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KOROVKIN TYPE APPROXIMATION FOR DOUBLE SEQUENCES VIA μ-STATISTICAL $\mathcal{A}$-SUMMATION PROCESS IN MODULAR SPACES

SELİN ÇINAR AND KAMİL DEMİRCİ

ABSTRACT. In the present paper, we extend the Korovkin type approximation theorem via $\mu-$statistical $\mathcal{A}$-summation process onto the double sequences of positive linear operators in a modular space. We apply our new result to bivariate Mellin-type operators in Orlicz spaces. Then we discuss the reduced results which are obtained by special choices and the matrix sequences.

1. INTRODUCTION

Summability theory often is the theory used in approximation theory to correct the lack of convergence of real or complex term sequences. Since there are many series (or sequences) where ordinary convergence fails, we consider more general types of convergence using the methods of summability theory. Some of these types of convergence are statistical convergence, $\mu-$statistical convergence (see more details, [12], [14], [15]). The concept of statistical convergence was presented by Fast [13] and this concept was studied by many authors. In partıcular, in [7] and [8] Connor stated two expansions of the concepts of statistical convergence using a complete $\{0,1\}$ valued measure μ defined on algebra of subsets of $\mathbb{N}$, natural numbers. The notion of statistical convergence was introduced for multiple sequences by Móricz [20]. Das and Bhunia in [10] introduced the notion of $\mu-$statistical convergence and convergence in $\mu-$density (following the line of Connor [7]) using a two valued measure μ defined on the algebra of subsets of $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ and mainly investigated the inter-relationship between these two concepts. Since the concept of statistical convergence is stronger than the classical meaning, it played an important role in the expansion of many theorems ([16], [17], [25]). Among the most important of these are Korovkin theorems which allow us to check the convergence with minumum computations. In this

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paper, our main purpose is to study a further generalization of the classical Korovkin theorem by considering a certain matrix summability process in the frame of $\mu-$statistical convergence in modular spaces for double sequences.

First, for the basis of this paper, we begin by recalling the concepts of Pringsheim convergence and statistical convergence for double sequences.

A double sequence $x=\{x_{i,j}\}$ of real numbers, is said to be convergent in the Pringsheim's sense if for each $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that $\left|x_{i,j}-L\right|<\varepsilon$ whenever $i,j>N$ and L is called the Pringsheim limit and denoted by $P-\lim\limits_{i,j}x_{i,j}=L$. More briefly, we will say that such an x is $P-$convergent to L ([28]).

The double sequence $x=\{x_{i,j}\}$ is statistically convergent to L provided that for every $\varepsilon>0$,

$$P-\lim_{m,n}\frac{1}{mn}\left|\left\{i\leq m,\ j\leq n:\ \left|x_{i,j}-L\right|\geq\varepsilon\right\}\right|=0,$$

where the vertical bars indicate the cardinality of the enclosed set. In that case we write $st_2-\lim\limits_{i,j}x_{i,j}=L$ (see [20]).

It can be easily seen that a $P-$convergent double sequence is statistically convergent to the same value but its converse is not always true. Also, it is crucial to state that a convergent single sequence needs to be bounded even though this necessity does not always hold for double sequences. A statistical convergent double sequence does not need to be bounded. For example, take into consideration the double sequence $x=\{x_{i,j}\}$ defined by

$$x_{i,j}=\begin{cases} ij & i \text{ and } j \text{ are squares}\\ 1, & \text{otherwise.}\end{cases}$$

Then, clearly $st_2-\lim\limits_{i,j}x_{i,j}=1$ but x is not $P-$convergent and also, it is not bounded. The characterization of the statistical convergence for double sequences is given in [20] as indicated below:

A double sequence $x=\{x_{i,j}\}$ is statistically convergent to L if and only if there exists a set $S\subset\mathbb{N}^2$ such that the natural density of S is 1 and

$$P-\lim_{\substack{i,j\to\infty\\ \text{and }(i,j)\in S}}x_{i,j}=L.$$

Let

$$A=\left(a_{k,l,i,j}\right),\ \ k,l,i,j\in\mathbb{N},$$

be a four-dimensional infinite matrix. The $A-$transform of $x=\{x_{i,j}\}$ denoted by $Ax:=\{(Ax)_{k,l}\}$ is defined by

$$(Ax)_{k,l}=\sum_{(i,j)\in\mathbb{N}^2}a_{k,l,i,j}x_{i,j},\quad k,l\in\mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(k,l)\in\mathbb{N}^2$.

Then the double sequence x is $A-$summable to L if the A-transform of x exists for all $k,l \in \mathbb{N}$ and convergent in the Pringsheim's sense i.e.,

$$P-\lim_{k,l} \sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j} x_{i,j} = L.$$

Recall that a four-dimensional matrix $A = \left(a_{k,l,i,j}\right)$ is said to be $RH-$regular if it maps every bounded $P-$convergent sequence into a $P-$convergent sequence with the same $P-$limit. The Robison-Hamilton conditions (see also [29]) state that a four-dimensional matrix $A = \left(a_{k,l,i,j}\right)$ is $RH-$regular if and only if

(i) $P-\lim\limits_{k,l} a_{k,l,i,j} = 0$ for each $(i,j) \in \mathbb{N}^2$,

(ii) $P-\lim\limits_{k,l} \sum\limits_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j} = 1$,

(iii) $P-\lim\limits_{k,l} \sum\limits_{i\in\mathbb{N}} \left|a_{k,l,i,j}\right| = 0$ for each $j \in \mathbb{N}$,

(iv) $P-\lim\limits_{k,l} \sum\limits_{j\in\mathbb{N}} \left|a_{k,l,i,j}\right| = 0$ for each $i \in \mathbb{N}$,

(v) $\sum\limits_{(i,j)\in\mathbb{N}^2} \left|a_{k,l,i,j}\right|$ is $P-$convergent for every $(k,l) \in \mathbb{N}^2$,

(vi) there exist finite positive integers A and B such that

$$\sum_{i,j>B} \left|a_{k,l,i,j}\right| < A$$

for every $(k,l) \in \mathbb{N}^2$. Now let $\mathcal{A} := \left\{A^{(m,n)}\right\} = \left\{a^{(m,n)}_{k,l,i,j}\right\}$ be a sequence of four-dimensional infinite matrices with non-negative real entries. For a given double sequence of real numbers, $x = \left\{x_{i,j}\right\}$ is said to be $\mathcal{A}-$summable to L if

$$P-\lim_{k,l} \sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} x_{i,j} = L$$

uniformly in m and n. If $A^{(m,n)} = A$, a four-dimensional infinite matrix, then $\mathcal{A}-$summability is the $A-$summability for that four-dimensional infinite matrix.
If $a^{(m,n)}_{k,l,i,j} = \frac{1}{kl}$, for $m \le i \le m+k-1$, $n \le j \le n+l-1$, $(m,n = 1,2,..)$ and $a^{(m,n)}_{k,l,i,j} = 0$ otherwise, then $\mathcal{A}-$summability reduces to almost convergence of double sequences introduced by Móricz [20]. Some results concerning the matrix summability method for double sequences may be reached in [16], [27], [31].

