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POINTWISE FOURIER INVERSION OF DISTRIBUTIONS ON PROJECTIVE SPACES

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ABSTRACT. Given a distribution T on the real, complex or quaternionic projective space we define, in analogy to the work of Łojasiewicz, the value of T at a point z of the projective space and we show that, if T has the value τ at z, then the Fourier-Laplace series of T at z is Abel-summable to τ.

1. INTRODUCTION.

Consider the periodic distribution T with period 2π defined by

$$T(\varphi) := \lim_{\varepsilon\to 0+} \int_{\varepsilon}^{2\pi-\varepsilon} \cot(t/2)\varphi(t)\,dt,$$

for all test functions φ (T is the principal value of $\cot(t/2)$). Its Fourier coefficients, given by $\mathcal{F}T(k) := T(\mathrm{e}^{-ikt})/2\pi$, are equal to $-i$ for $k>0$, 0 for $k=0$ and i for $k<0$. Hence the Fourier series of T,

$$\sum_{k\in\mathbb{Z}} \mathcal{F}T(k)\,\mathrm{e}^{ikt},$$

does not converge at any $t\in[-\pi,\pi]$; generally one only reads that it converges to T in the sense of distributions. In fact it is possible to reconstruct T from $\mathcal{F}T$ using pointwise convergence only (and no test functions); the Fourier series of T is Abel-summable to $\cot(t/2)$ at every $t\neq 0$:

$$\begin{aligned}
\lim_{r\to 1_-}\sum_{k\in\mathbb{Z}} r^{|k|}\mathcal{F}T(k)\,\mathrm{e}^{ikt} &= \lim_{r\to 1_-}(-i)\sum_{k=1}^{+\infty}(r\,\mathrm{e}^{it})^k + i\sum_{k=1}^{+\infty}(r\,\mathrm{e}^{-it})^k\\
&= \lim_{r\to 1_-}(-i)\frac{r\,\mathrm{e}^{it}}{1-r\,\mathrm{e}^{it}} + i\frac{r\,\mathrm{e}^{-it}}{1-r\,\mathrm{e}^{-it}}\\
&= \lim_{r\to 1_-}\frac{2r\sin t}{1+r^2-2r\cos t}\\
&= \cot(t/2).
\end{aligned}$$

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This result is general: Walter [8, p.146] proved that if a periodic distribution T in one variable has the value τ at a point t (in the sense of Łojasiewicz [5]), then the Fourier series of T at t is Cesàro- and hence Abel-summable to τ. A complete characterization for Fourier series and Fourier integrals on $\mathbb{R}$ was given by [7]. Note that the pointwise convergence or summability of expansions of distributions has been investigated with respect to other orthogonal systems, such as wavelets (see [4], [8], [9]).

In [2] we have generalized Walter's result to the spheres $\mathbb{S}^{n-1}$ ($n \geq 2$). For that we had to define the notion of value at a point for distributions on the sphere analogous to the one of Łojasiewicz; our definition only uses the Laplace-Beltrami operator and its iterates instead of more general differential operators. We have then been able to show that, if T has the value τ at $\xi \in \mathbb{S}^{n-1}$, the Fourier-Laplace series of T at ξ is Abel-summable to τ.

Here we will show in section 4 that from the result on the sphere we can obtain a similar result about the Fourier-Laplace expansion of distributions on real, complex and quaternionic projective spaces. In sections 2 and 3 we introduce the necessary tools on spheres and on projective spaces, respectively.

2. Pointwise Fourier inversion on the sphere.

The restriction to $\mathbb{S}^{n-1}$, the unit sphere in $\mathbb{R}^n$, of the non-radial part of the Laplace operator Δ on $\mathbb{R}^n$ is the *Laplace-Beltrami operator* on $\mathbb{S}^{n-1}$, $\Delta_{\mathbb{S}}$. It is self-adjoint with respect to the scalar product of $L^2(\mathbb{S}^{n-1}, d\sigma_{n-1})$ and commutes with rotations (we choose $d\sigma_{n-1}$ normalized).

