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PROJECTIVE CURVES WITH NICE NORMAL BUNDLES AND CONTAINING A PRESCRIBED SUBSET OF A HYPERPLANE

EDOARDO BALLICO

ABSTRACT. Fix a hyperplane $H \subset \mathbb{P}^n$, $n > 3$, and a finite set $S \subset H$. We give conditions on the integers d, g and $\sharp(S)$ such that there exists a smooth and connected curve $X \subset \mathbb{P}^n$ with $\deg(X) = d$, $p_a(X) = g$ and $S \subset X \cap H$. When $d = \sharp(S)$ we may take g up to order $2d/n$, $d \gg 0$, when S is in linear general position. We also prove the existence of X with $h^1(N_X(-1)) = 0$ if $n \geq 8$, g is odd and $2d \geq (n-3)g + n + 11$.

1. INTRODUCTION

Let $X \subset \mathbb{P}^n$ be a smooth, connected and non-degenerate curve. Fix a hyperplane $H \subset \mathbb{P}^n$. Set $d := \deg(X)$ and $g := p_a(X)$. Let N_X denote the normal bundle of X in $\mathbb{P}^n$. If $h^1(N_X) = 0$ the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^n)$ of $\mathbb{P}^n$ is smooth at $[X]$. Let Γ be the unique irreducible component of $\mathrm{Hilb}(\mathbb{P}^n)$ containing $[X]$. Fix a hyperplane $H \subset \mathbb{P}^n$. It is natural to ask the following question.

Question 1.1. *Fix a set $S \subset H$ such that $\sharp(S) \leq d$. Is there some $W \in \Gamma$ such that $S \subset W \cap H$ and no irreducible component of W is contained in H? Is it possible to find an irreducible W? A smooth W?*

There is an obvious necessary condition if $\sharp(S) \geq d - n + 3$: the linear span of S in H must have codimension at most $d - \sharp(S)$ in H. If this condition is satisfied and $g = 0$ the answer is yes (with W a smooth rational curve), even for certain zero-dimensional schemes ([7, Theorem 1.6]). We would like to raise similar questions in a range of d, g, n for which there are no curves with $h^1(N_X) = 0$ (although all the results proved in this paper are in the range when there are such curves), because when $n > 3$ if $h^1(N_X) = 0$ g has a linear upper bound in terms of d, while Castelnuovo's upper bound for the genus of curves in $\mathbb{P}^n$ is quadratic in d ([11, Theorem 3.7]).

Question 1.1 is natural, but we also had in mind a technical motivation. In many papers (e.g. in [6]) we needed to add a curve $E \subset H$ such that $X \cup E$ has certain

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properties. For instance, we need to add a line E such that $X \cup E$ is nodal and $p_a(X \cup E) = p_a(X) + 2$, i.e. we need X such that $X \cap H$ has 3 collinear points, say $\{p_1, p_2, p_3\}$, while $X \cap H \setminus \{p_1, p_2, p_3\}$ has the Hilbert function of a general union of $\deg(X) - 3$ points of H.

Assume that all $X \in \Gamma$ are smooth and non-degenerate curves of degree d and genus g. Fix an integer $\sigma > 0$ such that $\sigma \le d$ and consider sets $S \subset H$ with $\sharp(S) = \sigma$. We explain in Remark 2.3 the well-known fact that to answer Question 1.1 (as in Theorem 1.2 below) not just for a single set S with cardinality σ but for a set Δ of subsets of H with cardinality σ containing the general unions of σ points of H (maybe a different $W \in \Gamma$ for a different $S \in \Delta$) we also need a very strong numerical assumption. We recall that a zero-dimensional scheme $Z \subset \mathbb{P}^r$ is said to be *curvilinear* if its connected components have Zariski tangent spaces of dimension either 0 or 1. The scheme Z is curvilinear if and only if it is contained in the smooth locus of a curve. Any zero-dimensional subscheme of a smooth curve is curvilinear. Thus to find a smooth W such that $W \cap H \supseteq S$ we must require that S is curvilinear. In this case we extend [7, Theorem 1.6] to the case $g > 0$ (under certain assumptions on d, g, n). We prove the following result.

Theorem 1.2. *Fix integers $n \ge 3$, $g \ge g' \ge 0$, $b \ge 0$, $a \ge n$, $d \ge d'$. Set $s := \lfloor (d - d')/2 \rfloor$, $w := \lfloor g/(n+1) \rfloor$ and $w' := \lceil g/(n+1) \rceil$. Assume $d' \ge a + nw'$, $b - w' \le d' - a - nw'$ and $g - g' \le 2s + d - d'$. Fix a hyperplane $H \subset \mathbb{P}^n$, a curvilinear zero-dimensional scheme $A \subset H$ and a finite set $B \subset H$ such that $A \cap B = \emptyset$, $\deg(A) = a$, $\sharp(B) = b$, A spans H. Then there is a smooth, connected and non-degenerate curve $X \subset \mathbb{P}^n$ such that $A \cup B \subset X$, $\deg(X) = d$, $p_a(X) = g$ and $h^1(N_X(-A - B)) = 0$.*

By a theorem of Kleppe ([20, Theorem 1.8], [25]) the vanishing of the integer $h^1(N_X(-A - B))$ is important for the interpolation problem for general deformations of $A \cup B$ in H (or in $\mathbb{P}^n$).

Then we prove a stronger result ($A = \emptyset$, but $b = d$) under a geometrical assumption on the set $X \cap H$. A finite set $S \subset H$ is said to be in *linear general position* if any $S' \subseteq S$ spans a linear space of dimension $\min\{\sharp(S'), n-1\}$. If $\sharp(S) \le n$ S is in linear general position if and only if it is linearly independent, while if $\sharp(S) \ge n$ S is in linear general position if and only if any subset of X with cardinality n spans H. For a set $S \subset H$ to be in linear general position is an easy to check property with strong geometric consequences ([11, §7.12], [21, Lemma 1.1 and Corollary 1.6]). These sets are the main actors of [9]. We prove the following result.

Theorem 1.3. *Fix integers n, d and g such that $n \ge 3$, $g \ge 0$ and $d \ge n + gn/2$ (case g even) or $d \ge n + 1 + n(g-1)/2$ (case g odd). Let $H \subset \mathbb{P}^n$ be a hyperplane. Fix $S \subset H$ such that $\sharp(S) = d$ and S is in linear general position. Then there exists a smooth and connected curve $X \subset \mathbb{P}^n$ such that $\deg(X) = d$, $p_a(X) = g$, $S = X \cap H$ and $h^1(N_X) = 0$.*

Since points in Uniform Position in the sense of [11, Chapter 3] are in linear general position, Theorem 1.3 may be applied to them.

The cases $n=3$ and $n>3$ are quite different, even taking a general subset of H as S ([10, 20]; see Remark 2.2 for more details). For $n=3$ Theorem 1.3 is known when $2g \le d-2$ ([7, Theorem 2.1]).

Now we describe the third result of this paper. Assume $h^1(N_X(-1))=0$. Since $0=h^1(N_X(-1)) \ge h^1(N_X)$, the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^n)$ of $\mathbb{P}^n$ is smooth at $[X]$. Let Γ be the unique irreducible component of $\mathrm{Hilb}(\mathbb{P}^n)$ containing $[X]$. The assumption $h^1(N_X(-1))=0$ implies that for a general subset S of H with cardinality d there is $C \in \Gamma$ such that $C \cap H = S$. There is no information on how general the set S must be for the existence of some C. Nevertheless, we think that a statement for general $S \subset H$ is very interesting and many papers proved it under some restrictions on the triple (n,d,g) ([1, 2, 4, 5, 10, 15–17, 19, 20, 24, 25]). The cases $n=3$ and $n>3$ are very different (compare [10, 20] and [25, Theorem 5]). The following theorem is the third result of this paper.

