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SOME UNIFIED FORM OF OPEN SETS AND CONTINUITY IN IDEAL MINIMAL SPACES

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ABSTRACT. We introduce the notion of umIO(X)-structures determined by operators mInt, mCl, mInt* and mCl* on an ideal minimal space (X, m_X, I). By using umIO(X)-structures, we introduce and investigate a function $f : (X, m_X, I) \to (Y, \sigma)$ called *umI*-continuous. As special cases of *umI*-continuity, we obtain m-semi-I-continuity [11], m-pre-I-continuity [10], m-α-I-continuity [2], m-b-I-continuity [15], and m-β-I-continuity [9].

1. INTRODUCTION

The notion of ideal topological spaces was introduced in [13] and [23]. In [12], the authors obtained the further properties of ideal topological spaces.

The notions of minimal structures and minimal spaces are introduced in [20], as a generalization of topological spaces, and studied the notion of m-continuous functions on these spaces. Recently, Ozbakir and Yildirim [18] introduced the notion of an ideal minimal space. They defined and investigated the m-local functions, the $m^\star$-closure and the $m^\star$-minimal spaces. Quite recently, m-semi-I-open sets [11], m-pre-I-open sets [10], m-α-I-open sets [2], m-b-I-open sets [15], and m-β-I-open sets [9] in an ideal minimal space have been introduced and investigated. And by using these open sets, some kind of continuous functions from an ideal minimal space to a topological space are defined and investigated.

In this paper, we define the unified form of these open sets and the related continuous functions and show that many properties of these open sets and continuous functions are derived from properties of minimal spaces and m-continuous functions established in [20].

2. PRELIMINARIES

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by Cl(A) and Int(A), respectively. We recall several properties of minimal structures and m-continuous functions.

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Definition 2.1. *Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily m_X of $\mathcal{P}(X)$ is called a minimal structure (briefly m-structure) on X* [19], [20] *if $\emptyset \in m_X$ and $X \in m_X$.*

By (X, m_X), we denote a nonempty set X with an m-structure m_X on X and call it an m-space. Each member of m_X is said to be m_X*-open* (briefly m-open) and the complement of an m_X-open set is said to be m_X*-closed* (briefly m-closed).

Definition 2.2. *Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the m_X-closure of A and the m_X-interior of A are defined in* [14] *as follows:*

(1) $mCl(A) = \cap\{F : A \subset F, X \setminus F \in m_X\}$,
(2) $mInt(A) = \cup\{U : U \subset A, U \in m_X\}$.

Lemma 2.1. [14] *Let X be a nonempty set and m_X an m-structure on X. For subsets A and B of X, the following properties hold:*

(1) $\mathrm{mCl}(X \setminus A) = X \setminus \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X \setminus A) = X \setminus \mathrm{mCl}(A)$,
(2) If $(X \setminus A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,
(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,
(4) If $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,
(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,
(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Definition 2.3. *An m-structure m_X on a nonempty set X is said to have property $\mathcal{B}$* [14] *if the union of any family of subsets belonging to m_X belongs to m_X.*

Lemma 2.2. [22] *Let X be a nonempty set and m_X an m-structure on X satisfying property $\mathcal{B}$. For a subset A of X, the following properties hold:*

(1) $A \in m_X$ *if and only if* $\mathrm{mInt}(A) = A$,
(2) A *is m_X-closed if and only if* $\mathrm{mCl}(A) = A$,
(3) $\mathrm{mInt}(A) \in m_X$ *and* $\mathrm{mCl}(A)$ *is m_X-closed.*

Lemma 2.3. [19] *Let (X, m_X) be an m-space and A a subset of X. Then $x \in$* $\mathrm{mCl}(A)$ *if and only if $U \cap A \neq \emptyset$ for each $U \in m_X$ containing x.*

Definition 2.4. *A function $f : (X, m_X) \to (Y, \sigma)$ is said to be m-continuous at $x \in X$* [20], *where (Y, σ) is a topological space, if for each open set V containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$. The function f is said to be m-continuous if it has this property at each $x \in X$.*

Theorem 2.1. [20] *For a function $f : (X, m_X) \to (Y, \sigma)$, where m_X has property $\mathcal{B}$, the following properties are equivalent:*