A *spherical harmonic of degree l on $\mathbb{S}^{n-1}$* ($l \in \mathbb{N}_0$) is the restriction to $\mathbb{S}^{n-1}$ of a polynomial on $\mathbb{R}^n$ which is harmonic and homogeneous of degree l. We write $\mathcal{SH}_l(\mathbb{S}^{n-1})$ the set of spherical harmonics of degree l. Every non zero element of $\mathcal{SH}_l(\mathbb{S}^{n-1})$ is an eigenfunction of $\Delta_{\mathbb{S}}$ with eigenvalue $-l(n+l-2)$. Let $(E_1^l, \dots, E_{d_l}^l)$ be an orthonormal basis of $\mathcal{SH}_l(\mathbb{S}^{n-1})$. The function $Z_l(\zeta, \eta) := \sum_{j=1}^{d_l} E_j^l(\zeta)\overline{E_j^l(\eta)}$ is called the *zonal of degree l*. For all ζ, $\eta \in \mathbb{S}^{n-1}$, $Z_l(\zeta, \eta) = Z_l(\eta, \zeta) \in \mathbb{R}$ and

$$Z_l(\rho\zeta, \eta) = Z_l(\zeta, \rho^{-1}\eta) \tag{2.1}$$

if $\rho \in O(n)$ [6, lemma 2.8 p.143].

We write $\mathcal{D}(\mathbb{S}^{n-1})$ the set of test functions and $\mathcal{D}'(\mathbb{S}^{n-1})$ the set of *distributions* on $\mathbb{S}^{n-1}$. The support of $T \in \mathcal{D}'(\mathbb{S}^{n-1})$ is written $\operatorname{supp} T$. The *Fourier-Laplace series* of a distribution T on $\mathbb{S}^{n-1}$ is $\sum_{l=0}^{+\infty} \Pi_l(T)$, where $\Pi_l(T)(\zeta) := T[\eta \mapsto Z_l(\zeta, \eta)]$ for $\zeta \in \mathbb{S}^{n-1}$; this series converges to T in the sense of distributions. In [2] we introduced the following:

Definition 2.1. *A distribution $T \in \mathcal{D}'(\mathbb{S}^{n-1})$* has the value $\tau \in \mathbb{C}$ in $\zeta \in \mathbb{S}^{n-1}$ *if there exist $p \in \mathbb{N}_0$, $F \in C(\mathbb{S}^{n-1})$ and $f \in C^{2p}(\mathbb{S}^{n-1})$ such that*

(1) *in the sense of distributions $T = \Delta_{\mathbb{S}}^p F$ on a neighbourhood of ζ;*

(2) $F(\eta) = f(\eta) + o[d(\zeta,\eta)^p]$ *for* $\eta \to \zeta$*;*
(3) $\Delta_{\mathbb{S}}^p f(\zeta) = \tau$.

We then obtained [2, theorem 3.1]:

Theorem 2.1. *Let* $T \in \mathcal{D}'(\mathbb{S}^{n-1})$*,* $\xi \in \mathbb{S}^{n-1}$ *and* $\tau \in \mathbb{C}$*. If* T *has the value* τ *in* ξ*, then*

$$\lim_{r\to 1_-} \sum_{l=0}^{+\infty} r^l\, \Pi_l(T)(\xi) = \tau.$$

3. Projective spaces.

Here we will write $\mathbb{K}$ for either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the algebra of quaternions) and let $d := \dim_{\mathbb{R}} \mathbb{K}$. We also define $U(\mathbb{K}) := \{k \in \mathbb{K} : \|k\| = 1\}$ and note dk the normalized Haar measure of $U(\mathbb{K})$.