Theorem 1.4. *For all integers $n \ge 8$, $g \ge 3$ and $2d \ge (n-3)g+n+11$, with g odd there is a smooth, connected and non-degenerate curve $X \subset \mathbb{P}^n$ such that $\deg(X)=d$, $p_a(X)=g$, $h^1(N_X(-1))=0$ and X has general moduli.*

We put the restriction " g odd " in the statement of Theorem 1.4, because for g even a statement similar to Theorem 1.4 holds [4, Theorem 1]. The last assertion of Theorem 1.4 means that the isomorphism classes of the curves X with $h^1(N_X(-1))=0$ produced in the proof of the theorem cover a non-empty open subset of the moduli space $\mathcal{M}_g$. The proof covers the case $g=1$, where however a stronger result is known and used in the proof of Theorem 1.4 ([8, Theorem 4.1]).

Theorem 1.4 (and the corresponding statement for even g proved in [4]) are almost optimal, because Ch. Walter proved that $2d \ge (n-3)g+4$ if any such X exists ([25, Theorem 5]). There are well-known triples (d,g,n) with $2d \ge (n-3)g+4$, $n \ge 4$, and without curves X with $h^1(N_X(-1))=0$, e.g. general canonical curves $C \subset \mathbb{P}^4$ of degree 8 and genus 5, which are the complete intersection of 3 quadric hypersurfaces and so $h^1(N_C(-1))=3$. For $n=4$ all exceptional cases in the Brill-Noether range are classified and their geometry is explained in [19, Corollary 2]. We do not know how much Walter's bound $2d \ge (n-3)g+4$ is optimal for $n>3$. In the same paper he pointed out not only the quoted example of the canonical curves of $\mathbb{P}^4$, but that for very low g there are no integral and non-degenerate curves $C \subset \mathbb{P}^n$ with degree d, arithmetic genus g and $2d$ very near to $(n-3)g+4$ by Castelnuovo's upper bound for the genus ([11, Theorems 3.6, 3.11, 315]) and that for almost minimal degrees the existing curves are contained in many quadric hypersurfaces and hence their general hyperplane section has a nongeneral Hilbert function. For a fixed n only finitely many d,g are excluded for this reason and so we wonder if the following is true.

Question 1.5. *Fix an integer $n \geq 4$. Is there an integer $a(n) \geq 4$ such that for all integers d with $2d \geq (n-3)g + a(n)$ there is a smooth and non-degenerate curve $X \subset \mathbb{P}^n$ such that $\deg(X) = d$, $p_a(X) = g$ and $h^1(N_X(-1)) = 0$?*

Fix integers $n \geq 4$ and $0 < t < d$. Let $u(n,t,d) \in \mathbb{Z}$ be the minimum of all integers $(n+1)d - (n-3)g - (n-1)t$, where g has the following property. Let $S \subset H$ be any subset with $\sharp(S) = \sigma$ and S in linear general position. Then there exists a smooth, connected and non-degenerate $X \subset \mathbb{P}^n$ such that X is transversal to H, $X \cap H \supseteq S$, $\deg(X) = d$ and $p_a(X) = g$. We raise the following question (we have no idea on how to solve it).

Question 1.6. *Give upper and/lower bounds for the function $u(n,t,d)$.*

In Remark 2.3 we explain why to get a very strong and very interesting result for all t (even restricting to general subsets of H) it is not sufficient to take as $u(n,t,d)$ a function depending only on n.

For all $n \geq 4$ even the condition $h^1(N_X) = 0$ gives a linear upper bound for the genus of X in term of the degree of X. We wonder if with different techniques one can control the Hilbert function of a general hyperplane section outside this range. Fix an integer $n \geq 4$. For every integer $d \geq n$ let $e_n(d)$ (resp. $e'_n(d)$) be the maximal integer γ such that for all integers g with $0 \leq g \leq \gamma$ there is a smooth (resp. integral) non-degenerate curve $X \subset \mathbb{P}^n$ such that a general hyperplane section of X has the Hilbert function of a general subset of $\mathbb{P}^{n-1}$ with cardinality d, i.e. $h^0(H, I_{X\cap H,H}(t)) = \max\{0, \binom{n+t-1}{n-1} - d\}$ for all $t \in \mathbb{N}$. For every integer $d \geq n$ let $f_n(d)$ (resp. $f'_n(d)$) be the maximal integer g such that there is a smooth (resp. integral) non-degenerate curve $X \subset \mathbb{P}^n$ such that a general hyperplane section of X has the Hilbert function of a general subset of $\mathbb{P}^{n-1}$ with cardinality d.

Question 1.7. *Fix an integer $n \geq 4$. Is $e_n(d) = e'_n(d)$ for all (or almost all) d?*

$$\text{Is } \lim_{d\to+\infty} \frac{e_n(d)}{d} = \lim_{d\to+\infty} \frac{e'_n(d)}{d} = +\infty\ ?$$

$$\text{Is } \lim_{d\to+\infty} \frac{f_n(d)}{e_n(d)} = \lim_{d\to+\infty} \frac{f'_n(d)}{e'_n(d)} = 1\ ?$$

Theorems 1.2 and 1.3 concern non-general subsets of a hyperplane $H \subset \mathbb{P}^n$, $n \geq 4$. We think that the bounds obtained in this paper are far from optimal.

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2. Remarks and lemmas

Remark 2.1. Fix a reduced and connected curve $Y \subset \mathbb{P}^n$. Set $d := \deg(Y)$. Let $H \subset \mathbb{P}^n$ be a hyperplane containing no irreducible component of Y. Thus $Y \cap H$ is a zero-dimensional scheme and $\deg(Y \cap H) = d$. Since Y is reduced and connected, we have $h^1(I_Y) = 0$. The exact sequence

$$0 \to I_Y \to I_Y(1) \to I_{Y\cap H,H}(1) \to 0$$

shows that the zero-dimensional scheme $Y \cap H$ spans H. For any zero-dimensional scheme $E \subset H$ let $\langle E \rangle$ denote its linear span, i.e. the intersection of all hyperplanes of H containing E, with the convention $\langle E \rangle = H$ if there is no such hyperplane. If $F \subset E$ we have $\dim\langle E \rangle \le \dim\langle F \rangle + \deg(E) - \deg(F)$. Thus $\dim\langle S \rangle \ge n-1+\deg(S)-d$ for any zero-dimensional scheme $S \subseteq Y \cap H$.

Remark 2.2. Let $H \subset \mathbb{P}^n$ be a hyperplane. Let Γ be an irreducible component of the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^n)$ of $\mathbb{P}^n$ whose general element is a smooth and non-degenerate curve of degree d and genus g with $h^1(N_X) = 0$. Since $h^1(N_X) = 0$, we have $\dim\Gamma = (n+1)d + (n-3)(g-1)$. Hence if $(n+1)d + (n-3)(1-g) < d(n-1)$ for a general $S \subset H$ with $\sharp(S) = d$ there is no $X \in \Gamma$ such that $S = X \cap H$. If $n > 3$, this condition gives a linear upper bound for the maximum genus allowable. If $0 < t < d$ and $(n+1)d + (n-3)(1-g) < t(n-1)$, then the same holds for the inclusion in S of t general points of H. For $n = 3$ the upper bound for the genus for curves X satisfying $h^1(N_X(-2)) = 0$ (or $h^1(N_X(-1)) = 0$ or $h^1(N_X) = 0$) is of order $d^{3/2}$ ([10, 20]).

