SARAJEVO JOURNAL OF MATHEMATICS
Vol.16 (29), No.2, (2020), 179 – 186
DOI: 10.5644/SJM.16.02.04

THE STEINHAUS-WEIL PROPERTY: II. THE SIMMONS-MOSPAN CONVERSE

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In memory of Harry I. Miller (1939 - 2018)

ABSTRACT. In this second part of a four-part series (with Parts I, III, IV referring to [BinO4,5,6]), we develop (via Propositions 1, 2 and Theorems 1, 2) a number of relatives of the Simmons-Mospan theorem, a converse to the Steinhaus-Weil theorem (for another, see [BinO1], and for yet others [BinO3, §8.5]). In Part III [BinO5, Theorems 1, 2], we link this with topologies of Weil type.

1. A LEBESGUE DECOMPOSITION

We study the Simmons-Mospan converse of the Abstract. We use the notation and terminology of Part I [BinO4] and refer also to the longer arXiv version [BinO3] combining all four parts. In particular, we recall the following: $\mathcal{K}(G)$ denotes the compact subsets of G, a metric group, $\mathcal{M}(G)$ (and its subset $\mathcal{P}(G)$) denotes the family of regular σ-finite measures (respectively, probabilities) on G; for $\mathbf{t} = \{t_n\}$ a null sequence (i.e. $t_n \to 1_G$), $\sigma = \sigma(\mathbf{t})$ denotes the selective measure corresponding to $\mathbf{t}$, as guaranteed by the Subcontinuity Theorem of I (cf. Theorems 1 and 1_S of I), generated via amenability at 1 from probability measures μ_n which sum along $\mathbf{t}$ beyond n the Dirac point-masses with dyadic weights. Thus for K compact, σ is ('selectively') subcontinuous down an appropriate subsequence of $\mathbf{t}$. (Such subsequences mimic the admissible directions in the Cameron-Martin theory of Gaussian measures, cf. [Bog], [BinO2].) We make use of the *Mospan property* of a probability measure μ (I Prop. 6) relating the failure of the interior-point property that $1_G \in \mathrm{int}(K^{-1}K)$ for non-null compact K to the failure of subcontinuity of μ at K, that is: $0 = \mu_-(K) := \sup_{\delta>0} \inf\{\mu(Kt) : t \in B_\delta\}$ (with B_δ the ball of radius δ about 1_G).

We begin with definitions isolating left-handed components in Christensen's notion of Haar null sets [Chr1,2], and Solecki's left Haar null sets [Sol1,2]; whilst

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2010 *Mathematics Subject Classification.* Primary 22A10, 43A05; Secondary 28C10.

Key words and phrases. Steinhaus-Weil property, amenability at 1, measure subcontinuity, Simmons-Mospan theorem, Weil topology, interior-points property, Haar measure, Lebesgue decomposition, left Haar null, selective measure, Cameron-Martin space.

left-handedness is the preferred choice below, right-handed versions have analogous properties. As far as we are aware, the component notions in parts (ii)-(iv) here have not been previously studied. Below, G is a Polish group, unless otherwise stated.

Definition. **(**i) **Left μ-null:** For $\mu \in \mathcal{M}(G)$, say that $N \subseteq G$ is *left μ-null* ($N \in \mathcal{M}_0^{\mathrm{L}}(\mu)$) if it is contained in a universally measurable set $B \subseteq G$ such that

$$\mu(gB) = 0 \qquad (g \in G).$$

Thus a set $S \subseteq G$ is *left Haar null* ([Sol3] after [Chr1]) if it is contained in a universally measurable set $B \subseteq G$ that is left μ-null for some $\mu \in \mathcal{M}(G)$.

(ii) **Left μ-inversion:** For $\mu \in \mathcal{M}(G)$, say that $N \in \mathcal{M}_0^{\mathrm{L}}(\mu)$ is *left invertibly μ-null* ($N \in \mathcal{M}_0^{\text{L-inv}}(\mu)$) if

$$N^{-1} \in \mathcal{M}_0^{\mathrm{L}}(\mu),$$

so that N^{-1} is contained in a universally measurable set B^{-1} such that

$$\mu(gB^{-1}) = 0 \qquad (g \in G).$$

(iii) **Left μ-absolute continuity:** For $\mu, \nu \in \mathcal{M}(G)$, ν is *left absolutely continuous* w.r.t. μ ($\nu <^{\mathrm{L}} \mu$) if $\nu(N) = 0$ for each $N \in \mathcal{M}_0^{\mathrm{L}}(\mu)$, and likewise for the invertibility version: $\nu <^{\text{L-inv}} \mu$.

