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OSCILLATION THEOREMS FOR CONFORMABLE DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we investigate the oscillatory behavior of solutions of conformable differential equations of the form

$$D_\alpha\left(a(t)D_\alpha y(t)\right)+q(t)y(t)=0,\ \ t\geq t_0,$$

using the Riccati transformation and the integral average method. Examples are given to illustrate the significance of the main results.

1. INTRODUCTION

In recent years fractional differential equations are used in different fields such as physics, biology, bio-medical sciences, etc., see for example [1, 6, 11, 12, 14, 20]. There are many definitions for fractional derivatives such as Riemann-Liouville, Caputo, Riesz, Riesz-Caputo, Weyl, Grunwald-Letnikov, and Chen derivative, etc. In [9], the authors have suggested a new fractional derivative called the conformable derivative, and this definition satisfies almost all properties of the usual derivative.

In this paper, we investigate the oscillatory behavior of solutions of conformable differential equation of the form

$$D_\alpha\left(a(t)D_\alpha y(t)\right)+q(t)y(t)=0,\ \ t\geq t_0, \tag{1.1}$$

where D_α denotes the conformable derivative of order α with $0<\alpha\leq 1$. We assume that

(H_1) $a(t)\in C^1([t_0,\infty),\mathbb{R}_+)$, and $q(t)\in C([t_0,\infty),\mathbb{R}_+)$;
(H_2) $y(t)$ and $y'(t)$ are differentiable functions on $\mathbb{R}$.

A nontrivial solution $y(t)$ of differential equation (1.1) is said to be oscillatory if it has arbitrarily large zeros: nonoscillatory otherwise. The equation (1.1) is oscillatory if all its solutions are oscillatory.

If $\alpha=1$, then we have the following second order differential equation

$$(a(t)y'(t))'+q(t)y(t)=0. \tag{1.2}$$

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In the literature, there are many papers dealing with the oscillatory behavior of solutions of equation (1.2), see for example [7, 10, 18].

As a new research area, the oscillatory behavior of fractional differential equations received great attention by many authors, see for example [2–5, 8, 13, 17, 19] and the references cited there in. From the review of literature one can see that most of the oscillation criteria are obtained for fractional differential equations with a forcing term, and very few oscillatory results are available for fractional differential equations without a forcing term [2].

Therefore, in this paper, we obtain some new oscillation criteria for equation (1.1) using the Riccati transformation technique. By applying some new properties of this derivative, the classical oscillation problem of equation (1.2) can be extended to equation (1.1). The obtained results improve and complement the result obtained in the literature for fractional differential equations. Examples are given to show the importance of our main results.

2. PRELIMINARIES

In this section, we shall present some basic results on conformable derivatives. First we shall start with the definition;

Definition 2.1. *[9][Conformable Derivative] Let $f : [t_0, \infty) \to \mathbb{R}$, and $t \geq t_0 > 0$. Then the conformable derivative of f of order α is defined by*

$$D_\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon(t - t_0)^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in (0, 1]$.

Some Properties of Conformable Derivative

Lemma 2.1. *[9] Let $\alpha \in (0, 1]$ and f and g be $\alpha-$differentiable at a point $t \geq t_0$. Then*

(P_1) $D_\alpha(t^p) = pt^{p-\alpha}$, *for all* $p \in \mathbb{R}$;
(P_2) $D_\alpha(\lambda) = 0$, *for all constant functions* $f(t) = \lambda$;
(P_3) $D_\alpha(fg) = fD_\alpha(g) + gD_\alpha(f)$;
(P_4) $D_\alpha\left(\frac{f}{g}\right) = \frac{gD_\alpha(f) - fD_\alpha(g)}{g^2}$;
(P_5) *If f is differentiable, then*
$D_\alpha f(t) = t^{1-\alpha} f'(t)$.

Lemma 2.2. *Let $0 < \alpha \leq 1$. Then the conformable integral of order α starting at t_0 is defined by*

$$I_\alpha f(t) = \int_{t_0}^{t} (s - t_0)^{\alpha - 1} f(s) ds := \int_{t_0}^{t} f(s) d_\alpha(s, t_0).$$

If the conformable integral of a given function f exists, we can f is $\alpha-$ integrable.