Now, we recall some definitions and notations required for this paper.

Throughout the paper μ will denote a complete $\{0,1\}$ valued finite additive measure defined on an algebra Γ of subsets $\mathbb{N}^2$ that contains all subsets of $\mathbb{N}^2$ that are contained in the union of a finite number of rows and columns of $\mathbb{N}^2$ and $\mu(K) = 0$ if K is contained in the union of a finite number of rows and columns of $\mathbb{N}^2$.

Das and Bhunia in [10] also introduced the following two definitions:

Definition 1.1. *[10] A double sequence $x=\{x_{ij}\}$ of real numbers is said to be $\mu-$statistically convergent to $L\in\mathbb{R}$ if and only if for any $\varepsilon>0$,*

$$\mu\left(\left\{(i,j)\in\mathbb{N}^2:\ \left|x_{i,j}-L\right|\geq\varepsilon\right\}\right)=0.$$

Definition 1.2. *[10] A double sequence $x=\{x_{ij}\}$ of real numbers is said to be convergent to $L\in\mathbb{R}$ in $\mu-$density if there exists an $K\in\Gamma$ with $\mu(K)=1$ such that $x=\{x_{ij}\}_{i,j\varepsilon A}$ is convergent to L.*

If C_μ and C_μ^* denote respectively the sets of all double sequences which are $\mu-$statistically convergent and convergent in $\mu-$density then [7] (see also [9]) it is easy to prove that C_μ^* is a dense subset of C_μ which again is closed in l_2^∞ (the set of all bounded double sequences of real numbers endowed with the sup metric).

If $B=\left(b_{k,l,i,j}\right)$ is a nonnegative $RH-$regular summability matrix, then B can be used to generate a measure as follows:

for each $k,l\in\mathbb{N}$, set

$$\mu_{k,l}(K)=\sum_{(i,j)\in\mathbb{N}^2}b_{k,l,i,j}\chi_K(i,j)$$

for each $(i,j)\in\mathbb{N}^2$ and $K\subseteq\mathbb{N}^2$. Let

$$\Gamma:=\left\{K\subseteq\mathbb{N}^2: P-\lim_{k,l}\mu_{k,l}(K)=0 \text{ or } P-\lim_{k,l}\mu_{k,l}(K)=1\right\}.$$

Define $\mu_B:\Gamma\to\{0,1\}$ by

$$\mu_B(K)=P-\lim_{k,l}\mu_{k,l}(K)=P-\lim_{k,l}\sum_{(i,j)\in\mathbb{N}^2}b_{k,l,i,j}\chi_K(i,j)\,. \tag{1.1}$$

Then μ_B and Γ satisfy the requirements of the preceding definitions. Obviously, μ_B-statistical convergence coincides with the notion of $B-$statistical convergence for double sequences. If B is the double Cesáro matrix of order one, then μ_B-statistical convergence is equivalent to statistical convergence for double sequences.

Now, we recall some definitions and notations on modular spaces.

Let $I=[a,b]$ be a bounded interval of the real line $\mathbb{R}$ provided with the Lebesgue measure. Then, let $X\left(I^2\right)$ we denote the space of all real-valued measurable functions on $I^2=[a,b]\times[a,b]$ provided with equality *a.e.* As usual, let $C\left(I^2\right)$ denote the space of all continuous real-valued functions, and $C^\infty\left(I^2\right)$ denote the space of all infinitely differentiable functions on I^2. A functional

$$\rho:X\left(I^2\right)\to[0,+\infty]$$

is called a modular on $X\left(I^2\right)$ provided that the following conditions hold:

(i) $\rho(f)=0$ if and only if $f=0$ *a.e.* in I^2,
(ii) $\rho(-f)=\rho(f)$ for every $f\in X\left(I^2\right)$,
(iii) $\rho(\alpha f+\beta g)\leq\rho(f)+\rho(g)$ for every $f,g\in X(I^2)$ and for any $\alpha,\beta\geq0$ with $\alpha+\beta=1$.

A modular ρ is said to be $N-$quasi convex if there exists a constant $N \geq 1$ such that $\rho(\alpha f + \beta g) \leq N\alpha\rho(Nf) + N\beta\rho(Ng)$ holds for every $f, g \in X\left(I^2\right)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. In particular, if $N = 1$, then ρ is called convex. A modular ρ is said to be $N-$quasi semiconvex if there exists a constant $N \geq 1$ such that $\rho(af) \leq Na\rho(Nf)$ holds for every $f \in X\left(I^2\right)$ and $a \in (0,1]$. It is clear that every $N-$quasi convex modular is $N-$quasi semiconvex. Bardaro et. al. introduced and worked through the above two concepts in [4, 6].

We now present some obtained vector subspaces of $X(I^2)$ via a modular ρ as follows:

The modular space $L^{\rho}\left(I^2\right)$ generated by ρ is defined by

$$L^{\rho}\left(I^2\right) := \left\{ f \in X(I^2) : \lim_{\lambda \to 0^+} \rho(\lambda f) = 0 \right\}$$

and the space of the finite elements of $L^{\rho}\left(I^2\right)$ is given by

$$E^{\rho}\left(I^2\right) := \left\{ f \in L^{\rho}\left(I^2\right) : \rho(\lambda f) < +\infty \text{ for all } \lambda > 0 \right\}.$$

Observe that if ρ is $N-$quasi semiconvex, then the space

$$\left\{ f \in X\left(I^2\right) : \rho(\lambda f) < +\infty \text{ for some } \lambda > 0 \right\}$$

coincides with $L^{\rho}\left(I^2\right)$. The notions about modulars are introduced in [21] and widely discussed in [4] (see also [18, 22]).