(1) f is m-continuous;
(2) $f^{-1}(V)$ is m_X-open for every open set V of Y;
(3) $f^{-1}(F)$ is m_X-closed for every closed set F of Y;

(4) $\mathrm{mCl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$ *for every subset* B *of* Y*;*
(5) $f(\mathrm{mCl}(A)) \subset \mathrm{Cl}(f(A))$ *for every subset* A *of* X*;*
(6) $f^{-1}(\mathrm{Int}(B)) \subset \mathrm{mInt}(f^{-1}(B))$ *for every subset* B *of* Y*.*

For a function $f : (X, m_X) \to (Y, \sigma)$, we define $D_m(f)$ as follows:

$$D_m(f) = \{x \in X : f \text{ is not } m\text{-continuous at } x\}.$$

Theorem 2.2. [21] *For a function* $f : (X, m_X) \to (Y, \sigma)$*, where* m_X *has property* $\mathcal{B}$*, the following properties hold:*

$$\begin{aligned}
D_m(f) &= \bigcup_{G \in \sigma} \{f^{-1}(G) \setminus \mathrm{mInt}(f^{-1}(G))\} \\
&= \bigcup_{B \in \mathcal{P}(Y)} \{f^{-1}(\mathrm{Int}(B)) \setminus \mathrm{mInt}(f^{-1}(B))\} \\
&= \bigcup_{B \in \mathcal{P}(Y)} \{\mathrm{mCl}(f^{-1}(B)) \setminus f^{-1}(\mathrm{Cl}(B))\} \\
&= \bigcup_{A \in \mathcal{P}(X)} \{\mathrm{mCl}(A) \setminus f^{-1}(\mathrm{Cl}(f(A)))\} \\
&= \bigcup_{F \in \mathcal{F}} \{\mathrm{mCl}(f^{-1}(F)) \setminus f^{-1}(F)\},
\end{aligned}$$

where $\mathcal{F}$ *is the family of closed sets of* (Y, σ)*.*

3. Ideal topological spaces

Definition 3.1. *A nonempty collection* I *of subsets of a set* X *is called an ideal on* X [12] *if it satisfies the following two conditions:*

(1) $A \in I$ *and* $B \subset A$ *implies* $B \in I$*,*
(2) $A \in I$ *and* $B \in I$ *implies* $A \cup B \in I$*.*

An m-space (X, m_X) with an ideal I on X is called an ideal m-space and is denoted by (X, m_X, I).

Definition 3.2. *Let* (X, m_X, I) *be an ideal m-space. For any subset* A *of* X*,* $A^\star_m(I, m_X) = \{x \in X : U \cap A \notin I \text{ for every } U \in m(x)\}$*, where* $m(x) = \{U \in m_X : x \in U\}$*, is called the minimal local function of* A *with respect to* m_X *and* I [18]*. Hereafter* $A^\star_m(I, m_X)$ *is simply denoted by* $A^\star_m$*.*

Definition 3.3. *Let* (X, m_X, I) *be an ideal m-space. The set operator* $mCl^\star$ *called the minimal* $\star$*-closure* [18] *is defined as follows:* $mCl^\star(A) = A \cup A^\star_m$ *for every subset* A *of* X*. Let* $m^\star_X = \{U \subset X : \mathrm{mCl}^\star(X \setminus U) = X \setminus U\}$*. Then* $m^\star_X$ *is an m-structure on* X *which is finer than* m_X *and is called the* $m^\star$*-structure. The* $m^\star$*-interior is defined as follows:* $mInt^\star(A) = X \setminus \mathrm{mCl}^\star(X \setminus A)$ *for every subset* A *of* X*.*

Lemma 3.1. [18] *Let* (X, m_X, I) *be an ideal m-space and A, B be subsets of* X*. Then the following properties hold:*

(1) $\mathrm{mInt}^\star(A) \subset A \subset \mathrm{mCl}^\star(A)$,
(2) $\mathrm{mCl}^\star(X) = \mathrm{mInt}^\star(X) = X$ *and* $\mathrm{mInt}^\star(\emptyset) = \mathrm{mCl}^\star(\emptyset) = \emptyset$,
(3) $A \subset B$ *implies* $\mathrm{mCl}^\star(A) \subset \mathrm{mCl}^\star(B)$ *and* $\mathrm{mInt}^\star(A) \subset \mathrm{mInt}^\star(B)$,
(4) $\mathrm{mCl}^\star(A) \cup \mathrm{mCl}^\star(B) \subset \mathrm{mCl}^\star(A \cup B)$.