Lemma 2.1. *Let $Y \subset \mathbb{P}^n$ be a smooth and connected curve and $Z \subset Y$ a zero-dimensional scheme. Assume $h^1(N_Y(-Z)) = 0$. Fix $o \in \mathbb{P}^n \setminus Y$ and a line $L \subset \mathbb{P}^n$ such that $o \in L$, $\sharp(Y \cap L) = 1$, $L \cap Z = \emptyset$ and L is not tangent to Y. Then $Y \cup L$ is smoothable in a family of curves containing $Z \cup \{o\}$ and $h^1(N_{Y\cup L}(-Z-o)) = 0$.*

Proof. Set $\{q\} := Y \cap L$. Since $N_L(-o) \cong \mathcal{O}_L$, L intersects quasi-transversally Y, $\sharp(Y \cap L) = 1$, $o \notin Y$, and $L \cap Z = \emptyset$, $N_{Y\cup L}(-Z-o)_{|Y}$ is obtained from $N_Y(-Z)$ making a positive elementary transformation at q and $N_{Y\cup L}(-Z-o)_{|L}$ is obtained from $N_L(-o)$ making a positive elementary transformation at q ([13, §2], [23]). Thus $h^1(N_{Y\cup L}(-Z-o)_{|Y}) = 0$ and $N_{Y\cup L}(-Z-o)_{|L}$ is a direct sum of one line bundle of degree 1 and $n-2$ line bundles of degree 0. Thus $h^1(N_{Y\cup L}(-Z-o)_{|L}) = 0$ and the restriction map $H^0(N_{Y\cup L}(-Z-o)_{|L}) \to H^0(N_{Y\cup L}(-Z-o)_{|\{q\}})$ is surjective. The Mayer-Vietoris exact sequence

$$\begin{aligned} 0 &\to N_{Y\cup L}(-Z-o) \to N_{Y\cup L}(-Z-o)_{|Y} \oplus N_{Y\cup L}(-Z-o)_{|L} \\ &\to N_{Y\cup L}(-Z-o)_{|Y\cap L} \to 0 \end{aligned} \tag{2.1}$$

gives $h^1(N_{Y\cup L}(-Z-o)) = 0$. Set $d := \deg(Y)$, $g := p_a(Y)$ and $z := \deg(Z)$. Since $h^1(N_Y(-Z)) = 0$, the family $\mathcal{B} \subset \mathrm{Hilb}(\mathbb{P}^n)$ formed by all smooth curves containing Z is smooth and of dimension $(n+1)d + (3-n)(g-1) - z(n-1)$ at $[Y]$ ([20]). Let $\mathcal{A}$ denote the closed subset of $\mathrm{Hilb}(\mathbb{P}^n)$ formed by the nodal curves containing $Z \cup \{o\}$. Since $h^1(N_{Y\cup L}(-Z-o)) = 0$, $\deg(Y \cup L) = d+1$, $p_a(Y \cup L) = g$ and $\deg(Z \cup \{o\}) = z+1$, $\mathcal{A}$ is smooth and of dimension $\dim_{[Y]}\mathcal{B} + 2$ at $Y \cup L$. The singular elements of $\mathcal{A}$ near $Y \cup L$ are formed by lines through o meeting a curve near Y. Such a family of singular curves has dimension $\dim_{[Y]}\mathcal{B} + 1$ and hence it does not contain a neighborhood of $[Y \cup L]$ in $\mathcal{A}$. $\square$

Lemma 2.2. *Let $Y \subset \mathbb{P}^n$, $n \geq 3$, be a smooth and connected curve and $Z \subset Y$ a zero-dimensional scheme. Assume $h^1(N_Y(-Z)) = 0$. Fix non-negative integers x, y such that $0 \leq x + y \leq n + 3$ and $y \leq n$. Assume the existence of a rational normal curve $L \subset \mathbb{P}^n$ such that L intersects quasi-transversally Y, $L \cap Z = \emptyset$ and $\sharp(Y \cap L) = x$. Fix a set $E \subset L \setminus Y \cap L$ such that $\sharp(E) = y$. Then $h^1(N_{Y \cup L}(-Z - E)) = 0$ and $Y \cup L$ is smoothable in a family of curves of $\mathbb{P}^n$ containing $Z \cup E$.*

Proof. The normal bundle N_L of Y in $\mathbb{P}^n$ is a direct sum of $n-1$ line bundles of degree $n+2$. By assumption the curve $Y \cup L$ is nodal. If $x = 0$, then $Y \cup L$ is smooth with two connected components and the lemma is trivial. Thus we may assume $x > 0$. By [13] or [23] the restriction $N_{Y \cup L|Y}$ (resp. $N_{Y \cup L|L}$) to Y (resp. L) of the normal bundle $N_{Y \cup L}$ of $Y \cup L$ is obtained from N_Y (resp. N_L) making x positive elementary transformations, one for each point of $Y \cap L$. Thus $h^1(N_{Y \cup L|Y}) = h^1(N_{Y \cup L|L}) = 0$ and the restriction map $H^0(N_{Y \cup L|L}) \to H^0(N_{Y \cup L|E})$ is surjective. The exact sequence (2.1) with E instead of o gives $h^1(N_{Y \cup L}(-Z - E)) = 0$.

Now we prove that $Y \cup L$ is smoothable in a family of curves of $\mathbb{P}^n$ containing $Z \cup E$. Set $z := \deg(Z)$, $d := \deg(Y)$ and $g := p_a(Y)$. Since $h^1(N_Y(-Z)) = 0$, the only irreducible component, Γ, of the set of all curves near Y containing Z has dimension $(n+1)d + (3-n)(1-g) - (n-1)z$ ([20, Theorem 1.8]). Let $\mathcal{A}$ denote the closed subset of $\mathrm{Hilb}(\mathbb{P}^n)$ formed by the nodal curves containing $Z \cup E$. Since $h^1(N_{Y \cup L}(-Z - E)) = 0$, $\deg(Y \cup L) = d + n$, $p_a(Y \cup L) = g + x - 1$ and $\deg(Z \cup \{o\}) = z + y$, $\mathcal{A}$ is smooth and of dimension $\dim \Gamma + n(n+1) + (3-n)(x-1) - (n-1)y$ at $Y \cup L$ ([20, Theorem 1.8]). For any integer t such that $1 \leq t \leq x$ let $\mathcal{A}_t$ denote the set of all $X \in \mathcal{A}$ with exactly t nodes. Near $[Y \cup L]$ all elements of $\mathcal{A}_t$ are obtained fixing a subset $S \subset Y \cap L$ such that $\sharp(S) = t$, smoothing all nodes in $Y \cap L \setminus S$ and considering only deformations of $Y \cup L$ equisingular at each point of S. To conclude the proof of the lemma it is sufficient to prove that $\dim_{[Y \cup L]} \mathcal{A}_t < \dim_{[Y \cup L]} \mathcal{A}$ for all $t = 1, \dots, x$. The set of all rational normal curves containing E has dimension $(n-1)(n+3-y)$. A dimensional count gives that $\dim_{[Y \cup L]} \mathcal{A}_x = \dim_{[Y \cup L]} \mathcal{A} - x$. Thus a general element of $\mathcal{A}$ near $[Y \cup L]$ is irreducible. Fix an integer t such that $1 \leq t < x$. Fix any $q \in Y \cap L$ and call $\mathcal{A}(q)$ the set of all $A \in \mathcal{A}$ with a node near q. Since $h^1(N_L(-E - q)) = 0$, we set that locally around $[Y \cup L]$ in the space $\mathcal{A}$ smooth at q the set $\mathcal{A}(q)$ is given by a single local equation. Since $\dim_{[Y \cup L]} \mathcal{A}_x = \dim_{[Y \cup L]} \mathcal{A} - x$, all these equations are independent. We only need that all these equations are non-trivial, so that $\dim_{[Y \cup L]} \mathcal{A}_t < \dim_{[Y \cup L]} \mathcal{A}$ for $t > 0$. □

