic valued metric space,* $(x_n)$ *be any sequence in* $X$ *and* $x \in X$. *Then, the sequence* $(x_n)$ *converges to* $x$ *with respect to the metric* $d_{\mathbb{D}}$ *if and only if* $|d_{\mathbb{D}}(x_n, x)| \to 0$ *as* $n \to +\infty$.

*Proof.* Suppose that $(x_n)$ converges to $x$ with respect to the metric $d_{\mathbb{D}}$. For a chosen real number $c > 0$, let $\varepsilon = ce_1 + ce_2$. Then, $\varepsilon \in \mathbb{D}$ and $0 \precnsim \varepsilon$. Thus, there is a natural number $n_0$ such that $d_{\mathbb{D}}(x_n, x) \precnsim \varepsilon$ whenever $n \geq n_0$. In this case, $|d_{\mathbb{D}}(x_n, x)| < |\varepsilon| = c$ for all $n \geq n_0$. This implies that $|d_{\mathbb{D}}(x_n, x)| \to 0$ as $n \to +\infty$.

Conversely, suppose that $|d_{\mathbb{D}}(x_n, x)| \to 0$ as $n \to +\infty$. We claim that for a given hyperbolic number $\varepsilon = \varepsilon_1 e_1 + \varepsilon_2 e_2$ with $\varepsilon_1, \varepsilon_2 > 0$, there is a real number $\delta > 0$ such that for any $\alpha = \alpha_1 e_1 + \alpha_2 e_2 \in \mathbb{D}$

$$\alpha \prec \varepsilon \text{ whenever } |\alpha| < \delta.$$

In fact, set $\delta = \min\left\{\frac{\varepsilon_1}{2}, \frac{\varepsilon_2}{2}\right\}$ for every $0 \prec \varepsilon \in \mathbb{D}$. If $|\alpha| < \delta$, we obtain that $\alpha_1 < \varepsilon_1$ and $\alpha_2 < \varepsilon_2$, that is, $\alpha \prec \varepsilon$.

Also, for this $\delta$, there is a natural number $n_0$ such that $|d_{\mathbb{D}}(x_n, x)| < \delta$ for all $n \geq n_0$. This implies that $d_{\mathbb{D}}(x_n, x) \prec \varepsilon$ for all $n \geq n_0$. Thus, $(x_n)$ converges to $x$ with respect to the metric $d_{\mathbb{D}}$. The proof is completed. □

**Corollary 2.14.** *Let $(X, d_{\mathbb{D}})$ be a hyperbolic valued metric space, $(x_n)$ be any sequence in $X$ and $x \in X$. Then, the sequence $(x_n)$ converges to $x$ with respect to the metric $d_{\mathbb{D}}$ if and only if $d_{\mathbb{D}}(x_n, x) = |d_{\mathbb{D}}(x_n, x)|_k \to 0$ as $n \to +\infty$.*

*Proof.* The proof is clear from Lemma 2.3. □

**Lemma 2.15.** *Let $(X, d_{\mathbb{D}})$ be a hyperbolic valued metric space and $(x_n)$ be any sequence in $X$. Then, the sequence $(x_n)$ is a Cauchy sequence with respect to the metric $d_{\mathbb{D}}$ if and only if for all $m \in \mathbb{N}$, $|d_{\mathbb{D}}(x_n, x_{n+m})| \to 0$ as $n \to +\infty$.*

*Proof.* Suppose that $(x_n)$ is a Cauchy sequence with respect to the metric $d_{\mathbb{D}}$. For a chosen real number $c > 0$, let $\varepsilon = ce_1 + ce_2$. Then, $\varepsilon \in \mathbb{D}$ and $0 \precnsim \varepsilon$. Thus, there is a natural number $n_0$ such that $d_{\mathbb{D}}(x_n, x_k) \precnsim \varepsilon$ whenever $n, k \geq n_0$. Since there exists a natural number $m$ such that $k = m + n$ for each $k$ greater than $n$, we can write $d_{\mathbb{D}}(x_n, x_{n+m}) \precnsim \varepsilon$ for all $n \geq n_0$. Therefore, $|d_{\mathbb{D}}(x_n, x_{n+m})| < |\varepsilon| = c$ for all $n \geq n_0$. This implies that $|d_{\mathbb{D}}(x_n, x_{n+m})| \to 0$ as $n \to +\infty$.