Lemma 2.3. *[1] If $0 < \alpha \leq 1$ and $f \in C^1([t_0, \infty), \mathbb{R})$ then for all $t > t_0$, we have*

$$I_\alpha(D_\alpha f(t)) = f(t) - f(t_0)$$

and

$$D_\alpha(I_\alpha f(t)) = f(t).$$

3. Oscillation Results

In this section, we established some new oscillation conditions for the equation (1.1). In the following, for convenience we denote

$$\delta(t) = \int_t^\infty \frac{1}{a(s)} d_\alpha(s, t_0). \tag{3.1}$$

We study the oscillation of equation (1.1) under the following two cases;

$$\int_T^t \frac{1}{a(s)} d_\alpha(s, t_0) = \infty \text{ when } t \to \infty \tag{3.2}$$

and

$$\int_T^\infty \frac{1}{a(s)} d_\alpha(s, t_0) < \infty \text{ when } t \to \infty \tag{3.3}$$

where $T \geq t_0 + 1$. We begin with the following lemma.

Lemma 3.1. *Assume that condition* (3.2) *holds. If $y(t)$ is an eventually positive solution of equation* (1.1)*, then*

$$y(t) > 0,\ D_\alpha y(t) > 0,\ D_\alpha(a(t) D_\alpha y(t)) \leq 0.$$

Proof. Let $y(t)$ be an eventually positive solution of equation (1.1). Then from (1.1), we have

$$D_\alpha(a(t) D_\alpha y(t)) = -q(t) y(t) \leq 0 \tag{3.4}$$

for all $t \geq t_1 \geq t_0$. Since by (H_1) and (H_2), $a(t) D_\alpha y(t)$ is differentiable, we can write (3.4) as

$$t^{1-\alpha}(a(t) D_\alpha y(t))' = -q(t) y(t) \leq 0$$

or

$$a(t) D_\alpha y(t) \text{ is nonincreasing for all } t \geq t_1.$$

Therefore $a(t) D_\alpha y(t) > 0$ or $a(t) D_\alpha y(t) < 0$ for all $t \geq t_1$. Assume that $a(t) D_\alpha y(t) < 0$ for all $t \geq t_1$. Then for $t \geq t_1$, we have

$$a(t) D_\alpha y(t) \leq a(t_1) D_\alpha y(t_1) \leq -M$$

for some $M > 0$, and

$$D_\alpha y(t) \leq -\frac{M}{a(t)} \tag{3.5}$$

or

$$y(t) \leq -M \int_{t_1}^t \frac{1}{a(s)} d_\alpha(s, t_0).$$

Integrating the last inequality from t_1 to t and letting $t \to \infty$ we obtain a contradiction to the positivity of $y(t)$. Hence $D_\alpha y(t) > 0$ for all $t \geq t_1$. This completes the proof. □

Lemma 3.2. *Assume that condition* (3.3) *holds. If $y(t)$ is an eventually positive solution of equation* (1.1)*, then $y(t)$ satisfies one of the following two cases:*

(i) $y(t) > 0,\ D_\alpha y(t) > 0,\ D_\alpha(a(t)D_\alpha y(t)) \leq 0;$
(ii) $y(t) > 0,\ D_\alpha y(t) < 0,\ D_\alpha(a(t)D_\alpha y(t)) \leq 0.$

Proof. Let $y(t)$ be an eventually positive solution of equation (1.1). Then from (1.1), we have

$$D_\alpha(a(t)D_\alpha y(t)) = -q(t)y(t) \leq 0 \tag{3.6}$$

for all $t \geq t_1 \geq t_0$. Since by (H_1) and (H_2), $a(t)D_\alpha y(t)$ is differentiable, we can write (3.6) as

$$t^{1-\alpha}(a(t)D_\alpha y(t))' = -q(t)y(t) \leq 0$$

or

$$a(t)D_\alpha y(t) \text{ is nonincreasing for all } t \geq t_1.$$

Therefore $a(t)D_\alpha y(t) > 0$ or $a(t)D_\alpha y(t) < 0$ for all $t \geq t_1$. This completes the proof. □