Bardaro and Mantellini in [5] introduced some Korovkin type approximation theorems via the notions of modular convergence and strong convergence. Afterwards Karakuş et al. [17] investigated the modular Korovkin-type approximation theorem via statistical convergence and then, Orhan and Demirci [25] extended these types of approximations to the spaces of double sequences of positive linear operators.

Now we recall the convergence methods in modular spaces.

Definition 1.3. *[25] Let $\left\{f_{i,j}\right\}$ be a double function sequence whose terms belong to $L^{\rho}\left(I^2\right)$. Then, $\left\{f_{i,j}\right\}$ is modularly convergent to a function $f \in L^{\rho}\left(I^2\right)$ if*

$$P - \lim_{i,j} \rho\left(\lambda_0\left(f_{i,j} - f\right)\right) = 0 \text{ for some } \lambda_0 > 0. \tag{1.2}$$

Also, $\left\{f_{i,j}\right\}$ is statistically F-norm convergent (or, strongly convergent) to f if

$$P - \lim_{i,j} \rho\left(\lambda\left(f_{i,j} - f\right)\right) = 0 \text{ for every } \lambda > 0. \tag{1.3}$$

It is known from [21] that (1.2) and (1.3) are equivalent if and only if the modular ρ satisfies the Δ_2-condition, i.e.
there exists a constant $M > 0$ such that $\rho(2f) \leq M\rho(f)$ for every $f \in X(I^2)$.

Recently, Orhan and Demirci [24] have introduced the notion of $\mathcal{A}-$summation process on the one dimensional modular space $X(I)$ and then, Orhan and Kolay

[26] introduced $\mathcal{A}-$summation process for double sequences on a modular space. Now, the definition of the $\mathcal{A}-$summation process for double sequences as follows:

A sequence $\mathbb{T} := \{T_{i,j}\}$ of positive linear operators of D into $X(I^2)$ is called an $\mathcal{A}-$summation process on D if $\{T_{i,j}f\}$ is $\mathcal{A}-$summable to f (with respect to modular ρ) for every $f \in D$, i.e.,

$$P-\lim_{k,l} \rho\left(\lambda\left(A^{\mathbb{T}}_{k,l,m,n}(f)-f\right)\right)=0, \text{ uniformly in } m,n \tag{1.4}$$

for some $\lambda > 0$, where for all $k,l,m,n \in \mathbb{N}$, $f \in D$ the series

$$A^{\mathbb{T}}_{k,l,m,n}(f) := \sum_{(i,j)\in\mathbb{N}^2} a^{(m,n)}_{k,l,i,j} T_{i,j} f \tag{1.5}$$

is absolutely convergent almost everywhere with respect to Lebesgue measure and we denote the value of $T_{i,j}f$ at a point $(x,y) \in I^2$ by $T_{i,j}(f(u,v);x,y)$ or briefly, $T_{i,j}(f;x,y)$.

Our goal in the present work is to give the Korovkin theorem for double sequences of positive linear operators using the $\mu-$statistical $\mathcal{A}-$summation process on the modular space. Some results concerning summation processes in the space $L_p[a,b]$ of Lebesgue integrable functions on a compact interval may be founded in [23], [30].

It is required to give the following assumptions on a modular ρ :

A modular ρ is *monotone* if $\rho(f) \leq \rho(g)$ for $|f| \leq |g|$, ρ is said to be *finite* if $\chi_K \in L^{\rho}(I^2)$ whenever K is a measurable subset of I^2 such that $|K| < \infty$. If ρ is finite and, for every $\varepsilon > 0$, $\lambda > 0$, there exists a $\delta > 0$ such that $\rho(\lambda\chi_B) < \varepsilon$ for any measurable subset $B \subset I^2$ with $|B| < \delta$, then ρ is *absolutely finite* and if $\chi_{I^2} \in E^{\rho}(I^2)$, then ρ is *strongly finite.* A modular ρ is *absolutely continuous* provided that there exists an $\alpha > 0$ such that, for every $f \in X(I^2)$ with $\rho(f) < +\infty$, the following condition holds:

for every $\varepsilon > 0$ there is $\delta > 0$ such that $\rho(\alpha f \chi_B) < \varepsilon$ whenever B is any measurable subset of I^2 with $|B| < \delta$.

Observe now that (see [5, 6]) if a modular ρ is monotone and finite, then we have $C(I^2) \subset L^{\rho}(I^2)$. Similarly, if ρ is monotone and strongly finite, then $C(I^2) \subset E^{\rho}(I^2)$. Also, if ρ is monotone, absolutely finite and absolutely continuous, then $\overline{C^{\infty}(I^2)} = L^{\rho}(I^2)$ (See [3, 4, 19, 22] for more details).

2. Korovkin Type Theorems

Let ρ be a monotone and finite modular on $X(I^2)$. Assume that D is a set satisfying $C^{\infty}(I^2) \subset D \subset L^{\rho}(I^2)$ (Such a subset D can be constructed when ρ is monotone and finite, see [5]). Also, assume that $\mathbb{T} := \{T_{i,j}\}$ is a sequence of positive linear operators from D into $X(I^2)$ for which there exists a subset $X_{\mathbb{T}} \subset D$ with $C^{\infty}(I^2) \subset X_{\mathbb{T}}$ such that

$$\mu\left(\left\{(k,l)\in\mathbb{N}^2:\ \rho\left(\lambda\left(A^{\mathbb{T}}_{k,l,m,n}(h)\right)\right)\geqslant\varepsilon\right\}\right)\leq S\rho(\lambda h),\ \text{uniformly in } m,n, \tag{2.1}$$

holds for every $h\in X_{\mathbb{T}}$, $\lambda>0$ and for an absolute positive constant S.

We will use the test functions $f_r\ (r=0,1,2,3)$ defined by $f_0(x,y)=1,\ f_1(x,y)=x,\ f_2(x,y)=y$ and $f_3(x,y)=x^2+y^2$ throughout the paper.

We now prove our following main theorem.