Conversely, suppose that $|d_{\mathbb{D}}(x_n, x_{n+m})| \to 0$ as $n \to +\infty$. Then, for a given $0 \precnsim \varepsilon \in \mathbb{D}$, there is a real number $\delta > 0$ such that for any $\alpha = \alpha_1 e_1 + \alpha_2 e_2 \in \mathbb{D}$

$$\alpha \prec \varepsilon \text{ whenever } |\alpha| < \delta.$$

For this $\delta$, there is a natural number $n_0$ such that $|d_{\mathbb{D}}(x_n, x_{n+m})| < \delta$ for all $n \geq n_0$. This means that $d_{\mathbb{D}}(x_n, x_{n+m}) \prec \varepsilon$ for all $n \geq n_0$. Then, we obtain that $(x_n)$ is a Cauchy sequence with respect to the metric $d_{\mathbb{D}}$. □

**Corollary 2.16.** *Let $(X, d_{\mathbb{D}})$ be a hyperbolic valued metric space and $(x_n)$ be any sequence in $X$. Then, the sequence $(x_n)$ is a Cauchy sequence with respect to the metric $d_{\mathbb{D}}$ if and only if for all $m \in \mathbb{N}$, $d_{\mathbb{D}}(x_n, x_{n+m}) = |d_{\mathbb{D}}(x_n, x_{n+m})|_k \to 0$ as $n \to +\infty$.*

*Proof.* The proof is clear from Lemma 2.4. □

**Proposition 2.17.** *Let $(X, d_{\mathbb{D}})$ be a hyperbolic valued metric space. Then, we have the following statement :*

$$|d_{\mathbb{D}}(x,z) - d_{\mathbb{D}}(y,z)|_k \precsim d_{\mathbb{D}}(x,y)$$

*for all $x, y, z \in X$.*

*Proof.* We know $d_{\mathbb{D}}(x,z) \precsim d_{\mathbb{D}}(x,y) + d_{\mathbb{D}}(y,z)$ and $d_{\mathbb{D}}(y,z) \precsim d_{\mathbb{D}}(y,x) + d_{\mathbb{D}}(x,z)$ for any $x, y, z \in X$ . Thus, we have

$$-d_{\mathbb{D}}(y,x) \precsim d_{\mathbb{D}}(x,z) - d_{\mathbb{D}}(y,z) \precsim d_{\mathbb{D}}(x,y).$$

Let $d_{\mathbb{D}}(y,x)=\alpha_1 e_1+\alpha_2 e_2,\ d_{\mathbb{D}}(x,z)=\beta_1 e_1+\beta_2 e_2,\ d_{\mathbb{D}}(y,z)=\gamma_1 e_1+\gamma_2 e_2$ for any $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2\in\mathbb{R}^+\cup\{0\}$. In this case, we can write $|\beta_1-\gamma_1|\leq\alpha_1$ and $|\beta_2-\gamma_2|\leq\alpha_2$. This implies that

$$|d_{\mathbb{D}}(x,z)-d_{\mathbb{D}}(y,z)|_k=|\beta_1-\gamma_1|e_1+|\beta_2-\gamma_2|e_2\precsim\alpha_1 e_1+\alpha_2 e_2=d_{\mathbb{D}}(y,x).$$

The proof is completed. □

**Theorem 2.18.** *Let $(X,d_{\mathbb{D}})$ be a hyperbolic valued metric space, $(x_n)$ be any sequence in $X$, $x\in X$ and $x_n\overset{d_{\mathbb{D}}}{\to}x$ as $n\to+\infty$. Then, for every $y\in X$,*

$$|d_{\mathbb{D}}(x_n,y)-d_{\mathbb{D}}(x,y)|_k\to 0.$$

*Proof.* We know that $|d_{\mathbb{D}}(x_n,y)-d_{\mathbb{D}}(x,y)|_k\precsim d_{\mathbb{D}}(x_n,x)$ for every $y\in X$ and all $n\in\mathbb{N}$ by Proposition 2.5. Since $x_n\overset{d_{\mathbb{D}}}{\to}x$ *as* $n\to+\infty$, it follows that $|d_{\mathbb{D}}(x_n,y)-d_{\mathbb{D}}(x,y)|_k\to 0$ *as* $n\to+\infty$ which is what we want to see. Also, this means that $d_{\mathbb{D}}(x_n,y)\to d_{\mathbb{D}}(x,y)$. The proof is completed. □