(iv) **Left μ-singularity:** For $\mu, \nu \in \mathcal{M}(G), \nu$ is *left singular* w.r.t. μ (on B) ($\nu \perp^{\mathrm{L}} \mu$ (on B)) if B is a support of ν and $B \in \mathcal{M}_0^{\mathrm{L}}(\mu)$, and likewise $\nu \perp^{\text{L-inv}} \mu$.

Remark. For μ *symmetric*, since

$$_{g^{-1}}\mu(B) = \mu_g(B^{-1})$$

if B is *left* μ-null we may conclude only that B^{-1} is *right* μ-null. The 'inversion property', property (ii) above, is thus quite strong (though obvious in the abelian case).

Notice that each of $\mathcal{M}_0^{\mathrm{L}}(\mu)$ and $\mathcal{M}_0^{\text{L-inv}}(\mu)$ forms a σ-algebra (since $g\bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} gB_n$ and $g(\bigcup_{n\in\mathbb{N}} B_n)^{-1} = \bigcup_{n\in\mathbb{N}} gB_n^{-1}$). This implies the following left versions of the Lebesgue Decomposition Theorem (we need the second one below). The 'pedestrian' proof demonstrates that the Principle of Dependent Choice (DC) suffices, a further example that 'positive' results in measure theory follow from DC (as Solovay points out in [Solo, p. 31]).

Theorem LD. *For G a Polish group, $\mu, \nu \in \mathcal{M}(G)$, there are $\nu_{\mathrm{a}}, \nu_{\mathrm{s}} \in \mathcal{M}(G)$ with*

$$\nu = \nu_{\mathrm{a}} + \nu_{\mathrm{s}} \text{ with } \nu_{\mathrm{a}} <^{\mathrm{L}} \mu \text{ and } \nu_{\mathrm{s}} \perp^{\mathrm{L}} \mu,$$

and likewise, there are $\nu'_{\mathrm{a}}, \nu'_{\mathrm{s}} \in \mathcal{M}(G)$ with

$$\nu = \nu'_{\mathrm{a}} + \nu'_{\mathrm{s}} \text{ with } \nu_{\mathrm{a}} <^{\text{L-inv}} \mu \text{ and } \nu_{\mathrm{s}} \perp^{\text{L-inv}} \mu.$$

Proof. As the proof depends on σ-additivity, it will suffice to check the 'L' case. Write $G = \bigcup_{n\in\mathbb{N}} G_n$ with the G_n disjoint, universally measurable, and with each $\nu(G_n)$ finite (say, with all but one term σ-compact, and their complement ν-null). Put $s_n := \sup\{\nu(E) : E \subseteq G_n, E \in \mathcal{M}_0^{\mathrm{L}}(\mu)\}$. In $\mathcal{M}_0^{\mathrm{L}}(\mu)$, for each n with $s_n > 0$, choose $E_{n,m} \subseteq G_n$ with $\nu(E_{n,m}) > s_n - 1/m$, and put $B_n := \bigcup_{m\in\mathbb{N}} E_{n,m} \subseteq G_n$. Then the sets B_n are disjoint and lie in $\mathcal{M}_0^{\mathrm{L}}(\mu)$, as also does $B := \bigcup_{n\in\mathbb{N}} B_n$; moreover, $\nu(G_n\backslash B_n) = 0$ for each n. Put $A := G\backslash B$. Then $\nu(M) = 0$ for $M \in \mathcal{M}_0^{\mathrm{L}}(\mu)$ with $M \subseteq A$, since $A = \bigcup_{n\in\mathbb{N}}(G_n\backslash B_n)$. So $\nu_{\mathrm{a}} := \nu|A <^{\mathrm{L}} \mu$, and $\nu_{\mathrm{s}} := \nu|B\perp^{\mathrm{L}}\mu$, since $B \in \mathcal{M}_0^{\mathrm{L}}(\mu)$. □

Remark. A simpler argument rests on maximality: choose a maximal disjoint family $\mathcal{B}$ of universally measurable sets $M \in \mathcal{M}_0^{\mathrm{L}}(\mu)$ with finite positive $\nu(M)$; then, their union $B \in \mathcal{M}_0^{\mathrm{L}}(\mu)$ (as $\mathcal{B}$ would be countable, by the σ-finiteness of ν).

2. Discontinuity: the Simmons-Mospan Theorem

It is convenient to begin by repeating the gist of the Simmons-Mospan argument here, as it is short, despite its 'near perfect disguise', to paraphrase Loomis [Loo, p. 85]. The result follows from their use of the Fubini Theorem and the Lebesgue decomposition theorem of §1 above, but here we stress the dependence on the Fubini Null Theorem (Part I §1 – Fubini's Theorem for null sets) and on left μ-inversion. We revert to the Weil left-sided convention and associated KK^{-1} usage.