Definition 3.4. *Let (X, m_X, I) be an ideal m-space. A subset A of X is said to be*

(1) m-α-I-open [2] if $A \subset \mathrm{mInt}(\mathrm{mCl}^\star(\mathrm{mInt}(A)))$,
(2) m-semi-I-open [11] if $A \subset \mathrm{mCl}^\star(\mathrm{mInt}(A))$,
(3) m-pre-I-open [10] if $A \subset \mathrm{mInt}(\mathrm{mCl}^\star(A))$,
(4) m-b-I-open [15] if $A \subset \mathrm{mInt}(\mathrm{mCl}^\star(A)) \cup \mathrm{mCl}^\star(\mathrm{mInt}(A))$,
(5) m-β-I-open [9] if $A \subset \mathrm{mCl}(\mathrm{mInt}(\mathrm{mCl}^\star(A)))$,
(6) weakly m-semi-I-open if $A \subset \mathrm{mCl}^\star(\mathrm{mInt}(\mathrm{mCl}(A)))$,
(7) weakly m-b-I-open if $A \subset \mathrm{mCl}(\mathrm{mInt}(\mathrm{mCl}^\star(A))) \cup \mathrm{mCl}^\star(\mathrm{mInt}(\mathrm{mCl}(A)))$,
(8) strongly m-β-I-open if $A \subset \mathrm{mCl}^\star(\mathrm{mInt}(\mathrm{mCl}^\star(A)))$,
(9) m-semi⋆-I-open if $A \subset \mathrm{mCl}(\mathrm{mInt}^\star(A))$,
(10) m-β⋆-I-open if $A \subset \mathrm{mCl}(\mathrm{mInt}^\star(\mathrm{mCl}(A)))$,
(11) m-pre⋆-I-open if $A \subset \mathrm{mInt}^\star(\mathrm{mCl}(A))$.

If $(X, m_X, I) = (X, \tau, I)$, where τ is a topology, then by Definition 3.4 we obtain the definitions of α-*I*-open [8], semi-*I*-open [8], pre-*I*-open [3], *b*-*I*-open [16], β-*I*-open [8], weakly semi-*I*-open [6], weakly *b*-*I*-open [16], strongly β-*I*-open [7], semi⋆-*I*-open [5], β⋆-*I*-open [4], and pre⋆-*I*-open [4].

The family of all *m*-α-*I*-open (resp. *m*-semi-*I*-open, *m*-pre-*I*-open, *m*-*b*-*I*-open, *m*-β-*I*-open, weakly *m*-semi-*I*-open, weakly *m*-*b*-*I*-open, strongly *m*-β-*I*-open, *m*-semi⋆-*I*-open, *m*-β⋆-*I*-open, *m*-pre⋆-*I*-open) sets in an ideal minimal space (X, m_X, I) is denoted by mαIO(X) (resp. mSIO(X), mPIO(X), mBIO(X), mβIO(X), WmSIO(X), WmBIO(X), SmβIO(X), mS⋆IO(X), mβ⋆IO(X), mP⋆IO(X)).

Definition 3.5. *By umIO(X), we denote each one of the families mαIO(X), mSIO(X), mPIO(X), mBIO(X), mβIO(X), WmSIO(X), WmBIO(X), SmβIO(X)), mS⋆IO(X), mβ⋆IO(X), mP⋆IO(X).*

Lemma 3.2. *Let (X, m_X, I) be an ideal m-space. Then* umIO*(X) is an m-structure on X and has property $\mathcal{B}$.*

Proof. By Lemmas 2.1(3) and 3.1(2), umIO(X) is an *m*-structure on X. It follows from Lemmas 2.1(4) and 3.1(3) that umIO(X) has property $\mathcal{B}$. □

Remark 3.1. It is shown in Theorem 2.17 of [2] (resp. Theorem 3.12 of [11], Theorem 3.7 of [10], Proposition 3.12 of [15], Theorem 3.11 of [9]) that mαIO(X) (resp. mSIO(X), mPIO(X), mBIO(X), mβIO(X)) has property $\mathcal{B}$.