Theorem 3.1. *Assume that (H_1), (H_2), $q(t) \in C([t_0,\infty),\mathbb{R}_+)$ and condition* (3.2) *hold. If there exists a real valued function $\rho(t) \in C^1([t_0,\infty),\mathbb{R})$ such that*

$$\limsup_{t\to\infty} \int_T^t \left(\rho(s)q(s) - \frac{a(s)(D_\alpha\rho(s))^2}{4\rho(s)}\right) d_\alpha(s,t_0) = \infty \tag{3.7}$$

for $T \geq t_0 + 1$, then every solution of equation (1.1) *is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1.1) on the interval $[t_1,\infty)$. Without loss of generality one can assume that $y(t) > 0$ on $[t_1,\infty)$, and by Lemma 3.1, we have $D_\alpha y(t) > 0$ for all $t \geq t_1$. Define

$$w(t) = \frac{\rho(t)a(t)D_\alpha y(t)}{y(t)}. \tag{3.8}$$

Then $w(t) > 0$ for $t \geq t_1$. Differentiating (3.8) α times with respect to t, and then using (1.1) and the properties of conformable derivative, we have

$$\begin{aligned}
D_\alpha w(t) &= D_\alpha\left(\frac{\rho(t)a(t)D_\alpha y(t)}{y(t)}\right) \\
&= \frac{y(t)D_\alpha(\rho(t)a(t)D_\alpha y(t)) - \rho(t)a(t)D_\alpha y(t)D_\alpha y(t)}{(y(t))^2} \\
&= \frac{D_\alpha(\rho(t))a(t)D_\alpha y(t)}{y(t)} + \frac{\rho(t)D_\alpha(a(t)D_\alpha y(t))}{y(t)} - \frac{\rho(t)a(t)D_\alpha y(t)D_\alpha y(t)}{(y(t))^2} \\
&\leq -\rho(t)q(t) + \frac{D_\alpha\rho(t)}{\rho(t)}w(t) - \frac{1}{\rho(t)a(t)}w^2(t).
\end{aligned}$$

By using the inequality $Au - Bu^2 \leq \frac{A^2}{4B}$, with $A = \frac{D_\alpha \rho(t)}{\rho(t)}$, and $B = \frac{1}{\rho(t)a(t)}$, we obtain

$$D_\alpha w(t) \quad \leq \quad -\rho(t)q(t) + \frac{a(t)(D_\alpha \rho(t))^2}{4\rho(t)}.$$

Integrating the last inequality from T to t, we obtain

$$\int_T^t \left(\rho(s)q(s) - \frac{a(s)(D_\alpha \rho(s))^2}{4\rho(s)} \right) d_\alpha(s,t_0) < w(T).$$

This inequality holds for $T = t_0 + 1$, which means

$$\int_{t_0+1}^t \left(\rho(s)q(s) - \frac{a(s)(D_\alpha \rho(s))^2}{4\rho(s)} \right) d_\alpha(s,t_0) < w(t_0+1).$$

Taking limit supremum as $t \to \infty$, the right hand side is bounded, which is a contradiction to (3.7). This completes the proof of the theorem. □

Theorem 3.2. *Assume that* (H_1), (H_2), $q(t) \in C([t_0,\infty),\mathbb{R}_+)$ *and condition* (3.3) *hold. If*

$$\limsup_{t\to\infty} \int_T^t \left(\delta(s)q(s) - \frac{1}{4a(s)\delta(s)} \right) d_\alpha(s,t_0) = \infty \tag{3.9}$$

for $T \geq t_0 + 1$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of equation (1.1) on the interval $[t_1,\infty)$. Without loss of generality we may assume that $y(t) > 0$ for all $t \geq t_1$ and case(i) and (ii) of Lemma 3.2 hold. First consider case(i) of Lemma 3.2. Define

$$w(t) = \frac{a(t)D_\alpha y(t)}{y(t)}.$$

Then $w(t) > 0$ and

$$\begin{aligned} D_\alpha w(t) &= \frac{D_\alpha(a(t)D_\alpha y(t))}{y(t)} - \frac{a(t)D_\alpha y(t) D_\alpha y(t)}{y^2(t)} \\ &\leq -q(t). \end{aligned}$$

Integrating the last inequality from t_1 to t, we have

$$w(t) \leq -\int_{t_1}^t q(s) d_\alpha(s,t_0). \tag{3.10}$$

From condition (3.9), we have

$$\begin{aligned} \limsup_{t\to\infty} \int_T^t \left(\delta(s)q(s) - \frac{1}{4a(s)\delta(s)} \right) d_\alpha(s,t_0) &\leq \limsup_{t\to\infty} \int_T^t \delta(s)q(s) d_\alpha(s,t_0) \\ &\leq \delta(t) \int_T^t q(s) d_\alpha(s,t_0) = \infty, \end{aligned} \tag{3.11}$$

as $t \to \infty$. Using (3.11) and (3.10), we see that $\lim_{t\to\infty} w(t)$ is negative, which is a contradiction to the positivity of $w(t)$. Hence case (i) cannot happen.