**Definition 2.19.** Let $(X,d_{\mathbb{D}}^X)$ and $(Y,d_{\mathbb{D}}^Y)$ be two hyperbolic valued metric spaces and $f:X\to Y$ be a function. If for every $0\precnsim\varepsilon\in\mathbb{D}$ there exists $0\precnsim\delta\in\mathbb{D}$ such that

$$f(B_{\mathbb{D}}(x,\delta))\subset B_{\mathbb{D}}(f(x),\varepsilon),$$

*then we say that the function $f$ is hyperbolic continuous at $x\in X$.*

**Theorem 2.20.** *Let $(X,d_{\mathbb{D}}^X)$ and $(Y,d_{\mathbb{D}}^Y)$ be two hyperbolic valued metric spaces, $f:X\to Y$ be a function and $x\in X$. Then, the function $f$ is hyperbolic continuous at the point $x$ if and only if $f(x_n)\overset{d_{\mathbb{D}}^Y}{\to}f(x)$ for every sequence $(x_n)$ with $x_n\overset{d_{\mathbb{D}}^X}{\to}x$ as $n\to+\infty$.*

*Proof.* Let $f$ be hyperbolic continuous at the point $x\in X$. Then, for every $0\precnsim\varepsilon\in\mathbb{D}$ there exists $0\precnsim\delta\in\mathbb{D}$ such that $f(B_{\mathbb{D}}(x,\delta))\subset B_{\mathbb{D}}(f(x),\varepsilon)$. Also, since $x_n\overset{d_{\mathbb{D}}^X}{\to}x$ *as* $n\to+\infty$, there exists a natural number $n_0$ such that $d_{\mathbb{D}}^X(x_n,x)<\delta$ for all $n\geq n_0$. In that case, $x_n\in B_{\mathbb{D}}(x,\delta)$ for all $n\geq n_0$, and so $f(x_n)\in B_{\mathbb{D}}(f(x),\varepsilon)$ for all $n\geq n_0$ by hypothesis. Thus, we obtain that $f(x_n)\overset{d_{\mathbb{D}}^Y}{\to}f(x)$ as $n\to+\infty$.

Conversely, suppose that $f(x_n)\overset{d_{\mathbb{D}}^Y}{\to}f(x)$ as $n\to+\infty$ for every sequence $(x_n)$ with $x_n\overset{d_{\mathbb{D}}^X}{\to}x$ *as* $n\to+\infty$. Assume that the function $f$ is not hyperbolic continuous at $x\in X$. Then, there is some $0\precnsim\varepsilon\in\mathbb{D}$ such that

$$f(B_{\mathbb{D}}(x,\delta))\subsetneq B_{\mathbb{D}}(f(x),\varepsilon)$$

for every $0\precnsim\delta\in\mathbb{D}$. Since the hyperbolic valued metric space $(X,d_{\mathbb{D}}^X)$ is first countable, each point $x$ in $X$ has a countable neighbourhood basis. For each $x\in X$ we can take the neighbourhood basis of $x$, $B_{\mathbb{D}}^x$ to be the set of all hyperbolic open balls centered at $x$ with radius $\frac{1}{n}e_1+\frac{1}{n}e_2$

$$B_{\mathbb{D}}^x=\left\{B_{\mathbb{D}}\left(x,\frac{1}{n}e_1+\frac{1}{n}e_2\right):n\in\mathbb{N}\right\}.$$

