SARAJEVO JOURNAL OF MATHEMATICS
Vol.17 (30), No.2 (2021), 167 – 173
DOI: 10.5644/SJM.17.02.04

CENTRALIZERS OF SEMIPRIME INVERSE SEMIRINGS

SONIA DOG, D. MARY FLORENCE, R. MURUGESAN AND P. NAMASIVAYAM

ABSTRACT. Let S be a 2-torsion free semiprime inverse semiring such that all elements of the form $x+x'$ are in the center of S. We prove that any additive mapping $F\colon S\to S$ satisfying the condition $2F(xyx)+F(x)yx'+x'yF(x)=0$ is a centralizer.

1. INTRODUCTION

By a *semiring* $(S,+,\cdot)$ we mean a nonempty set S with two binary operations $+$ and $\cdot$ (called addition and multiplication) such that the multiplication is distributive with respect to the addition, $(S,+)$ is a semigroup with neutral element 0, and $(S,\cdot)$ is a semigroup with zero 0, i.e., $0a=a0=0$ for all $a\in S$. If a semigroup $(S,\cdot)$ is commutative, then we say that a semiring S is commutative. A semiring S is *semiprime* if $xSx=0$ implies $x=0$. It is 2-*torsion free* if $2x=0$ is possible only for $x=0$.

A semiring S is *additively inverse* (shortly: *inverse*), if for every $a\in S$ there exists a uniquely determined element $a'\in S$ such that

$$a+a'+a=a \quad \text{and} \quad a'+a+a'=a'. \tag{1.1}$$

Then, according to [7], for all $a,b\in S$ we have

$$(ab)'=a'b=ab',\ \ (a+b)'=b'+a',\ \ a'b'=ab,\ \ (a')'=a,\ \ 0'=0. \tag{1.2}$$

Moreover, the following implication is valid

$$a+b=0 \quad \textit{implies} \quad b=a' \ \ \textit{and} \ \ a+a'=0. \tag{1.3}$$

An inverse semiring S with commutative addition such that all elements of the form $x+x'$ are in the center $Z(S)$ of S is called an *MA-semiring*. *MA*-semirings were introduced in [8] and studied in various directions by many authors (see for example [1], [2], [3], [4], [9], [10] and [11]).

2010 *Mathematics Subject Classification.* 16Y60, 16W25.
Key words and phrases. Inverse semirings; MA-semirings; additive mappings; semiprime; 2-torsion free semirings.
The second author was supported by the found no. 12039 of Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.

Nowadays, various types of semirings have many natural applications to the theory of automata, formal languages, optimization theory, differential geometry, and other branches of applied mathematics (for details see [5] or [6]). A crucial role in the investigation of semirings is played by the *commutator* $[x,y]=xy+y'x$ (defined in inverse semirings) and various types of additive mappings $d\colon S\to S$ (i.e., endomorphisms of the additive semigroup of S). One such mapping is the *left centralizer* of S defined as an additive mapping $F\colon S\to S$ such that $F(xy)=F(x)y$ for all $x,y\in S$. A *right centralizer* is defined as an additive mapping F with the property $F(xy)=xF(y)$ for all $x,y\in S$. A mapping that is a left and right centralizer is called a *centralizer*. These mappings, sometimes under the name multipliers, are often studied in nonunital rings of functional analysis, which are usually constructed from locally compact groups.

Properties of centralizers in semirings are described in [10] and [11]. In [12] conditions under which an additive mapping is a centralizer are studied. Centralizers, and other additive mappings play an important role in rings since in some cases they enforce the commutativity of rings or coincide with other important mappings. Therefore, finding simple conditions under which additive mappings are centralizers is a useful activity for those who study rings and for those who study semirings.

Results obtained for semirings are in some sense similar to the results proved for rings, but the proofs are not similar. Methods that work for rings are not good for semirings. Therefore, the results proved for semirings are an essential generalization of the results proved for rings.

Motivated by the fact that in a 2-torsion free semiprime ring R any additive mapping F such that $2F(xyx)=F(x)yx+xyF(x)$ holds for all $x,y\in R$ is a centralizer (cf. [13]) we prove an analogous results for 2-torsion free semiprime MA-semirings.

2. Our result

We start with simple lemmas that will be used in the proof of our main theorem.

Lemma 2.1. *In any inverse semiring*

(i) $[x,y]=xy+(yx)'=xy+yx'$,

(ii) $[x,y]'=[x,y']=[x',y]=[y,x]$,

(iii) $[x',y']=[x,y]$,

(iv) $[x,yx]=[x,y]x$,

(v) $[x,y]=0$ *implies* $xy=yx$.