Definition 3.6. *Let (X, m_X, I) be an ideal m-space. For a subset A of X,* $\mathrm{umCl_I}(A)$ *and* $\mathrm{umInt_I}(A)$ *are defined as follows:*

(1) $\mathrm{umCl_I}(A) = \cap\{F : A \subset F, X \setminus F \in \mathrm{umIO}(X)\}$,

(2) $\mathrm{umInt_I}(A) = \cup\{U : U \subset A, U \in \mathrm{umIO}(X)\}$.

Lemma 3.3. *Let (X, m_X, I) be an ideal m-space and A, B subsets of X. Then the following properties hold:*

(1) $A \in \mathrm{umIO}(X)$ *if and only if* $\mathrm{umInt_I}(A) = A$,
(2) $\mathrm{umInt_I}(\emptyset) = \emptyset$ *and* $\mathrm{umInt_I}(X) = X$,
(3) If $A \subset B$, then $\mathrm{umInt_I}(A) \subset \mathrm{umInt_I}(B)$,
(4) $\mathrm{umInt_I}(A) \subset A$,
(5) $\mathrm{mInt_I}(\mathrm{mInt_I}(A)) = \mathrm{mInt_I}(A)$,
(6) $x \in \mathrm{umInt_I}(A)$ *if and only if there exists* $U \in \mathrm{umIO}(X)$ *such that* $x \in U \subset A$.

Proof. Since $\mathrm{umIO}(X)$ is an m-structure with property $\mathcal{B}$, this follows easily from Lemmas 2.1, 2.2 and 3.1. □

Remark 3.2. By Lemma 3.3, we obtain Theorem 2.23 of [2], Theorems 3.21 of [11], Theorem 3.13 of [10] and Theorem 3.20 of [9].

Lemma 3.4. *Let (X, m_X, I) be an ideal m-space and A, B subsets of X. Then the following properties hold:*

(1) $(X \setminus A) \in \mathrm{umIO}(X)$ *if and only if* $\mathrm{umCl_I}(A) = A$,
(2) $\mathrm{umCl_I}(\emptyset) = \emptyset$, $\mathrm{umCl_I}(X) = X$,
(3) If $A \subset B$, then $\mathrm{umCl_I}(A) \subset \mathrm{umCl_I}(B)$,
(4) $A \subset \mathrm{umCl_I}(A)$,
(5) $\mathrm{mCl_I}(\mathrm{mCl_I}(A)) = \mathrm{mCl_I}(A)$,
(6) $x \in \mathrm{umCl_I}(A)$ *if and only if* $A \cap U \neq \emptyset$ *for every* $U \in \mathrm{umIO}(X)$ *containing x,*
(7) $\mathrm{umCl_I}(X \setminus A) = X \setminus \mathrm{umInt_I}(A)$ *and* $\mathrm{umInt_I}(X \setminus A) = X \setminus \mathrm{umCl_I}(A)$.

Proof. This follows easily from Lemmas 2.1, 2.2, 2.3 and 3.1. □

Remark 3.3. By Lemma 3.4, we obtain Theorems 2.26, 2.28 and 2.30 of [2], Theorems 3.23, 3.24 and 3.25 of [11], Theorems 3.15, 3.16 and 3.17 of [10] and Theorems 3.22, 3.23 and 3.24 of [9].

4. *umI*-CONTINUOUS FUNCTIONS

Definition 4.1. *A function $f : (X, m_X, I) \to (Y, \sigma)$ is said to be m-α-I-continuous* [2] *(resp. m-semi-I-continuous* [11]*, m-pre-I-continuous* [10]*, m-b-I-continuous* [15]*, m-β-I-continuous [9], weakly m-semi-I-continuous, weakly m-b-I-continuous, strongly m-β-I-continuous, m-semi*-I-continuous, m-$\beta^\star$-I-continuous, m-pre*-I-continuous) if $f^{-1}(V)$ is m-α-I-open (resp. m-semi-I-open, m-pre-I-open, m-b-I-open, m-β-I-open, weakly m-semi-I-open, weakly m-b-I-open, strongly m-β-I-open, m-semi*-I-open, m-$\beta^\star$-I-open, m-pre*-I-open) in (X, m_X, I) for every open set V of (Y, σ).*

Definition 4.2. *A function $f : (X, m_X, I) \to (Y, \sigma)$ is said to be umI-continuous if $f : (X, \mathrm{umIO}(X)) \to (Y, \sigma)$ is m-continuous.*

By Theorem 2.1, we obtain the following characterizations of *umI*-continuous functions.