Lemma 2.3. *Let $H \subset \mathbb{P}^n$, $n \geq 2$, be a hyperplane. Fix $o \in \mathbb{P}^n \setminus H$ and a rational normal curve $D \subseteq H$. Let $\ell_o : \mathbb{P}^n \setminus \{o\} \to H$ denote the linear projection from o. For any integral curve $Y \subset X$, Y not a line, such that o is a smooth point of Y let $\ell(Y)$ denote the closure of $\ell_o(Y \setminus \{o\})$ in H. Let Δ denote the set of all rational normal curves $Y \subset \mathbb{P}^n$ such that $o \in Y$ and $\ell(Y) = D$. Then $\Delta \neq 0$ and Δ is a non-empty irreducible algebraic variety of dimension $n+2$.*

Proof. Note that in the case $n = 2$ we have $D = H$ and so Δ is the set of all smooth conics of $\mathbb{P}^2$ containing o. Thus the lemma is trivial when $n = 2$ and so we may assume $n > 2$.

Let F_{n-1} denote the Hirzebruch surface with a section, h, of its ruling with self-intersection $h^2 = 1-n$ ([12, §V.2]). We take h and a fiber f of the ruling π of F_{n-1} as a basis of $\mathrm{Pic}(F_{n-1}) \cong \mathbb{Z}^2$. We have $h\cdot(h+(n-1)f) = 0$ and $\pi_*(\mathcal{O}_{F_{n-1}}(h)) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1-n)$. Thus the projection formula gives $h^0(\mathcal{O}_{F_{n-1}}(h+xf)) = 2x+3-n$ for all $x \geq n-2$. We get that the complete linear system $|\mathcal{O}_{F_{n-1}}(h+(n-1)f)|$ induces a morphism $\phi : F_{n-1} \to \mathbb{P}^n$ which is an embedding outside h, $\phi(h)$ is a point, o', and $\phi(F_{n-1})$ is a degree $n-1$ cone with vertex o' over a rational normal curve of a hyperplane $H' \subset \mathbb{P}^n$ such that $o' \notin H'$. Up to a projective transformation we may assume $o' = o$ and $H' = H$. Fixing D and o is equivalent to fixing the cone $\phi(F_{n-1})$ (here we use that $n > 2$). The irreducible elements of $|\mathcal{O}_{F_{n-1}}(h+(n-1)f)|$ are projectively equivalent to D. Since $(h+nf)\cdot(h+(n-1)f) = n$, we get that Δ is the set of all irreducible (or equivalently, all smooth) elements of $|\mathcal{O}_{F_{n-1}}(h+nf)|$. Thus $\Delta \neq \emptyset$, Δ is irreducible and $\dim\Delta = n+2$. □

Lemma 2.4. *Take o, D, Δ as in the statement of Lemma 2.3 and the linear projection ℓ_o and the cone $\phi(F_{n-1})$ as in the proof of Lemma 2.3. Fix a finite set $B \subset \phi(F_{n-1}) \setminus \{o\}$ such that $\sharp(B) = n+2$, no two points of B are contained in the same line of $\phi(F_{n-1})$ and B is in linear general position. Then there exists $Y \in \Delta$ such that $B \subset Y$.*

Proof. Since $o \notin B$, there is a unique $E \subset F_{n-1} \setminus h$ such that $\phi(E) = B$. Since $\dim|\mathcal{O}_{F_{n-1}}(h+nf)| = n+2$, there is $C \in |\mathcal{I}_E(h+nf)|$. We need to check that C is smooth and irreducible (indeed it is even unique). This is true, because the singular elements of $|\mathcal{O}_{F_{n-1}}(h+nf)|$ have the following description. One type are the reducible curves of the form $F \cup Y'$ with Y' a smooth element of $|\mathcal{O}_{F_{n-1}}(h+(n-1)f)|$ and $F \in |\mathcal{O}_{F_{n-1}}(f)|$. Since $\phi(F)$ is a line of $\phi(F_{n-1})$, it contains at most one element of B. Since $\phi(Y')$ is a hyperplane section of $\phi(F_{n-1})$, it contains at most n elements of B. Thus $F \cup Y' \not\supseteq B$.

The other type of singular elements of $|\mathcal{O}_{F_{n-1}}(h+nf)|$ are of the form $h \cup G$ with $G \in |\mathcal{O}_{F_{n-1}}(nf)|$. Since $\phi(h) = \{o\}$ and each line of $\phi(F_{n-1})$ contains at most 1 element of B, at most finitely many $\phi(h \cup G)$ contain an element of B. □

Lemma 2.5. *Let $Y \subset \mathbb{P}^r$, $r \geq 2$, be an integral and non-degenerate curve. Assume that Y is not a rational normal curve. Let $\mathcal{A}$ denote the set of all subsets $S \subset Y$ such that $\sharp(S) = r+3$ and S is in linear general position. For each $S \in \mathcal{A}$ let C_S denote the only rational normal curve of $\mathbb{P}^r$ containing S. Set $\mathcal{B} := \cap_{S\in\mathcal{A}} C_S$. Then $\mathcal{B} = \emptyset$.*

Proof. The case $r = 2$ is trivial, because every non-empty open subset of $|\mathcal{O}_{\mathbb{P}^2}(2)|$ has no base points. Now assume $r \geq 3$ and that the lemma is true in $\mathbb{P}^{r-1}$. Since Y is integral and not a rational normal curve, we have $Y \not\subseteq \mathcal{B}$. Hence $o \notin \mathcal{B}$ for a general $o \in Y$. Let $\ell : \mathbb{P}^r \setminus \{o\} \to \mathbb{P}^{r-1}$ denote the linear projection from o. Let

$D \subset \mathbb{P}^r$ be the closure of $\ell(X \setminus \{o\}$ in $\mathbb{P}^{r-1}$. If D is not a rational normal curve of $\mathbb{P}^{r-1}$ we may use the inductive assumption. If D is a rational normal curve, then $\ell_{|X\setminus\{o\}}$ is not birational onto its image and this does not occur for a general $o \in Y$. □

Lemma 2.6. *Let $Y \subset \mathbb{P}^r$, $r \geq 3$, be an integral and non-degenerate curve. Fix $q \in \mathbb{P}^r \setminus Y$ and call $\mathcal{A}$ the set of all subsets $S \subset Y$ such that $\sharp(S) = r+2$ and $S \cup \{q\}$ is in linear general position. For each $S \in \mathcal{A}$ let C_S denote the only rational normal curve of $\mathbb{P}^r$ containing $S \cup \{q\}$. Set $\mathcal{B} := \cap_{S\in\mathcal{A}} C_S$. Then $\mathcal{B} = \{q\}$.*