The proofs are available in [8].

Lemma 2.2. *Let S be a semiprime inverse semiring and $a,b,c\in S$ such that $axb+bxc=0$ holds for all $x\in S$. Then $(a+c)xb=0$ for all $x\in S$.*

Lemma 2.2 is proved in [12] for MA-semirings, but it is also valid for inverse semirings. The proof is the same as in [12].

Lemma 2.3. *In any MA-semiring the following identities*

$$[xy,z] = x[y,z] + [x,z]y \quad \text{and} \quad [x,yz] = y[x,z] + [x,y]z, \tag{2.1}$$

$$[x,z](x^2 + (x^2)') = [x,z]x(x + x') \tag{2.2}$$

are valid.

Proof. The Jacobi identities (2.1) are proved in [8]. The proof of (2.2) is straightforward. □

Lemma 2.4. *In a* 2*-torsion free semiprime MA-semiring any additive mapping satisfying the identity* $F(x^2) + F(x)x' = 0$ *is a left centralizer.*

Proof. See the proof of Theorem 2.1 in [12]. □

In a similar way we can prove

Lemma 2.5. *In a* 2*-torsion free semiprime MA-semiring any additive mapping satisfying the identity* $F(x^2) + x'F(x) = 0$ *is a right centralizer.*

Theorem 2.1. *If an additive mapping* F *defined on a* 2*-torsion free semiprime MA-semiring* S *satisfies*

$$2F(xyx) + F(x)yx' + x'yF(x) = 0 \tag{2.3}$$

for all $x, y \in S$*, then it is a centralizer.*

Proof. The proof will be divided into three parts. First we will show that $2F(x^2) + F(x)x' + x'F(x) = 0$, then that $[F(x), x] = 0$ and finally we use these facts to prove that F is a centralizer.

1. By putting $x = x + z$ in (2.3) we obtain

$$2F(xyz + zyx) + F(x)yz' + F(z)yx' + x'yF(z) + z'yF(x) = 0, \tag{2.4}$$

which for $z = x^2$ gives

$$2F(xyx^2 + x^2yx) + F(x)yxx' + xx'yF(x) + F(x^2)yx' + x'yF(x^2) = 0. \tag{2.5}$$

Replacing y with $xy + yx$ in (2.3), we get

$$2F(x^2yx + xyx^2) + F(x)xyx' + F(x)yxx' + xx'yF(x) + x'yxF(x) = 0. \tag{2.6}$$

From (2.5), after application of (1.3), we deduce

$$2F(xyx^2 + x^2yx) + F(x)yxx' + x'xyF(x) = F(x^2)yx + xyF(x^2).$$

which together with (2.6) implies

$$(F(x^2) + F(x)x')yx + xy(F(x^2) + x'F(x)) = 0.$$

Taking $a = F(x^2) + F(x)x'$, $x = y$, $b = x$, $c = F(x^2) + x'F(x)$ and using Lemma 2.2, we get

$$\left(2F(x^2) + F(x)x' + x'F(x)\right)yx = 0.$$

This can be written in more useful form as $H(x)yx = 0$, where

$$H(x) = 2F(x^2) + F(x)x' + x'F(x).$$

Thus $xH(x)yxH(x) = 0$, which implies $xH(x) = 0$. In a similar way, putting $y = xyH(x)$ in $H(x)yx = 0$, we obtain $H(x)x = 0$. Hence $H(x+y)(x+y) = 0$. Therefore,

$$H(x)y + G(x,y)x + H(y)x + G(x,y)y = 0, \tag{2.7}$$

where

$$G(x,y) = 2F(xy+yx) + F(x)y' + F(y)x' + x'F(y) + y'F(x) = H(x+y).$$

Applying (1.3) to (2.7), we obtain

$$H(x)y + G(x,y)x = H(y)x' + G(x,y)y' = \big(H(y)x + G(x,y)y\big)'$$

because $H(x') = H(x)$ and $G(x',y) = G(x,y)'$. Therefore,

$$2\big(H(x)y + G(x,y)x\big) = \big(H(x)y + G(x,y)x\big) + \big(H(y)x + G(x,y)y\big)' = 0.$$

Since S is a 2-torsion free, $H(x)y + G(x,y)x = 0$ must hold, whence, after multiplication on the right by $H(x)$, we get $H(x)yH(x) = 0$. This implies that $H(x) = 0$ for all $x \in S$. Consequently, $G(x,y) = H(x+y) = 0$.