Theorem 2.1. *Let $\mathcal{A}=\left\{A^{(m,n)}\right\}$ be a sequence of four dimensional infinite non-negative real matrices and let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular on $X\left(I^2\right)$. Let $\mathbb{T}:=\left\{T_{i,j}\right\}$ be a sequence of positive linear operators from D into $X\left(I^2\right)$ satisfying (2.1) for each $f\in D$. Suppose that*

$$\mu\left(\left\{(k,l)\in\mathbb{N}^2:\ \rho\left(\lambda\left(A^{\mathbb{T}}_{k,l,m,n}(f_r)-f_r\right)\right)\geqslant\varepsilon\right\}\right)=0,\ \text{uniformly in } m,n \tag{2.2}$$

for every $\lambda>0$ and $r=0,1,2,3$. Now let f be any function belonging to $L^{\rho}\left(I^2\right)$ such that $f-g\in X_{\mathbb{T}}$ for every $g\in C^{\infty}\left(I^2\right)$. Then we have

$$\mu\left(\left\{(k,l)\in\mathbb{N}^2:\ \rho\left(\lambda_0\left(A^{\mathbb{T}}_{k,l,m,n}(f)-f\right)\right)\geqslant\varepsilon\right\}\right)=0,\ \text{uniformly in } m,n \tag{2.3}$$

for some $\lambda_0>0$.

Proof. We first claim that

$$\mu\left(\left\{(k,l)\in\mathbb{N}^2:\ \rho\left(\eta\left(A^{\mathbb{T}}_{k,l,m,n}(g)-g\right)\right)\geqslant\varepsilon\right\}\right)=0,\ \text{uniformly in } m,n \tag{2.4}$$

for every $g\in C(I^2)\cap D$ and $\eta>0$ where

$$A^{\mathbb{T}}_{k,l,m,n}(g)=\sum_{(i,j)\in\mathbb{N}^2}a^{(m,n)}_{k,l,i,j}T_{i,j}g.$$

To see this assume that g belongs to $C\left(I^2\right)\cap D$ and η is any positive number. By the continuity of g on I^2 and from the linearity and positivity of the operators $T_{i,j}$, we can easily see that (see, for instance [25]), for a given $\varepsilon>0$, there exists a number $\delta>0$ such that for all $(u,v),(x,y)\in I^2$

$$|g(u,v)-g(x,y)|<\varepsilon+\frac{2M}{\delta^2}\left\{(u-x)^2+(v-y)^2\right\}$$

where $M:=\sup\limits_{(x,y)\in I^2}|g(x,y)|$. Since $T_{i,j}$ is linear and positive, we get

$$\left|\sum_{(i,j)\in\mathbb{N}^2}a^{(m,n)}_{k,l,i,j}T_{i,j}(g;x,y)-g(x,y)\right|=\left|\sum_{(i,j)\in\mathbb{N}^2}a^{(m,n)}_{k,l,i,j}T_{i,j}(g(.,.)-g(x,y);\ x,y)\right.$$
$$\left.+g(x,y)\left(\sum_{(i,j)\in\mathbb{N}^2}a^{(m,n)}_{k,l,i,j}T_{i,j}(f_0;x,y)-f_0(x,y)\right)\right|$$

$$
\begin{aligned}
&\leq \sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(\left|g\left(.,.\right)-g\left(x,y\right)\right|;\ x,y\right) \\
&\quad +\left|g\left(x,y\right)\right|\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_0;x,y\right)-f_0\left(x,y\right)\right| \\
&\leq \sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(\varepsilon+\frac{2M}{\delta^2}\left\{\left(.-x\right)^2+\left(.-y\right)^2\right\};x,y\right) \\
&\quad +\left|g\left(x,y\right)\right|\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_0;x,y\right)-f_0\left(x,y\right)\right| \\
&= \varepsilon+\left(\varepsilon+M\right)\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_0;x,y\right)-f_0\left(x,y\right)\right| \\
&\quad +\frac{2M}{\delta^2}\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_3;x,y\right)-f_3(x,y)\right| \\
&\quad +\frac{4M}{\delta^2}\left(\left|f_1(x,y)\right|\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_1;x,y\right)-f_1(x,y)\right|\right. \\
&\quad \left.+\left|f_2(x,y)\right|\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_2;x,y\right)-f_2(x,y)\right|\right) \\
&\quad +\frac{2M}{\delta^2}\left|f_3(x,y)\right|\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_0;x,y\right)-f_0(x,y)\right|
\end{aligned}
$$

for every $x,y\in I$ and $m,n\in\mathbb{N}$. Therefore, from the last inequality we get

$$
\begin{aligned}
&\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(g;x,y\right)-g\left(x,y\right)\right| \\
&\leq \varepsilon+\left(\varepsilon+M+\frac{4Mc}{\delta^2}\right)\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_0;x,y\right)-f_0\left(x,y\right)\right| \\
&\quad +\frac{4Mc}{\delta^2}\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_1;x,y\right)-f_1\left(x,y\right)\right| \\
&\quad +\frac{4Mc}{\delta^2}\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_2;x,y\right)-f_2\left(x,y\right)\right| \\
&\quad +\frac{2M}{\delta^2}\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}\left(f_3;x,y\right)-f_3\left(x,y\right)\right|
\end{aligned}
$$

where $c := \max\{|f_1(x,y)|, |f_2(x,y)|\}$. So, if we denote
$K := \max\left\{\varepsilon + M + \frac{4Mc^2}{\delta^2}, \frac{4Mc}{\delta^2}, \frac{2M}{\delta^2}\right\}$,

$$\begin{aligned}\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(g;x,y) - g(x,y)\right| &\le \varepsilon + K\left\{\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_0;x,y) - f_0(x,y)\right|\right.\\ &+\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_1;x,y) - f_1(x,y)\right| + \left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_2;x,y) - f_2(x,y)\right|\\ &\left.+\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_3;x,y) - f_3(x,y)\right|\right\}.\end{aligned}$$

Hence, we obtain, for $\eta > 0$, that

$$\begin{aligned}&\eta\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(g;x,y) - g(x,y)\right|\\ &\le \eta\varepsilon + \eta K\left\{\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_0;x,y) - f_0(x,y)\right|\right.\\ &\quad+\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_1;x,y) - f_1(x,y)\right| + \left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_2;x,y) - f_2(x,y)\right|\\ &\quad\left.+\left|\sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f_3;x,y) - f_3(x,y)\right|\right\}\end{aligned}$$

Now we apply the modular ρ to both-sides of the above inequality, since ρ is monotone, we get

$$\begin{aligned}\rho\left(\eta\left(A_{k,l,m,n}^{\mathbb{T}}(g) - g\right)\right) \le \rho&\left(\eta\varepsilon + \eta K\left(A_{k,l,m,n}^{\mathbb{T}}(f_0) - f_0\right) + \eta K\left(A_{k,l,m,n}^{\mathbb{T}}(f_1) - f_1\right)\right.\\ &\left.+\eta K\left(A_{k,l,m,n}^{\mathbb{T}}(f_2) - f_2\right) + \eta K\left(A_{k,l,m,n}^{\mathbb{T}}(f_3) - f_3\right)\right).\end{aligned}$$