Theorem 4.1. *For a function* $f : (X, m_X, I) \to (Y, \sigma)$, *the following properties are equivalent:*

(1) f *is umI-continuous;*
(2) $f^{-1}(V)$ *is umI-open in* X *for every open set* V *of* Y*;*
(3) $f^{-1}(F)$ *is umI-closed in* X *for every closed set* F *of* Y*;*
(4) $\mathrm{umCl_I}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$ *for every subset* B *of* Y*;*
(5) $f(\mathrm{umCl_I}(A)) \subset \mathrm{Cl}(f(A))$ *for every subset* A *of* X*;*
(6) $f^{-1}(\mathrm{Int}(B)) \subset \mathrm{umInt_I}(f^{-1}(B))$ *for every subset* B *of* Y*.*

Remark 4.1. If umIO(X) is mαIO(X) (resp. mSIO(X), mPIO(X), mBIO(X), mβIO(X)), then by Theorem 4.1 we obtain Theorem 3.4 of [2] (resp. Theorem 4.3 of [11], Theorem 4.5 of [10], Propositions 4.3 and 4.4 of [15], Theorem 4.4 of [9]). We note that thre are some small mistakes in Proposition 4.4 of [15].

For a function $f : (X, m_X, I) \to (Y, \sigma)$, we define $D_{umI}(f)$ as follows:

$$D_{umI}(f) = \{x \in X : f \text{ is not } umI\text{-continuous at } x\}.$$

By Theorem 2.2, we obtain the following properties:

Theorem 4.2. *For a function* $f : (X, m_X, I) \to (Y, \sigma)$, *the following properties hold:*

$$\begin{aligned}
D_{umI}(f) &= \bigcup_{G\in\sigma} \{f^{-1}(G) \setminus \mathrm{umInt_I}(f^{-1}(G))\} \\
&= \bigcup_{B\in\mathcal{P}(Y)} \{f^{-1}(\mathrm{Int}(B)) \setminus \mathrm{umInt_I}(f^{-1}(B))\} \\
&= \bigcup_{B\in\mathcal{P}(Y)} \{\mathrm{umCl_I}(f^{-1}(B)) \setminus f^{-1}(\mathrm{Cl}(B))\} \\
&= \bigcup_{A\in\mathcal{P}(X)} \{\mathrm{umCl_I}(A) \setminus f^{-1}(\mathrm{Cl}(f(A)))\} \\
&= \bigcup_{F\in\mathcal{F}} \{\mathrm{umCl_I}(f^{-1}(F)) \setminus f^{-1}(F)\},
\end{aligned}$$

where $\mathcal{F}$ *is the family of closed sets of* (Y, σ).

5. Some properties of *umI*-continuity

Since the study of *umI*-continuity is reduced to the study of *m*-continuity, the properties of *umI*-continuous functions follow from the properties of *m*-continuous functions in [20].

Definition 5.1. *An m-space* (X, m_X) *is said to be m-*T_2 [20] *if for each distinct points* $x, y \in X$, *there exist* $U, V \in m_X$ *containing* x *and* y, *respectively, such that* $U \cap V = \emptyset$.

Definition 5.2. *An ideal m-space (X, m_X, I) is said to be umI-T_2 if the m-space $(X, \mathrm{umIO}(X))$ is m-T_2.*

Hence, an ideal m-space (X, m_X, I) is *umI*-T_2 if for each distinct points $x, y \in X$, there exist $U, V \in \mathrm{umIO}(X)$ containing x and y, respectively, such that $U \cap V = \emptyset$.

Remark 5.1. Let (X, m_X, I) be an ideal m-space. If umIO(X) = mαIO(X) (resp. mSIO(X), mPIO(X), mBIO(X)), then by Definition 5.2 we obtain the definition of m-α-I-T_2 [2] (resp. m-semi-I-T_2 [11], m-pre-I-T_2 [10], bm-T_2 [15]).