Proof. Let $\ell : \mathbb{P}^r \setminus \{q\} \to \mathbb{P}^r$ denote the linear projection from q. If $\ell(Y)$ is not a rational normal curve we may apply Lemma 2.5 to $\ell(Y) \subset \mathbb{P}^{r-1}$. Now assume that D is a rational normal curve. Since o is a smooth point of Y, we have $\deg(Y) - 1 = x\deg(D)$, where x is the degree of the rational map $Y \dashrightarrow D$ induced by $\ell_{|Y\setminus\{o\}}$. Since Y is not a rational normal curve, we get $x \geq 2$, i.e. $\ell_{|Y\setminus\{o\}}$ is not birational onto its image. This possibility may occur only for finitely many $o \in Y$, contradicting the generality of Y. □

Lemma 2.7. *Let $Y \subset \mathbb{P}^n$, $n \geq 2$, be an integral and non-degenerate curve. Fix a finite set $S \subset \mathbb{P}^n$ in linear general position and set $x := \sharp(S \cap Y)$ and $y := \sharp(S \cap (\mathbb{P}^n \setminus Y))$. Assume $x + y \leq n+2$. If Y is a rational normal curve assume $y > 0$. Let Γ denote the set of all $A \subset Y$ such that $\sharp(A) = n+3-x-y$, $A \cap S = \emptyset$ and $A \cup S$ is in linearly general position. For any $A \in \Gamma$ let C_A denote the unique rational normal curve of $\mathbb{P}^n$ containing A. Then $\cap_{A\in\Gamma} C_A = S$.*

Proof. Since Y is integral and spans $\mathbb{P}^n$, we have $\Gamma \neq \emptyset$. Increasing y if necessary we may assume $x + y = n+2$.

(a) Assume $n = 2$. Any set $F \subset \mathbb{P}^2$ such that $\sharp(F) = 4$ and no 3 of the points of F are collinear is the complete intersection of 2 smooth conics and a general $C \in |I_F(2)|$ is smooth and transversal to Y (note that in this case if $\deg(Y) > 0$ we have $\sharp(Y \cap C) > 4$).

(b) Assume $n > 2$ and that the lemma is true in $\mathbb{P}^{n-1}$.

(b1) Assume $x > 0$. Fix $o \in S \cap Y$ and let $\ell_o : \mathbb{P}^n \setminus \{o\} \to \mathbb{P}^{n-1}$ denote the linear projection from o. Set $S' := \ell_o(S \setminus \{o\})$. Let $T \subset \mathbb{P}^{n-1}$ denote the closure of $\ell_o(Y \setminus \{o\})$ in $\mathbb{P}^{n-1}$. T is an integral and non-degenerate curve of $\mathbb{P}^{n-1}$. Since S is in linear general position, we have $\sharp(S') = n+1$ and S' is in linearly general position. We have $\sharp(S' \cap T) = x-1$ and $\sharp(S' \cap (\mathbb{P}^{n-1} \setminus T)) = y = n+2-x$. Assume $\cap_{A\in\Gamma} C_A \neq S$ and fix $q \in \cap_{A\in\Gamma} C_A \setminus S$. Since $o \in S$ and $q \notin S$, the point $q' = \ell_o(q)$ is well-defined.

First assume that either $y > 0$ or that T is not a rational normal curve of $\mathbb{P}^{n-1}$. By the inductive assumption there is a rational normal curve $D \subset \mathbb{P}^{n-1}$ such that D contains S' and a point q'' of $D \setminus S'$. To apply Lemma 2.4 it is sufficient to observe that q'' is the image of a point of $Y \setminus \{o\}$.

Now assume $y=0$ and that T is a rational normal curve. In this case we assumed that Y is not a rational normal curve. In this case T_0 is an injective (at least for $n\geq 4$) projection of a degree n cone $J\subset\mathbb{P}^{n+1}$. With the notation of the proof of Lemma we have $J=\phi(F_n)$, where $\phi: F_n\to\mathbb{P}^{n+1}$ is the morphism induced by the linear system $|\mathcal{O}_{F_n}(h+nf)|$. Let $D\subset F_n$ be the curve such that Y is an injective linear projection of $\phi(D)\subset J$. Take positive integers a,b such that $D\in|\mathcal{O}_{F_n}(ah+bf)|$ with $a>0$ and $b\geq na$. Since $o\in Y$, $\phi(D)$ contains the vertex of J, i.e. $b>na$. We have $x=n+2$ and to apply the inductive assumption we may take any other point of $S\cap Y$. We see that Y cannot be contained in $n+2$ cones like T_0, concluding the proof in this case.

(b2) Assume $x=0$. We have $y=n+2$. We fix $o\in S$, consider the linear projection from o and use the inductive assumption. Since $y\geq 2$ in $\mathbb{P}^{n-1}$ we do not need to distinguish the case in which $\ell_o(Y)$ is a rational normal curve to apply the inductive assumption. □

Lemma 2.8. *Let $Y\subset\mathbb{P}^n$, $n\geq 3$, be an integral and non-degenerate curve. Assume that Y is not a rational normal curve. Let $E\subset Y$ be a general subset of Y with cardinality $n+3$. Let $D\subset\mathbb{P}^n$ be the only rational normal curve containing E. Then D meets Y quasi-transversally and $Y\cap D=E$.*

Proof. Since Y is integral and non-degenerate and $E\subset Y$ is general, there is one and only one rational normal curve $D\subset Y$ containing E. Since E is general in Y, no point of E is a singular point of Y. Since Y is not a rational normal curve, $Z:=D\cap Y$ (scheme-theoretic intersection) is a zero-dimensional scheme. We need to prove that $Z=E$ as schemes. It is sufficient to prove that $Z=E$ for a specific set E (of course, in linearly general position, otherwise D is not defined). We use induction on n. Set $d:=\deg(Y)$. Fix $q\in E$ and let $\ell_q:\mathbb{P}^n\setminus\{q\}\to\mathbb{P}^{n-1}$ denote the linear projection from q. Set $G:=E\setminus\{q\}$. Since $E\subset Y$ is general, q is a general point of Y. Hence $\ell_{q|Y\setminus\{q\}}$ is birational onto its image whose closure, W, in $\mathbb{P}^{n-1}$ is an integral and non degenerate curve of degree $d-1$. Since E is in linear general position, $A:=\ell_q(G)$ is a subset of $\mathbb{P}^{n-1}$ with cardinality $n+2$ and in linear general position. Thus there is a unique rational normal curve $C\subset\mathbb{P}^{n-1}$ containing A. For a fixed q we may move G among the subsets of Y with cardinality $n+2$. Thus A is a general subset of W with cardinality $n+2$. Let $T\subset\mathbb{P}^n$ be the cone with vertex q and C as a basis. Since T is a cone and a minimal degree surface, its minimal desingularization $u: F_{n-1}\to T$ is the Hirzebruch surface with a minimal degree section, h, with self-intersection $1-n$. Set $G':=u^{-1}(G)$. Since q is a smooth point of Y, $\ell_{q|Y\setminus\{q\}}$ extends to a surjective and birational morphism $\mu: Y\to W$. Since (after fixing q) G is general in Y, we have $G=\mu^{-1}(A)$ and μ is a local isomorphism at each point of G. We will take as D a curve $u(D')$ with $D'\in|\mathcal{O}_{F_n}(h+nf)|$ and $G'\subset D'$. Since $h^0(\mathcal{O}_{F_n}(h+n))=(n+1)+2$, there is at least one such D'. Every irreducible element of $X\in|\mathcal{O}_{F_n}(h+nf)|$ is smooth and $u(X)$ is a rational normal curve containing q. Since G is in linearly general position, we have $h^0(F_{n-1}, I_{G'}(h+nf)) = 1$ and the only element of $|I_{G'}(h+nf)|$ is irreducible. Thus $D = u(D')$.