In that case, $f\left(B_{\mathbb{D}}\left(x,\frac{1}{n}e_1+\frac{1}{n}e_2\right)\right)\subsetneq B_{\mathbb{D}}\left(f\left(x\right),\varepsilon\right)$ for all $n\in\mathbb{N}$. Hence, we can find a point $x_n\in X$ with $x_n\in B_{\mathbb{D}}\left(x,\frac{1}{n}e_1+\frac{1}{n}e_2\right)$, but $f\left(x_n\right)\notin B_{\mathbb{D}}\left(f\left(x\right),\varepsilon\right)$ for each $n\in\mathbb{N}$. This implies that $d_{\mathbb{D}}\left(x_n,x\right)\precsim\!\!\!\!/\;\frac{1}{n}e_1+\frac{1}{n}e_2$ and $d_{\mathbb{D}}\left(f\left(x_n\right),f\left(x\right)\right)\not\prec\varepsilon$ for each $n\in\mathbb{N}$. Then, $x_n\overset{d_{\mathbb{D}}^X}{\to}x$ *as* $n\to+\infty$, but $\left(f\left(x_n\right)\right)$ is not convergent to $f\left(x\right)$ with respect to the metric $d_{\mathbb{D}}^Y$. This yields a contradiction and so $f$ is hyperbolic continuous at the point $x\in X$. This completes the proof. □

**Theorem 2.21.** *The following statements are true for* $\alpha\in\mathbb{D}$ *:*

(i) $\alpha^0=1$.

(ii) *If* $\alpha\in\mathbb{D}^+$ , $\alpha\neq1$ *and* $1-\alpha$ *is invertible, then*

$$1+\alpha+\alpha^2+...+\alpha^n=\frac{1-\alpha^{n+1}}{1-\alpha} \tag{2.1}$$

*for all* $n\in\mathbb{N}$.

(iii) *If* $\alpha\in\mathbb{D}^+$ *and* $\alpha\prec1$, *then* $0\precsim\alpha^n\prec1$ *for all* $n\in\mathbb{N}$ *and* $\alpha^n\to0$.

*Proof.* (i) Let $\alpha=\alpha_1e_1+\alpha_2e_2$. Then, $\alpha^0=\alpha_1^0e_1+\alpha_2^0e_2=1e_1+1e_2=1$.

(ii) For $n=0$ and $n=1$, the proof is clear.

We assume that (2.1) holds for $n=k$. Now, we want to show that (2.1) holds for $n=k+1$. Consider

$$\begin{aligned}1+\alpha+\alpha^2+...+\alpha^k+\alpha^{k+1}&=\frac{1-\alpha^{k+1}}{1-\alpha}+\alpha^{k+1}\\&=\frac{1-\alpha^{k+1}}{1-\alpha}+\frac{\left(1-\alpha\right)\alpha^{k+1}}{1-\alpha}\\&=\frac{1-\alpha^{k+1}+\alpha^{k+1}-\alpha^{k+2}}{1-\alpha}\\&=\frac{1-\alpha^{k+2}}{1-\alpha}.\end{aligned}$$

Then, (2.1) is true for $n=k+1$. Thus, the mathematical induction principle completes the proof.

(iii) Let $\alpha=\alpha_1e_1+\alpha_2e_2$. Since $0\precsim\alpha=\alpha_1e_1+\alpha_2e_2\prec1=1e_1+1e_2$, we write $0\leq\alpha_1<1$ and $0\leq\alpha_2<1$. In this case, $0\leq\alpha_1^n<1$, $0\leq\alpha_2^n<1$ for all $n\in\mathbb{N}$ and so $\alpha_1^n\to0$, $\alpha_2^n\to0$ as $n\to+\infty$. On the other hand, since $\alpha^n=\alpha_1^ne_1+\alpha_2^ne_2$ for all $n\in\mathbb{N}$. Thus, $0\precsim\alpha^n\prec1$ for all $n\in\mathbb{N}$ and $\alpha^n\to0$.

Then, the proof of all parts of Theorem 2.5 is complete. □

## 3. Main Results

In this part, we first define the hyperbolic contraction mapping, then introduce two fixed point theorems for hyperbolic valued metric spaces and finally we give an example for main result.

**Definition 3.1.** Let $X$ be a set and $T$ be a mapping from $X$ to $X$. A fixed point of $T$ is a point $x\in X$ such that $Tx=x$.

**Definition 3.2.** Suppose that $(X, d_{\mathbb{D}})$ is a complete hyperbolic valued metric space, $T : X \to X$, $\alpha \in \mathbb{D}^{+}$ and $1-\alpha$ is invertible. The mapping $T$ is said to satisfy hyperbolic Lipschitz condition if

$$d_{\mathbb{D}}(Tx, Ty) \precsim \alpha d_{\mathbb{D}}(x, y)$$

*holds for all* $x, y \in X$. *If* $\alpha \prec 1$, *then the mapping* $T$ *is called a hyperbolic contraction mapping on* $X$.