So, we may write that

$$\begin{aligned}\rho\left(\eta\left(A_{k,l,m,n}^{\mathbb{T}}(g) - g\right)\right) &\le \rho(5\eta\varepsilon) + \rho\left(5\eta K\left(A_{k,l,m,n}^{\mathbb{T}}(f_0) - f_0\right)\right)\\ &+\rho\left(5\eta K\left(A_{k,l,m,n}^{\mathbb{T}}(f_1) - f_1\right)\right)\\ &+\rho\left(5\eta K\left(A_{k,l,m,n}^{\mathbb{T}}(f_2) - f_2\right)\right)\\ &+\rho\left(5\eta K\left(A_{k,l,m,n}^{\mathbb{T}}(f_3) - f_3\right)\right).\end{aligned}$$

Since ρ is N-quasi semiconvex and strongly finite, we have, assuming $0<\varepsilon\le 1$,

$$\begin{aligned}\rho\left(\eta\left(A^{\mathbb{T}}_{k,l,m,n}(g)-g\right)\right) \le N\varepsilon\rho(5\eta\varepsilon) &+ \rho\left(5\eta K\left(A^{\mathbb{T}}_{k,l,m,n}(f_0)-f_0\right)\right)\\ &+\rho\left(5\eta K\left(A^{\mathbb{T}}_{k,l,m,n}(f_1)-f_1\right)\right)\\ &+\rho\left(5\eta K\left(A^{\mathbb{T}}_{k,l,m,n}(f_2)-f_2\right)\right)\\ &+\rho\left(5\eta K\left(A^{\mathbb{T}}_{k,l,m,n}(f_3)-f_3\right)\right).\end{aligned}$$

For a given $\varepsilon^*>0$, choose an $\varepsilon\in(0,1]$ such that $N\varepsilon\rho\left(\frac{5\eta N}{\sigma}\right)<\varepsilon^*$. Now we define the following sets:

$$G_\eta := \left\{k : \rho\left(\eta\left(A^{\mathbb{T}}_{k,l,m,n}(g)-g\right)\right)\ge\varepsilon^*\right\},$$

$$G_{\eta,r} := \left\{k : \rho\left(5\eta K\left(A^{\mathbb{T}}_{k,l,m,n}(f_r)-f_r\right)\right)\ge\frac{\varepsilon^*-N\varepsilon\rho\left(\frac{5\eta N}{\sigma}\right)}{4}\right\}, r=0,1,2,3.$$

Then, it is easy to see that $G_\eta\subseteq\bigcup\limits_{r=0}^{3}G_{\eta,r}$. So, we can write that

$$\begin{aligned}&\mu\left(\left\{(k,l)\in\mathbb{N}^2 :\ \rho\left(\eta\left(A^{\mathbb{T}}_{k,l,m,n}(g)-g\right)\right)\geqslant\varepsilon\right\}\right)\\ &\le\sum_{r=0}^{3}\mu\left(\left\{(k,l)\in\mathbb{N}^2 :\ \rho\left(\eta\left(A^{\mathbb{T}}_{k,l,m,n}(f_r)-f_r\right)\right)\geqslant\varepsilon\right\}\right).\end{aligned}$$

Using the hypothesis (2.2), we get

$$\mu(G_\eta)=0,$$

which proves our claim (2.4). Observe that (2.4) also holds for every $g\in C^\infty(I^2)$. Now let $f\in L^\rho\left(I^2\right)$ satisfy $f-g\in X_{\mathbb{T}}$ for every $g\in C^\infty\left(I^2\right)$. Since $\left|I^2\right|<\infty$ and ρ is strongly finite and absolutely continuous, it can be seen that ρ is also absolutely finite on $X(I^2)$ (see [3]). Using these properties of the modular ρ, it is known from [4, 19] that the space $C^\infty(I^2)$ is modularly dense in $L^\rho\left(I^2\right)$, i.e., there exists a sequence $(g_{k,l})\subset C^\infty\left(I^2\right)$ such that

$$P-\lim_{k,l}\rho\left(3\lambda_0^*\left(g_{k,l}-f\right)\right)=0 \text{ for some } \lambda_0^*>0.$$

This means that, for every $\varepsilon>0$, there is a positive number $k_0=k_0(\varepsilon)$ so that

$$\rho\left(3\lambda_0^*\left(g_{k,l}-f\right)\right)<\varepsilon \quad \text{for every } k,l\ge k_0. \tag{2.5}$$

On the other hand, by the linearity and positivity of the operators $T_{i,j}$, we may write that

$$\lambda_0^* \left| \sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f;x,y) - f(x,y) \right| \leq \lambda_0^* \left| \sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f-g_{k_0,k_0};x,y) \right|$$
$$\leq \lambda_0^* \left| \sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(f-g_{k_0,k_0};x,y) \right|$$
$$+\lambda_0^* \left| \sum_{(i,j)\in\mathbb{N}^2} a_{k,l,i,j}^{(m,n)} T_{i,j}(g_{k_0,k_0};x,y) - g_{k_0,k_0}(x,y) \right| + \lambda_0^* \left| g_{k_0,k_0}(x,y) - f(x,y) \right|,$$

holds for every $x,y \in I$ and $m,n \in \mathbb{N}$. Applying the modular ρ and moreover considering the monotonicity of ρ, we have

$$\begin{aligned} \rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right) &\leq \rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f-g_{k_0,k_0})\right)\right) \\ &\quad+\rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(g_{k_0,k_0})-g_{k_0,k_0}\right)\right) \\ &\quad+\rho\left(3\lambda_0^*(g_{k_0,k_0}-f)\right). \end{aligned} \tag{2.6}$$

Then, it follows from (2.5) and (2.6) that

$$\begin{aligned} \rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right) &\leq \varepsilon+\rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f-g_{k_0,k_0})\right)\right) \\ &\quad+\rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(g_{k_0,l_0})-g_{k_0,k_0}\right)\right). \end{aligned} \tag{2.7}$$