(a) First assume $n = 3$. In this case C is a smooth conic. By Bertini's theorem a general conic is transversal to W. Since A is general in W, C may be seen (even after fixing W) as a general conic. Thus C is transversal to W. Since $G = \mu^{-1}(A)$ and $\mu : Y \to W$ is a local isomorphism at each point of Y, D and Y meet quasi-transversally and $D \cap Y = E$.

(b) Now assume $n > 3$ and that the lemma is true in $\mathbb{P}^{n-1}$ for all non-degenerate curves, different from the rational normal curve of $\mathbb{P}^{n-1}$. Thus C and W intersect quasi-transversally and $C \cap W = A$ since $G = \mu^{-1}(A)$ and $\mu : Y \to W$ is a local isomorphism at each point of Y. Thus D and Y meet quasi-transversally and $D \cap Y = E$. □

Lemma 2.9. *Let $Y \subset \mathbb{P}^n$, $n \geq 3$, be a smooth, connected and non-degenerate curve. Set $d' := \deg(Y)$ and $g' := p_a(Y)$. Fix a finite set $S \subset X$ and integers $d \geq d'$, $g \geq g'$ and set $s := \lfloor (d-d')/n \rfloor$. Assume $g - g' \leq 2s + d - d' - sn$ and $h^1(N_Y(-S)) = 0$; if $g = g' + 2s$ and $d = d' + sn$ assume that $d' \neq n$. Then there exists a smooth, connected and non-degenerate curve $X \subset \mathbb{P}^n$ such that $X \supset S$, $\deg(X) = d$, $p_a(X) = g$ and $h^1(N_X(-S)) = 0$.*

Proof. We may assume $(d,g) \neq (d',g')$, i.e. $d > d'$. In steps (a), (b) and (c) we will silently use the following observation. Let $\pi : \Pi \to \mathbb{P}^n$ denote the blowing up of S. Let Y' denote the strict transform of Y. Since Y is smooth at each point of S, π induces an isomorphism between Y' and Y and this isomorphism induces an isomorphism between $N_Y(-S)$ and $N_{Y',\Pi}$. For any smooth curve $L \subset \mathbb{P}^n$ such that $L \cap S = \emptyset$ set $L' := \pi^{-1}(L)$. Since $S \cap L = \emptyset$, π induces an isomorphism between Y' and Y and this isomorphism induces an isomorphism between N_L and $N_{L',\Pi}$. If $L' \cup Y'$ is smoothable inside Π, then π shows the existence of a smoothing of $L \cup Y$ with a family of curves containing S. Since π induces an isomorphism between $N_{Y \cup L}(-S)$ and $N_{Y' \cup L',\Pi}$, to prove that $h^1(N_{Y \cup L}(-S)) = 0$ (and then to conclude by the semicontinuity theorem for cohomology) it is sufficient to prove that $h^1(N_{Y' \cup L'}) = 0$.

(a) Assume $d = d' + 1$ and $g = g'$. We take as X a smoothing with fixed S of $Y \cup L$, where L is a general line meeting Y at exactly one point. The proof that $Y \cup L$ is smoothable among curves fixing S is easier than the one of Lemma 2.1.

(b) Assume $d = d' + 1$ and $g = g' + 1$. We take as X a smoothing with fixed S of $Y \cup L$, where L is a general secant line of Y. The proof that $Y \cup L$ is smoothable among curves fixing S is similar (inside Π) to the proof of Lemma 2.1.

(c) Assume $d = d' + n$, $g = g' + n + 2$ and $d' \neq n$. Let $E \subset Y$ be a general subset with cardinality $n + 3$. In particular E is in linear general position and $E \cap S = \emptyset$. Let $L \subset \mathbb{P}^n$ a general rational normal curve containing E. Since $d' \neq n$ we have $L \neq Y$. As in the proof of Lemma 2.8 we see that $L \cap S = \emptyset$, $L \cap Y = E$ and that $Y \cup L$ is nodal. Set $X' := Y' \cup L'$ and $F := \pi^{-1}(E)$. Consider the Mayer-Vietoris exact sequence of the normal bundle of X' in Π:

$$0 \to N_{X',\Pi} \to N_{X',\Pi|Y'} \oplus N_{X',\Pi|L'} \to N_{X',\Pi|F} \to 0. \quad (2.2)$$

The rank $n-1$ vector bundle $N_{X',\Pi|Y'}$ (resp. $N_{X',\Pi|L'}$) on Y' (resp. on L') is obtained from $N_{Y',\Pi}$ (resp. $N_{L'|\Pi}$) making $n+3$ positive elementary transformations, one for each point of F ([**?**, §2]). Since $h^1(N_{Y',\Pi}) = 0$, we have $h^1(N_{X',\Pi|Y'}) = 0$. Since N_L is a direct sum of $n-1$ line bundles of degree $n+2$ and $L \cap S = \emptyset$, $N_{L',\Pi}$ is a direct sum of $n-1$ line bundles of degree $n+2$. Thus $N_{X',\Pi|L'}$ is a direct sum of line bundles of degree at least $n+2$. Thus $h^1(N_{X',\Pi|L}) = 0$ and $h^1(N_{X',\Pi|L}(-F)) = 0$. Thus the restriction map $H^0(L', N_{X',\Pi|L'} \to H^0(F, N_{X',\Pi|F})$ is surjective. From (2.2) we get $h^1(N_{X',\Pi}) = 0$. To see that X' is smoothable it is sufficient to observe that $h^1(N_{L',\Pi}(-F)) = 0$ ([13, Th. 4.1 and Rem. 4.1.1]).

(d) Assume $d = d' + n$ and $g = g + +n + 1$. Adapt the proof of part (c) to this easier case taking E with $\sharp(E) = n+2$.

(e) If $g - g' \le s + d - d'$ we apply s times step (d), then $g - g' - s(n+1)$ times step (b) and then $d - d' - g + g' - s$ times step (a). If $s + d - d' < g - g' \le 2s + d - d'$ we apply several times step (c) and then if necessary steps (a) and (b). □

Remark 2.3. Let $H \subset \mathbb{P}^n$, $n \ge 4$, be a hyperplane. Let Γ be a family of smooth, connected and non-degenerate curves whose closure in Hilb$(\mathbb{P}^n)$ is an irreducible component of Hilb$(\mathbb{P}^n)$. Set $d := \deg(X)$ and $g := p_a(X)$. Fix a positive integer $\sigma \le d$ and call Δ the set of subsets of H with cardinality σ. Fix $S \in \Delta$ and let W_S denote the set of $X \in W$ such that $X \cap H \supseteq S$. Assume $W_S \neq \emptyset$. J. Kleppe proved that for each $X \in W_S$ the vector space $H^0(N_X(-S))$ is the Zariski tangent space of W_S at $[X]$, while $H^1(N_X(-S))$ may be used as an obstruction space ([20, Theorem 1.8], [25]). Hence if $W_S \neq \emptyset$ we have $h^0(N_X(-S)) \ge 0$. We have $\chi(N_X(-S)) = (n+1)d + (3-n)(g-1) - (n-1)\sigma$. In many cases (but not in all cases!) we have $h^0(N_X(-S)) = h^0(N_X) - (n-1)\sigma$. Thus the inequality