**Theorem 3.3.** *Let* $(X, d_{\mathbb{D}})$ *be a complete hyperbolic valued metric space and* $T$ *be a hyperbolic contraction mapping on* $X$. *Then,* $T$ *has a unique fixed point.*

*Proof.* Let $x_0$ be any point in $X$. We define the iterative sequence $(x_n)$,

$$\begin{aligned} &x_0, \\ Tx_0 &= x_1, \\ Tx_1 &= TTx_0 = T^2x_0 = x_2, \\ &\vdots \\ T^n x_0 &= x_n, \end{aligned}$$

and we show that the sequence $(x_n)$ is a Cauchy sequence with respect to the metric $d_{\mathbb{D}}$. For all $m, n \in \mathbb{N}$ if $m < n$, then

$$\begin{aligned} d_{\mathbb{D}}(x_m, x_n) &= d_{\mathbb{D}}(T^m x_0, T^n x_0) \\ &= d_{\mathbb{D}}\left(T^m x_0, T^m T^{n-m} x_0\right) \\ &\precsim \alpha^m d_{\mathbb{D}}\left(x_0, T^{n-m} x_0\right) \\ &= \alpha^m d_{\mathbb{D}}(x_0, x_{n-m}) \\ &\precsim \alpha^m \{d_{\mathbb{D}}(x_0, x_1) + d_{\mathbb{D}}(x_1, x_2) + \ldots + d_{\mathbb{D}}(x_{n-m-1}, x_{n-m})\} \\ &= \alpha^m \left\{d_{\mathbb{D}}(x_0, x_1) + d_{\mathbb{D}}(Tx_0, Tx_1) + \ldots + d_{\mathbb{D}}\left(T^{n-m-1}x_0, T^{n-m-1}x_1\right)\right\} \\ &\precsim \alpha^m \left\{d_{\mathbb{D}}(x_0, x_1) + \alpha d_{\mathbb{D}}(x_0, x_1) + \ldots + \alpha^{n-m-1} d_{\mathbb{D}}(x_0, x_1)\right\} \\ &= \alpha^m d_{\mathbb{D}}(x_0, x_1)\left\{1 + \alpha + \ldots + \alpha^{n-m-1}\right\} \\ &= \alpha^m d_{\mathbb{D}}(x_0, x_1)\frac{1-\alpha^{n-m}}{1-\alpha} \\ &\prec \alpha^m d_{\mathbb{D}}(x_0, x_1)\frac{1}{1-\alpha}. \end{aligned}$$

Since $d_{\mathbb{D}}(x_0, x_1) \in \mathbb{D}^{+}$ is fixed and $\alpha^m \prec 1$, we can make $\alpha^m d_{\mathbb{D}}(x_0, x_1)\frac{1}{1-\alpha}$ as small as we want by taking $m$ sufficiently large. It follows that $(x_n)$ is a Cauchy sequence with respect to the metric $d_{\mathbb{D}}$. Since $(X, d_{\mathbb{D}})$ is a complete hyperbolic valued metric space, there exists a point $x \in X$ such that $x_n \overset{d_{\mathbb{D}}}{\to} x$ as $n \to +\infty$. In that case, $x = Tx$, otherwise $0 \precnsim d_{\mathbb{D}}(x, Tx)$ and we would have

$$\begin{aligned} d_{\mathbb{D}}(x, Tx) &\precsim d_{\mathbb{D}}(x, x_n) + d_{\mathbb{D}}(x_n, Tx) \\ &= d_{\mathbb{D}}(x, x_n) + d_{\mathbb{D}}\left(TT^{n-1}x_0, Tx\right) \\ &= d_{\mathbb{D}}(x, x_n) + d_{\mathbb{D}}(Tx_{n-1}, Tx) \\ &\precsim d_{\mathbb{D}}(x, x_n) + \alpha d_{\mathbb{D}}(x_{n-1}, x). \end{aligned}$$

Making $n \to +\infty$, one gets $d_{\mathbb{D}}(x, Tx) = 0$ which is a contradiction and hence $x = Tx$. This shows that $x$ is a fixed point of the mapping $T$.