Next consider case (ii) of Lemma 3.2. Define the function $v(t)$ by

$$v(t) = \frac{a(t)D_\alpha y(t)}{y(t)}. \tag{3.12}$$

Then $v(t) < 0$ for $t \geq t_1$. Since $a(t)D_\alpha y(t)$ is nonincreasing, we have

$$a(s)D_\alpha y(s) \leq a(t)D_\alpha y(t), \text{ for } s \geq t.$$

Dividing the last inequality by $a(s)$ and then integrating from t to l, we have

$$y(l) \leq y(t) + a(t)D_\alpha y(t) \int_t^l \frac{1}{a(s)} d_\alpha(s,t_0).$$

Letting $l \to \infty$ in the above inequality and using (3.1), we obtain

$$0 \leq y(t) + a(t)D_\alpha y(t)\delta(t), \ t \geq t_1. \tag{3.13}$$

From (3.12) and (3.13), we get

$$v(t)\delta(t) \geq -1, \ t \geq t_1. \tag{3.14}$$

Differentiating (3.12) α times with respect to t, we have

$$\begin{aligned} D_\alpha v(t) &= D_\alpha \left(\frac{a(t)D_\alpha y(t)}{y(t)} \right) \\ &= \left[\frac{y(t)D_\alpha(a(t)D_\alpha y(t)) - a(t)D_\alpha y(t)D_\alpha(y(t))}{(y(t))^2} \right]. \end{aligned} \tag{3.15}$$

By (3.15), the fact that $D_\alpha(a(t)D_\alpha y(t)) \leq 0$, (1.1) and (3.12), we obtain

$$D_\alpha v(t) \quad \leq \quad -q(t) - \frac{v^2(t)}{a(t)}. \tag{3.16}$$

Multiplying by $\delta(t)$ the above inequality and integrating the resulting inequality from t_1 to t, we see that

$$\delta(t)v(t)-\delta(t_1)v(t_1)+\int_{t_1}^t \frac{v(s)}{a(s)} d_\alpha(s,t_0) \leq -\int_{t_1}^t \delta(s)q(s)d_\alpha(s,t_0) - \int_{t_1}^t \frac{\delta(s)v^2(s)}{a(s)} d_\alpha(s,t_0)$$

which yields

$$\int_{t_1}^t \left[\delta(s)q(s) - \frac{1}{4a(s)\delta(s)} \right] d_\alpha(s,t_0) \leq 1 + \delta(t_1)w(t_1) < \infty$$

when using (3.14), which contradicts (3.9). This completes the proof of the theorem. □

4. EXAMPLES

In this section, we provide two examples to verify our main results.

Example 4.1. *Consider the following conformable differential equation*

$$D_{\frac{1}{2}}\left(\sqrt{t}D_{\frac{1}{2}}y(t)\right)+\frac{1}{\sqrt{t}}y(t)=0, t\geq 1. \tag{4.1}$$

Comparing with equation (1.1), we have $\alpha=\frac{1}{2}$, $a(t)=\sqrt{t}$ and $q(t)=\frac{1}{\sqrt{t}}$. It is easy to verify that all conditions of Theorem 3.1 are satisfied. Hence all solutions of equation (4.1) are oscillatory.

Example 4.2. *Consider the following conformable differential equation*

$$D_{\frac{1}{2}}\left(tD_{\frac{1}{2}}y(t)\right)+t^2y(t)=0, t\geq 1. \tag{4.2}$$

Compared with equation (1.1), we have $\alpha=\frac{1}{2}$, $a(t)=t$ and $q(t)=t^2$. It is easy to verify that all conditions of Theorem 3.2 are satisfied. Hence all solutions of equation (4.2) are oscillatory.

5. CONCLUSION

In this paper, we have obtained some new sufficient conditions for the oscillation of all solutions of the studied equation. The established results are new and complement the results reported for fractional differential equations.

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