For $x, y \in \mathbb{K}^{n+1} \setminus \{0\}$, we write $x \sim y$ if there exists $k \in \mathbb{K}^*$ such that $x = ky$ and let $[x]$ the equivalence class of x. The *projective space of dimension n on* $\mathbb{K}$ is $P^n(\mathbb{K}) := \mathbb{K}^{n+1} \setminus \{0\}/\sim$; it is a compact symmetric space of rank one (see [3]). Identifying $\mathbb{K}^{n+1}$ with $\mathbb{R}^{dn+d}$, we see that $P^n(\mathbb{K}) = \mathbb{S}^{dn+d-1}/\sim$. The connected component of the identity in the group of isometries of $P^n(\mathbb{K})$ is a group we denote $S\mathbb{K}(n+1)$; in fact $S\mathbb{K}(n+1) = SO(n+1)$, $SU(n+1)$ or $Sp(n+1)$ for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ respectively (see [1]). Moreover, the action of $S\mathbb{K}(n+1)$ on $P^n(\mathbb{K})$ is the one induced by the action on $\mathbb{S}^{dn+d-1}$ of $S\mathbb{K}(n+1)$ as a subgroup of $SO(dn+d)$. We also have, from the action of $U(\mathbb{K})$ on $\mathbb{K}^{n+1}$,

$$U(\mathbb{K}) < O(dn+d). \tag{3.1}$$

If g is a $U(\mathbb{K})$-invariant function on $\mathbb{S}^{dn+d-1}$, we can define a function $g{\downarrow}$ on $P^n(\mathbb{K})$ by $g{\downarrow}([\eta]) := g(\eta)$. Conversely, if f is a function on $P^n(\mathbb{K})$, we get, by putting $f{\uparrow}(\eta) := f([\eta])$, a $U(\mathbb{K})$-invariant function $f{\uparrow}$ on $\mathbb{S}^{dn+d-1}$ with $(f{\uparrow}){\downarrow} = f$. Now, given an arbitrary function g on $\mathbb{S}^{dn+d-1}$, we define a $U(\mathbb{K})$-invariant function $g^\sharp$ on $\mathbb{S}^{dn+d-1}$ by $g^\sharp(\eta) := \int_{U(\mathbb{K})} g(k\eta)\, dk$ (when g is $U(\mathbb{K})$-invariant, $g^\sharp = g$). We then put $g^\flat := (g^\sharp){\downarrow}$. If T is a distribution on $P^n(\mathbb{K})$, we let, for $\varphi \in \mathcal{D}(\mathbb{S}^{dn+d-1})$, $T{\uparrow}(\varphi) := T(\varphi^\flat)$. Then $T{\uparrow}$ is a distribution on $\mathbb{S}^{dn+d-1}$ of the same order as T and $\operatorname{supp} T{\uparrow} = \{\eta \in \mathbb{S}^{dn+d-1} : [\eta] \in \operatorname{supp} T\}$.

We write dp_n the unique normalized Radon measure on $P^n(\mathbb{K})$ which is $S\mathbb{K}(n+1)$-invariant. The link between dp_n and $d\sigma_{dn+d-1}$ is:

$$\int_{\mathbb{S}^{dn+d-1}} g(\zeta)\, d\sigma_{dn+d-1}(\zeta) = \int_{P^n(\mathbb{K})} g^\flat(z)\, dp_n(z)$$

for every $g \in \mathcal{D}(\mathbb{S}^{dn+d-1})$. Finally, we can define the Laplace-Beltrami operator Δ_P on $P^n(\mathbb{K})$ by $\Delta_P(f) := (\Delta_{\mathbb{S}}(f{\uparrow})){\downarrow}$, using (3.1) and the facts that $f{\uparrow}$ is $U(\mathbb{K})$-invariant and $\Delta_{\mathbb{S}}$ commutes with all rotations. Then Δ_P commutes with all elements of $S\mathbb{K}(n+1)$.

In analogy to the case of the sphere, we now introduce the following

Definition 3.1. *A distribution* $T \in \mathcal{D}'(P^n(\mathbb{K}))$ has the value $\tau \in \mathbb{C}$ in $z \in P^n(\mathbb{K})$ *if there exist* $q \in \mathbb{N}_0$, $F \in C(P^n(\mathbb{K}))$ *and* $f \in C^{2q}(P^n(\mathbb{K}))$ *such that*
(1) *in the sense of distributions* $T = \Delta_P^q F$ *on a neighbourhood of* z;
(2) $F(w) = f(w) + o[d(z,w)^q]$ *for* $w \to z$;
(3) $\Delta_P^q f(z) = \tau$.