So, taking the $\mu-$statistical limit as $(k,l)\in\mathbb{N}^2$ on both sides of (2.7) and using the fact that $g_{k_0,k_0}\in C^{\infty}(I^2)$ and $f-g_{k_0,k_0}\in X_{\mathbb{T}}$, we obtained from (2.1) that

$$\begin{aligned} &\mu\left(\left\{(k,l)\in\mathbb{N}^2:\ \rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right)\geqslant\varepsilon\right\}\right) \\ &\leq \varepsilon+\mu\left(\left\{(k,l)\in\mathbb{N}^2:\ \rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f-g_{k_0,k_0})\right)\right)\geqslant\varepsilon\right\}\right) \\ &\quad+\mu\left(\left\{(k,l)\in\mathbb{N}^2:\ \rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(g_{k_0,k_0})-g_{k_0,k_0}\right)\right)\geqslant\varepsilon\right\}\right), \end{aligned}$$

which gives

$$\begin{aligned} &\mu\left(\left\{(k,l)\in\mathbb{N}^2:\ \rho\left(\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(f)-f\right)\right)\geqslant\varepsilon\right\}\right) \\ &\leq \varepsilon(S+1)+\mu\left(\left\{(k,l)\in\mathbb{N}^2:\ \rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(g_{k_0,k_0})-g_{k_0,k_0}\right)\right)\geqslant\varepsilon\right\}\right). \end{aligned} \tag{2.8}$$

By (2.4), since

$$\mu\left(\left\{(k,l)\in\mathbb{N}^2:\ \rho\left(\eta\left(A_{k,l,m,n}^{\mathbb{T}}(g)-g\right)\right)=0\geqslant\varepsilon\right\}\right)=0,\ \text{uniformly in } m,n,$$

we get

$$\mu\left(\left\{(k,l)\in\mathbb{N}^2:\ \rho\left(3\lambda_0^*\left(A_{k,l,m,n}^{\mathbb{T}}(g_{k_0,k_0})-g_{k_0,k_0}\right)\right)\geqslant\varepsilon\right\}\right)=0,\ \text{uniformly in } m,n. \tag{2.9}$$

Combining (2.8) with (2.9), we conclude that

$$\mu\left(\left\{(k,l)\in\mathbb{N}^2:\ \rho\left(\lambda_0^*\left(A^{\mathbb{T}}_{k,l,m,n}(f)-f\right)\right)\geqslant\varepsilon\right\}\right)\leq\varepsilon(S+1).$$

Since $\varepsilon>0$ was arbitrary, furthermore, since $\rho\left(\lambda_0^*\left(A^{\mathbb{T}}_{k,l,m,n}(f)-f\right)\right)$ is non-negative for all $k,l,m,n\in\mathbb{N}$, we can easily see that

$$\mu\left(\left\{(k,l)\in\mathbb{N}^2:\ \rho\left(\lambda\left(A^{\mathbb{T}}_{k,l,m,n}(f)-f\right)\right)\geqslant\varepsilon\right\}\right)=0,\ \text{uniformly in } m,n,$$

which completes the proof. □

Theorem 2.2. *Let $\mathcal{A}=\left\{A^{(m,n)}\right\}$ be a sequence of four dimensional infinite non-negative real matrices. Let ρ and $\mathbb{T}:=\left\{T_{i,j}\right\}$ be the same as in Theorem 2.1. If ρ satisfies the Δ_2-condition, then the statements (i) and (ii) are equivalent:*

(i) $\mu\left(\left\{(k,l)\in\mathbb{N}^2:\ \rho\left(\lambda\left(A^{\mathbb{T}}_{k,l,m,n}(f_r)-f_r\right)\right)\geqslant\varepsilon\right\}\right)=0,$ *uniformly in m,n, for every $\lambda>0$ and $r=0,1,2,3$,*

(ii) $\mu\left(\left\{(k,l)\in\mathbb{N}^2:\ \rho\left(\lambda\left(A^{\mathbb{T}}_{k,l,m,n}(f)-f\right)\right)\geqslant\varepsilon\right\}\right)=0,$ *uniformly in m,n, for every $\lambda>0$ provided that f is any fuction belonging to $L^{\rho}\left(I^2\right)$ such that $f-g\in X_T$ for every $g\in C^{\infty}\left(I^2\right)$.*

3. Application

In this section, we give an example of positive linear operators which satisfies the conditions of Theorem 2.1.

Example 3.1. *Take $I=[0,1]$ and let $\varphi:[0,\infty)\to[0,\infty)$ be a continuous function for which the following conditions hold:*

- *φ is convex,*
- *$\varphi(0)=0$, $\varphi(u)>0$ for $u>0$ and $\lim\limits_{u\to\infty}\varphi(u)=\infty$.*

Hence, consider the functional ρ^{φ} on $X(I^2)$ defined by

$$\rho^{\varphi}(f):=\int_0^1\int_0^1\varphi(|f(x,y)|)\,dxdy\quad\text{for}\quad f\in X\left(I^2\right). \tag{3.1}$$

In this case, ρ^{φ} is a convex modular on $X\left(I^2\right)$, which satisfies all assumptions listed in Section 1 (see [5]). Let us consider the Orlicz space generated by φ as follows:

$$L^{\rho}_{\varphi}(I^2):=\left\{f\in X\left(I^2\right):\rho^{\varphi}(\lambda f)<+\infty\quad\text{for some } \lambda>0\right\}.$$

Then consider the following bivariate Mellin-type operator $\mathbb{M}:=\left\{M_{i,j}\right\}$ on the space $L^{\rho}_{\varphi}\left(I^2\right)$ which is defined by:

$$M_{i,j}(f;x,y)=\int_0^1\int_0^1 K_{i,j}(t_1,t_2)\,f(t_1x,t_2y)\,dt_1dt_2 \tag{3.2}$$

for $x,y,t_1,t_2\in I$ *and* $i,j\in\mathbb{N}$ *where* $K_{i,j}(t_1,t_2)$ *defined by*

$$K_{i,j}(t_1,t_2)=(i+1)(j+1)\,t_1^i t_2^j.$$

Observe that the operators $U_{i,j}$ *map the Orlicz space* $L_\varphi^\rho(I^2)$ *into itself. Recall that from Lemma16 in [2], we obtain that for every* $f\in L_\varphi^\rho(I^2)$ *and* $i,j\geq 2$

$$\rho^\varphi(U_{i,j}f)\leq 32\rho^\varphi(f).$$

Then, we know that, for any function $f\in L_\varphi^\rho(I^2)$ *such that* $f-g\in X_{\mathbf{U}}$ *for every* $g\in C^\infty(I^2)$, $(U_{i,j}f)$ *is modularly convergent to* f, *with the choice of* $X_{\mathbf{U}}:=L_\varphi^\rho(I^2)$.