Lemma 5.1. [20] *If $f : (X, m_X) \to (Y, \sigma)$ is an m-continuous injection and (Y, σ) is T_2, then (X, m_X) is m-T_2.*

Theorem 5.1. *If $f : (X, m_X, I) \to (Y, \sigma)$ is a umI-continuous injection and (Y, σ) is T_2, then X is umI-T_2.*

Proof. The proof follows from Definition 5.2 and Lemma 5.1. □

Remark 5.2. Let (X, m_X, I) be an ideal m-space. If umIO(X) = mαIO(X) (resp. mSIO(X), mPIO(X)), then by Theorem 5.1 we obtain Theorem 3.14 of [2] (resp. Theorem 4.11 of [11], Theorem 4.11 of [10]).

Definition 5.3. *A function $f : (X, m_X) \to (Y, \sigma)$ is said to have a strongly m-closed graph (resp. m-closed graph) [20] if for each $(x, y) \in (X \times Y) \setminus \mathrm{G}(f)$, there exist $U \in m_X$ containing x and $V \in \sigma$ containing y such that $[U \times \mathrm{Cl}(V)] \cap \mathrm{G}(f) = \emptyset$ (resp. $[U \times V] \cap \mathrm{G}(f) = \emptyset$).*

Definition 5.4. *A function $f : (X, m_X, I) \to (Y, \sigma)$ is said to have a strongly umI-closed graph (resp. umI-closed graph) if a function $f : (X, \mathrm{umIO}(X)) \to (Y, \sigma)$ has a strongly m-closed graph (resp. m-closed graph).*

Lemma 5.2. [20] *If $f : (X, m_X) \to (Y, \sigma)$ is an m-continuous function and (Y, σ) is T_2, then f has a strongly m-closed graph.*

Theorem 5.2. *If $f : (X, m_X, I) \to (Y, \sigma)$ is a umI-continuous function and (Y, σ) is T_2, then f has a strongly umI-closed graph.*

Proof. The proof follows from Definition 5.4 and Lemma 5.2.

Corollary 5.1. *If $f : (X, m_X, I) \to (Y, \sigma)$ is a umI-continuous function and (Y, σ) is T_2, then f has a umI-closed graph.*

Proof. The proof is obvious by Theorem 5.2. □

Remark 5.3. Let $f : (X, m_X, I) \to (Y, \sigma)$ be a *umI*-continuous function and (Y, σ) be T_2. If umIO(X) = mαIO(X) (resp. mSIO(X), mPIO(X)), then by Corollary 5.1 we obtain Theorem 3.10 of [2] (resp. Theorem 4.9 of [11], Theorem 4.8 of [10]).

Lemma 5.3. [20] *If $f : (X, m_X) \to (Y, \sigma)$ is a surjective function with a strongly m-closed graph, then (Y, σ) is T_2.*

Theorem 5.3. *If $f : (X, m_X, I) \to (Y, \sigma)$ is a surjective function with a strongly umI-closed graph, then (Y, σ) is T_2.*

Proof. The proof follows from Definition 5.4 and Lemma 5.3. □

Lemma 5.4. [20] *Let (X, m_X) be an m-space and m_X have property $\mathcal{B}$. If $f : (X, m_X) \to (Y, \sigma)$ is an injective m-continuous function with an m-closed graph, then (X, m_X) is m-T_2.*

Theorem 5.4. *If $f : (X, m_X, I) \to (Y, \sigma)$ is an injective umI-continuous function with a umI-closed graph, then X is umI-T_2.*

Proof. The proof follows from Definition 5.4, Lemma 5.4 and the fact that umIO(X) has property $\mathcal{B}$. □

Remark 5.4. Let $f : (X, m_X, I) \to (Y, \sigma)$ be an injective *umI*-continuous function with a *umI*-closed graph. If umIO(X) = mαIO(X) (resp. mSIO(X), mPIO(X)), then by Theorem 5.4 we obtain Theorem 3.15 of [2] (resp. Theorem 4.12 of [11], Theorem 4.12 of [10]).

Definition 5.5. *An m-space (X, m_X) is said to be m-connected* [20] *if X cannot be written as the union of two nonempty disjoint sets of m_X.*

Definition 5.6. *An ideal m-space (X, m_X, I) is said to be umI-connected if the m-space $(X, \mathrm{umIO}(X))$ is m-connected.*

Hence, the ideal *m*-space (X, m_X, I) is *umI*-connected if X cannot be written as the union of two nonempty disjoint sets of umIO(X).