The distribution T has the value τ in $[\eta] \in P^n(\mathbb{K})$ if and only if $T\uparrow$ has the value τ in $\eta \in \mathbb{S}^{dn+d-1}$. Moreover, if T is equal in the sense of distributions to a continuous function F on a neighbourhood of z, then T has the value $F(z)$ in z.

4. Fourier inversion on $P^n(\mathbb{K})$.

Given $T \in \mathcal{D}'(P^n(\mathbb{K}))$, $\Pi_l(T\uparrow) \in \mathcal{SH}_l(\mathbb{S}^{dn+d-1})$ is $U(\mathbb{K})$-invariant:

$$\begin{aligned}
\Pi_l(T\uparrow)(u\zeta) &= T\uparrow(\eta \mapsto Z_l(u\zeta,\eta)) \\
&= T(Z_l(u\zeta,\,.\,)^\flat) \\
&= T([\eta] \mapsto \int_{U(\mathbb{K})} Z_l(u\zeta,k\eta)\,dk) \\
&= T([\eta] \mapsto \int_{U(\mathbb{K})} Z_l(\zeta,u^{-1}k\eta)\,dk) \\
&= T([\eta] \mapsto \int_{U(\mathbb{K})} Z_l(\zeta,k\eta)\,dk) \\
&= T(Z_l(\zeta,\,.\,)^\flat) \;=\; \Pi_l(T\uparrow)(\zeta)
\end{aligned}$$

(where $u \in U(\mathbb{K})$, $\zeta \in \mathbb{S}^{dn+d-1}$), using (2.1) and (3.1) for the fourth equality. Hence we can define a function $\Xi_l(T)$ on $P^n(\mathbb{K})$ by $\Xi_l(T) := (\Pi_l(T\uparrow))\downarrow$. Since $\Pi_l(T\uparrow)$ is either 0 or an eigenfunction of $\Delta_\mathbb{S}$, $\Xi_l(T)$ is either 0 or an eigenfunction of Δ_P. Moreover, if $l \neq m$, $\Xi_l(T)$ and $\Xi_m(T)$ are orthogonal in $L^2(P^n(\mathbb{K}), dp_n)$. This justifies the name *Fourier-Laplace series of T* we give to $\sum_{l=0}^{+\infty} \Xi_l(T)$; this series converges to T in the sense of distributions:

$$\begin{aligned}
\lim_{N\to+\infty} \sum_{l=0}^{N} \Xi_l(T)(\varphi) &= \lim_{N\to+\infty} \sum_{l=0}^{N} \int_{P^n(\mathbb{K})} \Xi_l(T)(z)\varphi(z)\,dp_n(z) \\
&= \lim_{N\to+\infty} \sum_{l=0}^{N} \int_{\mathbb{S}^{dn+d-1}} \Pi_l(T\uparrow)(\zeta)\varphi\uparrow(\zeta)\,d\sigma_{dn+d-1}(\zeta) \\
&= \lim_{N\to+\infty} \sum_{l=0}^{N} \Pi_l(T\uparrow)(\varphi\uparrow) \\
&= T\uparrow(\varphi\uparrow) \;=\; T((\varphi\uparrow)^\flat) \;=\; T(\varphi)
\end{aligned}$$

if $\varphi \in \mathcal{D}(P^n(\mathbb{K}))$. From the above discussion and Theorem 2.1 we deduce:

Theorem 4.1. *Let $T \in \mathcal{D}'(P^n(\mathbb{K}))$, $z \in P^n(\mathbb{K})$ and $\tau \in \mathbb{C}$. If T has the value τ in z, then*

$$\lim_{r \to 1_-} \sum_{l=0}^{+\infty} r^l \, \Xi_l(T)(z) = \tau.$$

Remark 4.1. The theorem shows that if the value of T in z exists, it is unique.

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