$$(n-3)(g-1) \le (n+1)d - (n-1)\sigma \quad (2.3)$$

(equivalent to $h^1(N_X(-S)) = 0$) is often a necessary condition to have $W_S \neq \emptyset$ for a general $S \in \Delta$. When $\sigma = d$, we have $N_X(-S) \cong N_X(-1)$. Ch. Walter proved in this case that $h^0(N_X(-1)) \ge n+1$ and hence that if $h^1(N_X(-1)) = 0$ we have

$$(n-3)g + 4 \le 2d \quad (2.4)$$

([25, Theorem 5]). Just applying (2.4) in the case $\sigma = d-1$ gives an upper bound for g better than (2.3). But neither (2.3) nor the improved by $n+1$ bound hopefully obtained generalizing [25, Theorem 5] to some $\sigma < d$ (a task we do not know how to do) would be very good for low σ. For instance take any $\sigma \le n+1$. Hence in this case any S in linear general position is realized by any smooth and non-degenerate curve $Y \subset \mathbb{P}^n$. Hence for $\sigma \le n+1$, the maximal possible g is the maximal genus $\pi(n,d)$ of all smooth and non-degenerate degree d curves of $\mathbb{P}^n$. Since $\pi(n,d)$ is quadratic in d ([11, Theorems 3.7 and 3.11]) (2.3) is not satisfied when $\sigma \le n+1$. Somewhere between $n+1$ and d the upper bound for g must go from quadratic in d to linear in d, but we have no guess on this matter.

3. PROOFS OF THE THEOREMS

Proof of Theorem 1.4: Let $Y \subset \mathbb{P}^n$, $n \ge 4$, be a linearly normal elliptic curve. Thus Y is smooth, $p_a(Y) = 1$ and $\deg(Y) = n+1$.

Claim 1: We have $h^1(N_Y(-1)) = 0$.

Proof of Claim 1: We have $\deg(N_Y(-1)) = 2(n+1)$. By [8, Theorem 4.1] $N_Y(-1)$ is polystable. We have $\deg(N_Y(-1)) = \deg(N_Y) - (n+1)(n-1) = \deg T_{\mathbb{P}^n} - (n+1)(n-1) = 2(n+1) > 0$. The definition of semistability, implies $h^0(N_Y(-1)^\vee) = 0$. Since Y is an elliptic curve, duality implies $h^1(N_Y(-1)) = 0$.

Claim 2: Assume $n \ge 5$. For any 3 general points $p_1, p_2, p_3 \in \mathbb{P}^n$ and general lines $L_1, L_2, L_3 \subset \mathbb{P}^n$ such that $p_i \in L_i$ for all i there is a smooth linearly normal elliptic curve $Y \subset \mathbb{P}^n$ containing $\{p_1, p_2, p_3\}$ with L_i as its tangent line at each p_i.

Proof of Claim 2: By a theorem of Kleppe ([20, Theorem 1.8]) it is sufficient to prove that $h^1(N_Y(-Z)) = 0$, where Z is any zero-dimensional scheme of Y with $\deg(Z) = 6$. This is true by the semistablity of N_Y ([8, Theorem 4.1]), because $6(n-1) < (n+1)^2$, N_Y has degree $\deg T_{\mathbb{P}^n|Y} = (n+1)^2$ and rank $n-1$.

Then we continue the proof of the theorem as in the proof of [4, Theorem 1]. □

Proof of Theorem 1.2: Until step (d) we assume $d = d'$ and $g = g'$. When $g = 0$ it is sufficient to do the case $b = 0$, which is [7, Theorem 1.6]. Assume $g > 0$. We order the points $p_1, \dots, p_b$ of B. For any integer t let $\Gamma(t)$ denote the set of all $E \subset Y$ such that $\sharp(E) = t$. Fix a general $(S_1, \dots, S_w) \in \Gamma(n+2)^w$. We have $h^1(N_Y(-Z)) = 0$ for any degree $n+2$ effective divisor Z of Y by the possible splitting types of the normal bundles of smooth and non-degenerate rational curves ([22]). Thus for each $i \in \{1, \dots, w\}$ the generality of $S_i \in \Gamma(n+2)$ implies $b_i \notin S_i$ and that $S_i \cup \{b_i\}$ is in linear general position in $\mathbb{P}^n$. Thus there is a unique rational normal curve $C_i \subset \mathbb{P}^n$ containing $S_i \cup B_i$. We have $C_i \cap C_j = \emptyset$ for all $i, j \in \{1, \dots, w\}$ such that $i \ne j$ and $C_i \cap Y = S_i$ for all i by Lemma 2.6 and the generality of $(S_1, \dots, S_w)$.

(a) We first do the case $b = w$, $d = a + nw$ and $g = w(n+1)$.

Claim 1: For general S_i, $1 \le i \le w$, we have $b_j \notin C_i$ for all $j \ne i$ and the curve $E := Y \cup C_1 \cup \cdots \cup C_b$ is nodal and with $p_a(E) = w(n+1)$.

Proof of Claim 1: Since $C_i \cap S_i = S_i$ for all i, to prove that E is nodal with arithmetic genus $w(n+1)$ it is sufficient to prove that each C_i meets Y quasi-transversally . Assume the existence of $i \in \{1, \dots, w\}$ such that C_i is tangent to Y at some $p \in S_i$. A monodromy argument gives that C_i is tangent to Y at all points of S_i. Write $S_i = J \cup \{o\}$ with $\sharp(J) = n+1$ and take a general $q \in Y$. Since $J \cup \{q\} \cup \{b_i\}$ is in linear general position, there is a unique rational normal curve $C \supset J \cup \{q\} \cup \{b_i\}$. Since o is a limit of the family $\{q\}_{q \in Y}$, we may take all C_j, $j \ne i$, and C instead of $C_1, \dots, C_w$. By the generality of $C_1, \dots, C_w$, we see that

C is tangent to Y at all points of $J \cup \{q\}$. Thus $\deg(C \cap C_i) \geq 2(n+1)$. Since $2(n+1) \geq n+3$, we get $C = C_i$ for a general $q \in Y$, absurd.

(b) Assume $w' = w$. Increasing b if necessary we may assume $d = a+b+(n-1)w$. Take $B' := \{b_1, \dots, b_w\}$ and F as in step (a). The curve F is a smoothing (with fixed $A \cup B'$) of $Y \cup C_1 \cup \cdots \cup C_w$ with each C_i a rational normal curve. Since $g > 0$ and $w = w'$, we have $w > 0$. By step (a) F is smooth, connected and non-degenerate, $\deg(F) = a + wn$, $A \cup B' \subset F$ and $h^1(N_F(-A-B')) = 0$. Take general lines L_i, $b - nw + 1 \leq i \leq b$ containing b_i and meeting F.

Claim 2: We may take F so that each L_i meets F quasi-transversally at a unique point and $L_i \cap L_j = \emptyset$ for all $i \neq j$.