To prove the uniqueness of fixed point, let $x^*$ be another fixed point of $T$, that is, $Tx^* = x^*$. Then,

$$d_{\mathbb{D}}(x, x^*) = d_{\mathbb{D}}(Tx, Tx^*) \precsim \alpha d_{\mathbb{D}}(x, x^*) \prec d_{\mathbb{D}}(x, x^*)$$

and so $d_{\mathbb{D}}(x, x^*) = 0$. We obtain that $x^* = x$. □

**Theorem 3.4.** *Let $(X, d_{\mathbb{D}})$ be a complete hyperbolic valued metric space and $T : X \to X$ satisfy:*

$$d_{\mathbb{D}}(T^n x, T^n y) \precsim \alpha d_{\mathbb{D}}(x, y)$$

*for all $x, y \in X$, where $\alpha \in \mathbb{D}^+$, $1 - \alpha$ is invertible and $\alpha \prec 1$. Then, $T$ has a unique fixed point.*

*Proof.* By Theorem 3.1, $T^n$ has a unique fixed point, that is, there is a unique $x \in X$ such that $T^n x = x$. Since

$$d_{\mathbb{D}}(Tx, x) = d_{\mathbb{D}}(TT^n x, T^n x) = d_{\mathbb{D}}(T^n Tx, T^n x) \precsim \alpha d_{\mathbb{D}}(Tx, x) \prec d_{\mathbb{D}}(Tx, x),$$

we obtain that $d_{\mathbb{D}}(Tx, x) = 0$ and hence $Tx = x$.

To show the uniqueness of fixed point of the mapping $T$, let $x^*$ be another fixed point of $T$, that is, $Tx^* = x^*$. In this case, $T^n x^* = x^*$ and so, by hypothesis we obtain that

$$d_{\mathbb{D}}(x, x^*) = d_{\mathbb{D}}(T^n x, T^n x^*) \precsim \alpha d_{\mathbb{D}}(x, x^*) \prec d_{\mathbb{D}}(x, x^*)$$

and so $x^* = x$. □

The following example supports Theorem 3.1.

**Example 3.5.** *Let*

$$X_1 = \{\gamma = \gamma_1 e_1 + \gamma_2 e_2 \in \mathbb{D} : \gamma_1 = \gamma_2, \gamma_1 \geq 0\},$$
$$X_2 = \{\gamma = \gamma_1 e_1 + \gamma_2 e_2 \in \mathbb{D} : \gamma_1 = -\gamma_2, \gamma_1 \geq 0\}$$

*and $X = X_1 \cup X_2$. Define a mapping $d_{\mathbb{D}} : X \times X \to \mathbb{D}$ as*

$$d_{\mathbb{D}}(\alpha, \beta) = \begin{cases} \frac{7}{6}|\alpha_1 - \beta_1| e_1 + \frac{9}{8}|\alpha_1 - \beta_1| e_2, \alpha, \beta \in X_1 \\ \frac{5}{6}|\alpha_1 - \beta_1| e_1 + \frac{7}{8}|\alpha_1 - \beta_1| e_2, \alpha, \beta \in X_2 \\ \left(\frac{7}{6}\alpha_1 + \frac{5}{6}\beta_1\right) e_1 + \left(\frac{9}{8}\alpha_1 + \frac{7}{8}\beta_1\right) e_2, \alpha \in X_1, \beta \in X_2 \\ \left(\frac{5}{6}\alpha_1 + \frac{7}{6}\beta_1\right) e_1 + \left(\frac{7}{8}\alpha_1 + \frac{9}{8}\beta_1\right) e_2, \alpha \in X_2, \beta \in X_1 \end{cases},$$

*where $\alpha = \alpha_1 e_1 + \alpha_2 e_2, \beta = \beta_1 e_1 + \beta_2 e_2$, then $(X, d_{\mathbb{D}})$ is a complete hyperbolic valued metric space.*

*Consider a mapping $T$ on $X$ with $\gamma = \gamma_1 e_1 + \gamma_2 e_2$ as*

$$T\gamma = \begin{cases} \gamma_1 e_1 - \gamma_1 e_2, \gamma \in X_1 \\ \frac{\gamma_1}{2} e_1 + \frac{\gamma_1}{2} e_2, \gamma \in X_2 \end{cases}.$$