Let $K\in\Gamma$ *be such that* $\mu(K)=0$, *and* $\mathbb{N}^2\diagdown K$ *is infinite element set. Now define* $\{s_{i,j}\}$ *by*

$$s_{i,j}=\begin{cases}1, & (i,j)\in K\\ 0, & (i,j)\notin K.\end{cases} \tag{3.3}$$

Since $\mu\left(\mathbb{N}^2\diagdown K\right)=1$ *and* $P-\lim\limits_{(i,j)\in\mathbb{N}^2\diagdown K}s_{i,j}=0$, *now observe that*

$$\mu\left(\left\{(i,j)\in\mathbb{N}^2:\ \left|s_{ij}-0\right|\geqslant\varepsilon\right\}\right)=0.$$

Also, assume that $A:=\left\{A^{(m,n)}\right\}=\left\{a_{k,l,i,j}^{(m,n)}\right\}$ *is a sequence of four dimensional infinite matrices defined by* $a_{k,l,i,j}^{(m,n)}=\frac{1}{kl}$ *if* $1\leq i\leq k$, $1\leq j\leq l$, $(m,n=1,2,\ldots)$ *and* $a_{k,l,i,j}^{(m,n)}=0$ *otherwise. Then, using the operators* $U_{i,j}$, *we define the sequence of positive linear operators* $\mathbb{V}:=\{V_{i,j}\}$ *on* $L_\varphi^\rho(I^2)$ *as follows:*

$$V_{i,j}(f;x,y)=(1+s_{i,j})M_{i,j}(f;x,y)\ \text{for}\ f\in L_\varphi^\rho(I^2),\ x,y\in[0,1]\ \text{and}\ i,j\in\mathbb{N}. \tag{3.4}$$

We get, for every $h\in X_V:=L_\varphi^\rho(I^2)$, $\lambda>0$ *and for any positive constant* M, *that*

$$\mu\left(\left\{(k,l)\in\mathbb{N}^2:\ \rho\left(\lambda\left(A_{k,l,m,n}^{\mathbb{T}}(h)\right)\right)\geqslant\varepsilon\right\}\right)\leq M\rho(\lambda h),\ \text{uniformly in}\ m,n,$$

where $A_{k,l,m,n}^{\mathbb{V}}(h)=\sum\limits_{(i,j)\in\mathbb{N}^2}^{\infty}a_{k,l,i,j}^{(m,n)}V_{i,j}h$ *as in (1.5). Therefore the condition (2.1) works for our operators* $V_{i,j}$ *given by (3.4) with the choice of* $X_V=X_U=L_\varphi^\rho(I^2)$.

We now claim that

$$\mu\left(\left\{(k,l)\in\mathbb{N}^2:\ \rho\left(\lambda\left(A_{k,l,m,n}^{\mathbb{T}}(f_r)-f_r\right)\right)\geqslant\varepsilon\right\}\right)=0,\ uniformly\ in\ m,n;\ r=0,1,2,3. \tag{3.5}$$

Observe that $M_{i,j}(f_0;x,y)=1$, $M_{i,j}(f_1;x,y)\leq\frac{1}{i+2}+f_1(x,y)$, $M_{i,j}(f_2;x,y)\leq\frac{1}{j+2}+f_2(x,y)$ *and* $M_{i,j}(f_3;x,y)\leq\frac{2}{i+3}+\frac{2}{j+2}+f_3(x,y)$. *So, we can see,*

$$\begin{aligned}
&\rho^{\varphi}\left(\lambda\left(\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}(1+s_{i,j})-1\right)\right)\\
&\quad=\int_0^1\int_0^1\varphi\left(\left|\lambda\left(\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}(1+s_{i,j})-1\right)\right|\right)d_xd_y\\
&\quad=\varphi\left(\lambda\left(\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}(1+s_{i,j})-1\right)\right),
\end{aligned}$$

because of

$$\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}(1+s_{i,j})=\begin{cases}2 & (i,j)\in K\\ 1, & (i,j)\notin K\end{cases},\quad m,n=1,2,\ldots,$$

$\mu\left(\mathbb{N}^2\diagdown K\right)=1$ *and* $P-\lim_{(i,j)\in\mathbb{N}^2\diagdown K}\rho\left(\lambda\left(\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}(1+s_{i,j})-1\right)\right)=0$, *and also using continuity of* φ, *we get get*

$$\mu\left(\left\{(k,l)\in\mathbb{N}^2:\varphi\left(\lambda\left(\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}(1+s_{i,j})-1\right)\right)\geqslant\varepsilon\right\}\right)=0,\ \textit{uniformly in } m,n \tag{3.6}$$

and hence

$$\mu\left(\left\{(k,l)\in\mathbb{N}^2:\rho^{\varphi}\left(\lambda\left(A^{\mathbb{V}}_{k,l,m,n}(f_0)-f_0\right)\right)\geqslant\varepsilon\right\}\right)=0,\ \textit{uniformly in } m,n$$

which guarantees that (3.5) holds true for $r=0$. *Also, since*

$$\begin{aligned}
&\rho^{\varphi}\left(\lambda\left(A^{\mathbb{V}}_{k,l,m,n}(f_1)-f_1\right)\right)\\
&=\rho^{\varphi}\left(\lambda\left(\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}(1+s_{i,j})M_{i,j}(f_1;x,y)-x\right)\right)\\
&\leq\rho^{\varphi}\left(\lambda\left(\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}(1+s_{i,j})\left(\frac{1}{i+2}+f_1(x,y)\right)-f_1(x,y)\right)\right)\\
&=\rho^{\varphi}\left(\lambda\left(\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}(1+s_{i,j})\left(\frac{1}{i+2}+x\right)-x\right)\right)\\
&\leq\rho^{\varphi}\left(2\lambda\left(\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}\frac{(1+s_{i,j})}{i+2}\right)\right)+\rho^{\varphi}\left(2\lambda\left(\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}(1+s_{i,j})-1\right)\right).
\end{aligned}$$

We know that (3.6) is

$$\mu\left(\left\{(k,l)\in\mathbb{N}^2:\rho^{\varphi}\left(2\lambda\left(\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}\left(1+s_{i,j}\right)-1\right)\right)\geqslant\varepsilon\right\}\right)=0.$$

And because of

$$\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}\frac{\left(1+s_{i,j}\right)}{i+2}=\begin{cases}\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}\frac{2}{i+2}, & (i,j)\in K\\ \sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}\frac{1}{i+2}, & (i,j)\notin K\end{cases},\quad m,n=1,2,\ldots,$$