Proof of Claim 2: Let R_i, $b - nw + 1 \leq i \leq b$, be a general line containing b_i and intersecting C_w. Since $b_i \neq b_j$, any two meeting lines are coplanar and C_w spans $\mathbb{P}^n$, we have $R_i \cap R_j = \emptyset$ for all $i \neq j$. Since F is a smoothing of $Y \cup C_1 \cup \cdots \cup C_w$, it is sufficient to prove that for all i $R_i \cap C_h = \emptyset$ for all $h < w$, $R_i \cap Y = \emptyset$, $\sharp(R_i \cap C_w) = 1$, R_i meets quasi-transversally C_w. Fix i. Let T_0 be the cone with vertex b_i and Y as a basis. For $1 \leq h < w$ let T_h be the cone with vertex b_i and base C_h. To get all the statements it is sufficient to prove that in step (a) we may find C_w with the additional property that $C_w \nsubseteq \cup_{0 \leq h < w} T_w$. Assume that this is false and take $h \in \{0, \dots, w-1\}$ such that $C_w \subset T_h$. Since C_w contains a general point of C_{w-1}, varying C_w we see that $C_{w-1} \subset T_h$. Taking the first w (for some data) for which this occurs we get $h = w-1$. Assume either $w \geq 2$ or that Y is a rational normal curve. With these assumptions T_{w-1} is a degree n cone which is a linear projection from one point of $\mathbb{P}^{n+1}$ of a degree n cone $T \subset \mathbb{P}^{n+1}$ over a rational normal curve of $\mathbb{P}^n$ and the linear projection $\eta : T \to T_{w-1}$ is injective and an isomorphism outside the vertex of T. With the notation of the proof of Lemma 2.3 T is the image of the complete linear system $|\mathcal{O}_{F_n}(h+nf)|$ on the Hirzebruch surface F_n and C_{w-1}, C_w are isomorphic linear projections of two elements $A_1, A_2 \in |\mathcal{O}_{F_n}(h+nf)|$. Since $C_{w-1} \cup C_w$ is nodal and η is injective, we get $\sharp(C_{w-1} \cap C_w) = \deg(A_1 \cap A_2) = \mathcal{O}_{F_n}(h+nf) \cdot \mathcal{O}_{F_n}(h+nf) = n$, a contradiction. Now assume $w = 1$ and $\deg(Y) > n$. We see that T_0 is the cone with base C_1 and vertex b_i. Thus T_0 is an injective linear projection of the degree $n+1$ cone $T \subset \mathbb{P}^{n+1}$ just described. Since $\deg(Y) > n$ and $Y \subset T_0$, we see that Y is an isomorphic linear projection of a curve $D \in |\mathcal{O}_{F_n}(xh+bf)|$ with $x \geq 2$ and $b \geq xb$. We get $\deg(Y \cap C) \geq \mathcal{O}_{F_n}(xh+bf) \cdot \mathcal{O}_{F_n}(h+nf) = b \geq 2n > n+2$, a contradiction.

Applying $d - a - nw$ times Lemma 2.1 we get $h^1(N_G(-A-B)) = 0$, where $G = F \cup L_{b-w+1} \cup \cdots \cup L_b$. By Lemmas 2.1 and 2.2 we may deform G to a smooth curve in a family of curves containing $A \cup B$. Apply the semicontinuity theorem for cohomology.

(c) Assume $w' \neq w$. Thus $w' = w+1$. Take a smooth F as in step (a) (hence $\deg(F) = a + nw$, $p_a(F) = w(n+1)$ and $F \supset A \cup \{b_1, \dots, b_w\}$). First assume $b \leq w$. Increasing if necessary b we may assume $b = w$. Take E as in step (a) containing

$b_1,\dots,b_w$. Set $E'' := F\cup D$, where D is a general rational curve containing a general subset of F with cardinality $g-(n+1)w+1$. As in Claim 1 we get that $\sharp(F\cap D) = g-(n+1)w+1$ and F' is nodal. As in Lemmas 2.1 and 2.2 we see F' is smoothable in a family of curves of $\mathbb{P}^n$ containing $A\cup B$. Now assume $b > w$. By assumption we have $d \ge nw'+a$, $(n+1)w < g < (n+1)w'$ and $b-w' \le d-a-nw'$. Increasing if necessary b we may assume $d = a+b+(n-1)w'$. As in step (a) we take $F\cup C$ with C rational normal curve of $\mathbb{P}^n$, $\sharp(C\cap E) = g-(n+1)w+1$ and $b_{w+1}\in C$. Set $B' := \{b_1,\dots,b_{w+1}\}$. As in step (a) the curve $F\cup C$ is nodal, of degree $a+nw'$, $p_a(F\cup C) = g$, $F\cup C \supset A\cup B'$, $h^1(N_{F\cup C}(-A-B')) = 0$ and $F\cup C$ is smoothable in a family of curves containing $A\cup B'$. Call F' one such smooth curve. Then as in step (b) we add $b-w'$ general lines R_i, $w'+1\le i\le b$ with $b_i\in R_i$ and R_i intersecting F'. We conclude as in step (b) using F' instead of the curve F used in step (b).

(d) Assume $(d,g)\neq(d',g')$. Apply Lemma 2.9 with $S := A\cup B$. □

Proof of Theorem 1.3: Set $w := \lfloor g/2\rfloor$. Write $S = S_0\cup S_1\cdots\cup S_w\cup A$ with $\sharp(S_i) = n$ for $1\le i\le w$, $\sharp(S_0) = n$ if g is even, $\sharp(S_0) = n+1$ if g is odd, $S_i\cap S_j = \emptyset$ for all $i\neq j$, and $\sharp(A) = d-nw-\sharp(S_0)$.

(a) Assume g is even and $d = n+ng/2$. Let $C_0\subset\mathbb{P}^n$ be a rational normal curve such that $C_0\cap H = S_0$ (it exists, because any two subsets of H with cardinality n spanning H are projectively equivalent). For $1\le i\le w$ let C_i be a general rational normal curve of $\mathbb{P}^n$ containing S_i and with $\sharp(C_i\cap C_{i-1}) = 3$. As in the proof of Theorem 1.2 we see that $E := \cup_{i=0}^{w} C_i$ is a connected nodal curve of degree d and genus g

(b) Assume g is odd and $d = n+1+n(g-1)/2$. In this case we start taking as C_0 a smooth and linearly normal elliptic curve such that $C_0\cap H = S_0$ (it exists, because any two subsets of H with cardinality $n+1$ in linear general position are projectively equivalent). Then we continue as in the proof of Theorem 1.2.

(c) Assume either g is even and $d > n+ng/2$ or g is odd and $d > n+1+n(g-1)/2$, i.e. assume $A\neq\emptyset$. Take $E := C_0\cup C_1\cup\cdots\cup C_w$ as in step (a) or as in step (b), respectively. Order the points $p_1,\dots,p_z$, $z = \sharp(A)$, of A. As in Claim 2 in the proof of Theorem 1.2 take the union of E and z lines $L_1,\dots,L_z$ with L_i a general line containing p_i and intersecting C_w.

Note that if $X\cap H = S$ we have $N_X(-S) = N_X(-1)$. Hence the proof of Theorem 1.4 (or [4, Theorem 1]) gives $h^1(N_X(-S)) = 0$. □

Remark 3.1. For all $n\ge 8$, $g\ge 0$ even (resp. $g > 0$ odd) and $d\ge (n-3)g/2+n+3$ (resp. $d\ge (g-1)/2+n+4$) [3, Theorem 1] (resp. Theorem 1.4) there is a smooth, connected and non-degenerate curve $X\subset\mathbb{P}^n$ with degree d, genus g and $h^1(N_X(-1)) = 0$. The lower bounds for d arising in these theorems just come from their proofs (a game with linearly normal elliptic curves and rational normal curves possibly in lower dimensional linear subspaces).

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Edoardo Ballico
University of Trento
Department of Mathematics
38123 Trento, Italy
The author is a member of
GNSAGA of INdAM (Italy)
e-mail: *edoardo.ballico@unitn.it*