2. Now our intention is to prove that $[F(x),x] = 0$, i.e., $F(x)x' = x'F(x)$. Since $G(x,y) = H(x+y) = 0$, for $y = 2xyx$ we have

$$2F(2x^2yx + 2xyx^2) + 2F(x)xyx' + 2F(xyx)x' + x'2F(xyx) + 2xyx'F(x) = 0.$$

But $2F(xyx) = F(x)yx + xyF(x)$, by (2.3) and (1.3). So, the last expression, after application of (2.5), can be transformed to

$$2F(x^2yx + xyx^2) + F(x)xyx' + x'yxF(x) + xyF(x)x' + x'F(x)yx = 0. \tag{2.8}$$

From (2.3) and (1.3) we can also deduce that $2F(xyx) + F(x)yx' = xyF(x)$ and $2F(xyx) + x'yF(x) = F(x)yx$. So, (2.8) can be reduced to

$$F(x)yx^2 + x^2yF(x) + xyF(x)x' + x'F(x)yx = 0. \tag{2.9}$$

Thus for $y = yx$ we have

$$F(x)yx^3 + x^2yxF(x) + xyxF(x)x' + x'F(x)yx^2 = 0.$$

This, by (1.3),

$$F(x)yx^3 + x'F(x)yx^2 = x^2yx'F(x) + xyxF(x)x.$$

Now, multiplying (2.9) on the right side by x and using the last expression, we obtain

$$x^2y[F(x),x] + xy[F(x),x]x' = 0. \tag{2.10}$$

From this, by (1.3), we have

$$x^2y[F(x),x] = xy[F(x),x]x. \tag{2.11}$$

Putting $y = F(x)y$ in (2.10) we obtain

$$x^2F(x)y[F(x),x] + xF(x)y[F(x),x]x' = 0.$$

Multiplying (2.10) on the left by $F(x)$, we get

$$F(x)x^2y[F(x),x] + F(x)xy[F(x),x]x' = 0. \tag{2.12}$$

Adding these two expressions, we obtain

$$\begin{aligned}
0 &= (x^2F(x)y[F(x),x] + xF(x)y[F(x),x]x')' + F(x)x^2y[F(x),x] + F(x)xy[F(x),x]x' \\
&= [F(x),x^2]y[F(x),x] + [x,F(x)]y[F(x),x]x \\
&= x[F(x),x]y[F(x),x] + [F(x),x]xy[F(x),x] + [x,F(x)]y[F(x),x]x \\
&\overset{(2.12)}{=} x'F(x)xy[F(x),x] + x[F(x),x]y[F(x),x] + xF(x)y[F(x),x]x \\
&\overset{(2.11)}{=} x'F(x)xy[F(x),x] + x(F(x)x + x'F(x))y[F(x),x] + xF(x)xy[F(x),x] \\
&= (x + x' + x)F(x)xy[F(x),x] + xx'F(x)y[F(x),x] \\
&= xF(x)xy[F(x),x] + xx'F(x)y[F(x),x] \\
&= x(F(x)x + x'F(x))y[F(x),x] \\
&= x[F(x),x]y[F(x),x].
\end{aligned}$$

This means that $x[F(x),x] = 0$.

Replacing y by xy in (2.9), we get

$$F(x)xyx^2 + x^3yF(x) + x^2yF(x)x' + x'F(x)xyx = 0,$$

which by (1.3) gives

$$x^3yF(x) + x^2yF(x)x' = F(x)'xyx^2 + xF(x)xyx.$$

Now, multiplying (2.9) on the left by x and using the last expression, we get

$$xF(x)yx^2 + f(x)'xyx^2 + xF(x)xyx + xx'F(x)yx = 0,$$

that can be written as

$$[x,F(x)]yx^2 + x[F(x),x]yx = 0.$$

Since $x[F(x),x] = 0$ and $[x,y]' = [y,x]$, from the last expression it follows that $[F(x),x]'yx^2 = 0$, i.e. $[F(x),x]y(x^2)' = 0$. Thus, by (1.2),

$$[F(x),x]yx^2 = 0.$$

Replacing y by $yF(x)$ and adding $[F(x),x]y(x^2)'F(x) = 0$, we obtain

$$[F(x),x]yF(x)x^2 + [F(x),x]y(x^2)'F(x) = 0,$$

which is equivalent to $[F(x),x]y[F(x),x^2] = 0$. This, by (2.1), gives

$$[F(x),x]y([F(x),x]x + x[F(x),x]) = 0.$$

But $x[F(x),x]=0$, so $[F(x),x]y[F(x),x]x=0$. From this, replacing y by xy, we get $[F(x),x]xy[F(x),x]x=0$, whence applying the semiprimeness of S, we deduce $[F(x),x]x=0$.