$\mu\left(\mathbb{N}^2\diagdown K\right)=1$ *and* $P-\lim_{(i,j)\in\mathbb{N}^2\diagdown K}\rho\left(2\lambda\left(\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}\frac{1}{i+2}\right)\right)=0$, *also using continuity of* φ, *we get*

$$\mu\left(\left\{(k,l)\in\mathbb{N}^2:\rho^{\varphi}\left(2\lambda\left(\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}\frac{\left(1+s_{i,j}\right)}{i+2}\right)\right)\geqslant\varepsilon\right\}\right)=0.$$

We have

$$\mu\left(\left\{(k,l)\in\mathbb{N}^2:\rho^{\varphi}\left(\lambda\left(A_{k,l,m,n}^{\mathbb{V}}(f_1)-f_1\right)\right)\geqslant\varepsilon\right\}\right)=0,\ \textit{uniformly in } m,n.$$

So (3.5) holds true for $r=1$. *Similarly, we have*

$$\mu\left(\left\{(k,l)\in\mathbb{N}^2:\rho^{\varphi}\left(\lambda\left(A_{k,l,m,n}^{\mathbb{V}}(f_2)-f_2\right)\right)\geqslant\varepsilon\right\}\right)=0,\ \textit{uniformly in } m,n.$$

Finally, since

$$\begin{aligned}&\rho^{\varphi}\left(\lambda\left(A_{k,l,m,n}^{\mathbb{V}}(f_3)-f_3\right)\right)\\&=\rho^{\varphi}\left(\lambda\left(\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}(1+s_{i,j})M_{i,j}\left(f_3;x,y\right)-f_3\left(x,y\right)\right)\right)\\&\leq\rho^{\varphi}\left(\lambda\left(\sum_{i=1}^{k}\sum_{j=1}^{l}\frac{1}{kl}(1+s_{i,j})\left(\frac{2}{i+3}+\frac{2}{j+2}+x^2+y^2\right)-\left(x^2+y^2\right)\right)\right)\\&\leq\rho^{\varphi}\left(4\lambda\left(\frac{1}{kl}\sum_{i=1}^{k}\sum_{j=1}^{l}\left(1+s_{i,j}\right)\frac{2}{i+3}\right)\right)+\rho^{\varphi}\left(4\lambda\left(\frac{1}{kl}\sum_{i=1}^{k}\sum_{j=1}^{l}\left(1+s_{i,j}\right)-1\right)\right)\\&\quad+\rho^{\varphi}\left(4\lambda\left(\frac{1}{kl}\sum_{i=1}^{k}\sum_{j=1}^{l}\left(1+s_{i,j}\right)\frac{2}{j+3}\right)\right)+\rho^{\varphi}\left(4\lambda\left(\frac{1}{kl}\sum_{i=1}^{k}\sum_{j=1}^{l}\left(1+s_{i,j}\right)-1\right)\right).\end{aligned}$$

Hence we can easily see that

$$\mu\left(\left\{(k,l)\in\mathbb{N}^2:\rho^{\varphi}\left(\lambda\left(A_{k,l,m,n}^{\mathbb{V}}(f_3)-f_3\right)\right)\geqslant\varepsilon\right\}\right)=0,\ \textit{uniformly in } m,n.$$

So, our claim (3.5) holds true for each $r=0,1,2,3$. $\{V_{i,j}\}$ *satisfies all the hypothesis of Theorem 2.1 and we immediately see that,*

$$\mu\left(\left\{(k,l)\in\mathbb{N}^2 : \rho\left(\lambda\left(A^{\mathbb{T}}_{k,l,m,n}(f)-f\right)\right)\geqslant\varepsilon\right\}\right)=0, \text{ uniformly in } m,n,$$

on $I^2=[0,1]\times[0,1]$ *for all* $f\in L^{\rho}_{\varphi}(I^2)$.

4. CONCLUSIONS

Now, we give some reduced results showing the importance of Theorem 2.1 and Theorem 2.2 in approximation theory with special choices:

1. We know from the condition (1.1) that μ_B-sta tistical convergence and $B-$statistical convergence are equivalent. If B is the double Cesáro matrix, then from our Theorem 2.1 and Theorem 2.2 we immediately get the Korovkin type theorems of the statistical $\mathcal{A}-$summation process for double sequences on modular spaces given by Orhan and Kolay [26] and in addition, if one replaces the matrices $A^{(m,n)}$ by the identity matrix, then we get the statistical Korovkin type theorems for double sequences given by Demirci and Orhan [25].

2. If one replaces the matrices $A^{(m,n)}$ by the identity matrix, then from our Theorem 2.1 and Theorem 2.2 we immediately get the $\mu-$statistical Korovkin type theorems for double sequences on modular spaces.

3. As it is well-known $(X,\|.\|)$ is a normed space so then, $\rho(.)=\|.\|$ is a convex modular in X. So, by choosing $\rho(.)=\|.\|$, from our Theorem 2.1 and Theorem 2.2, the following reductions are obtained on normed spaces:

i. We get the $\mu-$statistical $\mathcal{A}-$summation process for double sequences on normed spaces by choosing $\rho(.)=\|.\|$.

ii. If we consider the condition (1.1), then we immediately get the $B-$statistical $\mathcal{A}-$summation process for double sequences on normed spaces and in addition, if one replaces the matrices $A^{(m,n)}$ by the identity matrix, we immediately get the $B-$statistical Korovkin type theorems for double sequences in the normed spaces in [11].

iii. If we take the double identity matrix, instead of B in condition (1.1), then we immediately get the $\mathcal{A}-$summation process for double sequences on normed spaces given by [16] and in addition, $A=A^{(m,n)}=\left\{a^{(m,n)}_{k,l,i,j}\right\}$ where $a^{(m,n)}_{k,l,i,j}=\frac{1}{kl}$, for $m\le i\le m+k-1$, $n\le j\le n+l-1$, $(m,n=1,2,..)$ and $a^{(m,n)}_{k,l,i,j}=0$ otherwise, we immediately get the almost Korovkin type approximation theorem for double sequences in [1].

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Selin Çınar
Sinop University
Osmaniye Mahallesi
Universite Caddesi No:52G
Turkey
e-mail: scinar@sinop.edu.tr
and
Kamil Demirci
Sinop University
Osmaniye Mahallesi
Universite Caddesi No:52G
Turkey
e-mail:kamild@sinop.edu.tr