*In that case,*

$$d_{\mathbb{D}}(T\alpha,T\beta)=\begin{cases}\frac{5}{6}\left|\alpha_1-\beta_1\right|e_1+\frac{7}{8}\left|\alpha_1-\beta_1\right|e_2,\alpha,\beta\in X_1\\ \frac{7}{12}\left|\alpha_1-\beta_1\right|e_1+\frac{9}{16}\left|\alpha_1-\beta_1\right|e_2,\alpha,\beta\in X_2\\ \left(\frac{5}{6}\alpha_1+\frac{7}{12}\beta_1\right)e_1+\left(\frac{7}{8}\alpha_1+\frac{9}{16}\beta_1\right)e_2,\alpha\in X_1,\beta\in X_2\\ \left(\frac{7}{12}\alpha_1+\frac{5}{6}\beta_1\right)e_1+\left(\frac{9}{16}\alpha_1+\frac{7}{8}\beta_1\right)e_2,\alpha\in X_2,\beta\in X_1\end{cases}$$

$$\prec\begin{cases}\left|\alpha_1-\beta_1\right|e_1+\left|\alpha_1-\beta_1\right|e_2,\alpha,\beta\in X_1\\ \frac{5}{7}\left|\alpha_1-\beta_1\right|e_1+\frac{7}{9}\left|\alpha_1-\beta_1\right|e_2,\alpha,\beta\in X_2\\ \left(\alpha_1+\frac{5}{7}\beta_1\right)e_1+\left(\alpha_1+\frac{7}{9}\beta_1\right)e_2,\alpha\in X_1,\beta\in X_2\\ \left(\frac{5}{7}\alpha_1+\beta_1\right)e_1+\left(\frac{7}{9}\alpha_1+\beta_1\right)e_2,\alpha\in X_2,\beta\in X_1\end{cases}$$

$$=\left(\frac{6}{7}e_1+\frac{8}{9}e_2\right)\begin{cases}\frac{7}{6}\left|\alpha_1-\beta_1\right|e_1+\frac{9}{8}\left|\alpha_1-\beta_1\right|e_2,\alpha,\beta\in X_1\\ \frac{5}{6}\left|\alpha_1-\beta_1\right|e_1+\frac{7}{8}\left|\alpha_1-\beta_1\right|e_2,\alpha,\beta\in X_2\\ \left(\frac{7}{6}\alpha_1+\frac{5}{6}\beta_1\right)e_1+\left(\frac{9}{8}\alpha_1+\frac{7}{8}\beta_1\right)e_2,\alpha\in X_1,\beta\in X_2\\ \left(\frac{5}{6}\alpha_1+\frac{7}{6}\beta_1\right)e_1+\left(\frac{7}{8}\alpha_1+\frac{9}{8}\beta_1\right)e_2,\alpha\in X_2,\beta\in X_1\end{cases}$$

$$=\left(\frac{6}{7}e_1+\frac{8}{9}e_2\right)d_{\mathbb{D}}(\alpha,\beta),$$

*that is,* $d_{\mathbb{D}}(T\alpha,T\beta)\prec\left(\frac{6}{7}e_1+\frac{8}{9}e_2\right)d_{\mathbb{D}}(\alpha,\beta)$ *holds for all* $\alpha,\beta\in X$. *Thus, the mapping* $T$ *is a hyperbolic contraction mapping since* $0\prec\frac{6}{7}e_1+\frac{8}{9}e_2\prec 1$ and $1-\left(\frac{6}{7}e_1+\frac{8}{9}e_2\right)=\frac{1}{7}e_1+\frac{1}{9}e_2$ is invertible. *Therefore,* $T$ *has a fixed point* $\gamma=0\in X$, *which is unique.*

## 4. Concluding Remarks

In this paper, fixed point theorems have been studied for hyperbolic valued metric spaces, as stated in Theorem 3.1 and Theorem 3.2. For the future, firstly, we will develop common fixed point theorems for hyperbolic valued metric spaces, later we will define hyperbolic valued $S-$ metric spaces, hyperbolic valued $b-$ metric spaces and hyperbolic valued $G-$ metric spaces and we will investigate fixed point theorems and common fixed point theorems for them.

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