Lemma 5.5. [20] *Let $f : (X, m_X) \to (Y, \sigma)$ be a function, where m_X has property $\mathcal{B}$. If f is an m-continuous surjection and (X, m_X) is m-connected, then (Y, σ) is connected.*

Theorem 5.5. *If $f : (X, m_X, I) \to (Y, \sigma)$ is a umI-continuous surjection and (X, m_X, I) is umI-connected, then (Y, σ) is connected.*

Proof. The proof follows from Definition 5.6, Lemma 5.5 and the fact that umIO(X) has property $\mathcal{B}$. □

Remark 5.5. Let $f : (X, m_X, I) \to (Y, \sigma)$ be a surjective *umI*-continuous function and (X, m_X, I) be *umI*-connected. If umIO(X) = mαIO(X) (resp. mSIO(X), mPIO(X)), then by Theorem 5.5 we obtain Theorem 3.17 of [2] (resp. Theorem 4.14 of [11], Theorem 4.14 of [10]).

Definition 5.7. *Let (X, m_X, I) be an ideal m-space. A subset K of X is said to be umI-compact relative to X if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of K by umI-open sets of X, there exists a finite subset Δ_0 of Δ such that $K \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in I$. The space (X, m_X, I) is said to be umI-compact if X is umI-compact relative to X.*

Definition 5.8. *Let (Y,σ,J) be an ideal topological space. A subset K of Y is said to be J-compact relative to Y if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of K by open sets of Y, there exists a finite subset Δ_0 of Δ such that $K \setminus \cup\{U_\alpha : \alpha \in \Delta_0\} \in J$. The space (Y,σ,J) is said to be J-compact if Y is J-compact relative to Y.*

It is known in [17] that if $f : X \to Y$ is a function and I is an ideal on X then $f(I)$ is an ideal on Y.

Theorem 5.6. *If $f : (X,m_X,I) \to (Y,\sigma,f(I))$ is a umI-continuous function and K is umI-compact relative to X, then $f(K)$ is $f(I)$-compact relative to Y.*

Proof. Let K be *umI*-compact relative to X and $\{V_\alpha : \alpha \in \Delta\}$ any cover of $f(K)$ by open sets of Y. For each $x \in K$, there exists an $\alpha(x) \in \Delta$ such that $f(x) \in V_{\alpha(x)}$. Since f is *umI*-continuous, there exists a *umI*-open set $U_{\alpha(x)}$ containing x such that $f(U_{\alpha(x)} \subset V_{\alpha(x)}$. Since $\{U_{\alpha(x)} : x \in K\}$ is a cover of K by *umI*-open sets of X, there exists a finite subset K_0 of K such that $K \setminus \cup\{U_{\alpha(x)} : x \in K_0\} = I_0$, where $I_0 \in I$; hence

$$f(K) \subset \cup\{f(U_{\alpha(x)} : x \in K_0\} \cup f(I_0) \subset \cup\{V_{\alpha(x)} : x \in K_0\} \cup f(I_0).$$

Therefore, we obtain $f(K) \setminus \cup\{V_{\alpha(x)} : x \in K_0\} \in f(I_0)$. This shows that $f(K)$ is $f(I)$-compact relative to Y. □

Corollary 5.2. *If $f : (X,m_X,I) \to (Y,\sigma,f(I))$ is a umI-continuous surjective function and (X,m_X,I) is umI-compact, then $(Y,\sigma,f(I))$ is $f(I)$-compact.*

Remark 5.6. Let $f : (X,m_X,I) \to (Y,\sigma)$ be a surjective *umI*-continuous function and (X,m_X,I) be *umI*-compact. If umIO(X) = mαIO(X) (resp. mSIO(X), mPIO(X)), then by Corollary 5.2 we obtain Theorem 3.22 of [2] (resp. Theorem 4.19 of [11], Theorem 4.19 of [10]).

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One of the referees gave the authors many comments to improve this paper and also suggested to state the reference [1] as a special example for this paper. Other referee pointed out that they should add (9), (10) and (11) in Definition 3.4 as generalizations of semi*-I-open [5], $\beta^\star$-I-open [4] and pre*-I-open [4] sets, respectively. Therefore, the authors sincerely thank them for their kindness.

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