Now, replacing x by $x+y$ in $x[F(x),x]=0$ and using fact that $z[F(z),z]=0$ for all $z\in S$, we obtain

$$x[F(x),y]+x[F(y),x]+x[F(y),y]+y[F(x),x]+y[F(x),y]+y[F(y),x]=0.$$

Since $F(x')=F(x)'$, the above, in view of Lemma 2.1, for $x=x'$ has the form

$$x[F(x),y]+x[F(y),x]+x'[F(y),y]+y[F(x),x]+y[F(x'),y]+y[F(y),x']=0.$$

Moreover, in view of (1.3) and Lemma 2.1, from the previous identity we obtain

$$x[F(x),y]+x[F(y),x]+y[F(x),x]=x'[F(y),y]+y[F(x'),y]+y[F(y),x'],$$

which together with the last identity gives

$$2(x[F(x),y]+x[F(y),x]+y[F(x),x])=0.$$

As S is 2-torsion free, the last implies

$$x[F(x),y]+x[F(y),x]+y[F(x),x]=0$$

whence, multiplying on the left by $[F(x),x]$, we obtain

$$[F(x),x]x[F(x),y]+[F(x),x]x[F(y),x]+[F(x),x]y[F(x),x]=0,$$

i.e., $[F(x),x]y[F(x),x]=0$ because $[F(x),x]x=0$. This gives $[F(x),x]=0$.

3. From $[F(x),x]=0$ it follows that $F(x)x'=x'F(x)$. Thus $0=H(x)=2F(x^2)+2F(x)x'$, and consequently,

$$F(x^2)+F(x)x'=F(x^2)+x'F(x)=0$$

because a semiring S is 2-torsion free.

Lemmas 2.4 and 2.5 complete the proof. □

REFERENCES

[1] Y. Ahmed, W.A. Dudek, Stronger Lie derivation on *MA*-semirings, Afrika Math., **31** (2020), 891 – 901.

[2] Y. Ahmed, W.A. Dudek, Generalized multiplicative derivations in inverse semirings, Ufa Math. J., **13** (2021), 110 – 118.

[3] Y. Ahmed, W.A. Dudek, M. Aslam, Additive mappings on *MA*-semirings, Asian-European J. Math., **14** (2021), 2150067.

[4] Y. Ahmed, W.A. Dudek, M. Aslam, Symmetric bi-derivations on Jacobi semirings, C. R. Acad. Bulare Sci., **74** (2021), 324 – 331.

[5] K. Glazek, A Guide to Literature on Semirings and their Applications in Mathematics and Information Sciences with Complete Bibliography, Kluwer Acad. Publ., Dodrecht, 2002.

[6] J.S Golan, The theory of semirings with applications in mathematics and theoretical computer science, John Wiley and sons, Inc., New York, 1992.

[7] P.H. Karvellas, Inversive semirings, J. Austral. Math. Soc. **18** (1974), 277 – 288.

[8] M.A. Javed, M. Aslam, M. Hussain, On condition (A_2) of Bandelt and Petrich for inverse semirings, Int. Math. Forum, **7** (2012), 2903 – 2914.
[9] M.A. Javed, M. Aslam, Some commutativity conditions in prime MA-semirings, Ars Combin. **114** (2014), 373 – 384.
[10] M. Nadeem, M. Aslam, On the generalization of Brešar theorems, Quasigroups and Related Systems, **24** (2016), 123 – 128.
[11] S. Shafiq, M. Aslam, Centralizers on semiprime MA-semirings, Quasigroups and Related Systems, **24** (2016), 269 – 276.
[12] S. Shafiq, M. Aslam and M.A. Javed, On centralizer of semiprime inverse semiring, Discussiones Math. General Algebra and Appl. **36** (2016), 71 – 84.
[13] J. Vukman, I. Kosi-ulbl, On centralizers of semiprime rings, Aequationes Math. **66** (2003), 277 – 283.

(Received: May 11, 2020)
(Revised: June 8, 2021)

Sonia Dog
22 Pervomayskaya str
39600 Kremenchuk
Ukraine
e-mail: *soniadog2@gmail.com*
and
D. Mary Florence
Department of Mathematics
Kanyakumari Community College
Mariagiri – 629153, Tamil Nadu
India
e-mail: *dmaryflorence@gmail.com*
and
R. Murugesan
Department of Mathematics
Thiruvalluvar College
Papanasam – 627425, Tamil Nadu
India
e-mail: *rmurugesa2020@yahoo.com*
and
P. Namasivayam
Department of Mathematics
The M.D.T Hindu College
Tirunelveli – 627010, Tamil Nadu
India
e-mail: *vasuhe2010@gmail.com*