Proposition 1 (Local almost nullity). *For G a Polish group, $\mu \in \mathcal{M}(G)$, $V \subseteq G$ open and $K \in \mathcal{K}(G) \cap \mathcal{M}_0^{\text{L-inv}}(\mu)$, so that $K, K^{-1} \in \mathcal{M}_0^{\mathrm{L}}(\mu)$:*
– for any $\nu \in \mathcal{M}(G)$, $\nu(tK) = 0$ for μ-almost all $t \in V$, and likewise $\nu(Kt) = 0$.

Proof. For ν invertibly μ-absolutely continuous (as in §1 above), the conclusion is immediate; for general ν this will follow from Theorem LD (§1), once we have proved the corresponding singular version of the assertion: that is the nub of the proof.

Thus, suppose that $\nu\perp^{\text{L-inv}}\mu$ on K. For $t \in V$ let $t = uw$ be any expression for t as a group product of $u, w \in G$, and note that $\mu(uK^{-1}) = 0$, as $K^{-1} \in \mathcal{M}_0^{\mathrm{L}}(\mu)$. Let H be the set

$$\bigcup\nolimits_{t\in V}(\{t\} \times tK),$$

here viewed as a union of vertical t-sections. We next express it as a union of u-horizontal sections and apply the Fubini Null Theorem (Th. FN, Part I §1).

Since $u = tk = uwk$ is equivalent to $w = k^{-1}$, the u-horizontal sections of H may now be rewritten, eliminating t, as

$$\{(t,u) : uw = t \in V, u \in tK = uwK\} = \{(uw,u) : uw \in V, uw \in uK^{-1}\}.$$

So H may now be viewed as a union of u-horizontal sections as

$$\bigcup\nolimits_{u\in G}(V \cap (uK^{-1})) \times \{u\}),$$

all of these u-horizontal sections being μ-null. By Th. FN, μ-almost all vertical t-sections of H for $t \in V$ are ν-null. As the assumptions on K are symmetric the right-sided version follows. □

The result here brings to mind the Dodos Dichotomy Theorem [Dod1, Th. A] for *abelian* Polish groups G: if an analytic set A is witnessed as Haar-null under one measure $\mu \in \mathcal{P}(G)$, then either A is Haar-null for quasi all $\nu \in \mathcal{P}(G)$ or else it is not Haar-null for quasi all such ν, i.e. if $A \in \mathcal{M}_0(\mu)$ (omitting the unnecessary superscript L), then either $A \in \mathcal{M}_0(\nu)$ for quasi all such ν, or $A \notin \mathcal{M}_0(\nu)$ for quasi all such ν (i.e. *quasi all* w.r.t. the Prokhorov-Lévy metric in $\mathcal{P}(G)$ [Dud, 11.3, cf. 9.2]). Indeed, [Dod2, Prop. 5] when A is σ-compact A is Haar-null for quasi all $\nu \in \mathcal{P}(G)$. The result is also reminiscent of [Amb, Lemma 1.1].

Before stating the Simmons-Mospan specialization to the Haar context and also to motivate one of the conditions in its subsequent generalizations, we cite (and give a direct proof of) the following known result (equivalence of Haar measure η and its inverse $\tilde{\eta}$), encapsulated in the formula

$$\tilde{\eta}(K) := \eta(K^{-1}) = \int_K d\eta(t)/\Delta(t) \qquad (K \in \mathcal{K}(G)),$$

exhibiting the direct connection between η and $\tilde{\eta}$ via the (positive) modular function Δ [HewR, 15.14], or [Hal, §60.5f]; this equivalence result holds more generally between any two probability measures when one is left and the other right *quasi-invariant* – see [Xia, Cor. 3.1.4]; this is related to a theorem of Mackey's [Mac], cf. the longer combined arXiv version [BinO3, §8.16]. As will be seen from the proof, in Lemma H below, there is no need to assume the group is separable: a compact metrizable subspace (being totally bounded) is separable.

Lemma H (cf. [Hal, §50(ff); §59 Th. D]). *In a locally compact metrizable group G, for $K \in \mathcal{K}(G)$, if $\eta(K) = 0$, then $\eta(K^{-1}) = 0$, and, by regularity, likewise for K measurable.*

Proof. Fix an η-null $K \in \mathcal{K}(G)$. As K is compact, the modular function Δ of G is bounded away from 0 on K, say by $M > 0$; furthermore, K is separable, so pick $\{d_n : n \in \mathbb{N}\}$ dense in K. Then for any $\varepsilon > 0$ there are two (finite) sequences $m(1), \ldots m(n) \in \mathbb{N}$ and $\delta(1), \ldots, \delta(n) > 0$ such that $\{B_{\delta(i)} d_{m(i)} : i \le n\}$ covers K and

$$M \sum\nolimits_{i \le n} \eta(B_{\delta(i)}) \le \sum\nolimits_{i \le n} \eta(B_{\delta(i)}) \Delta(d_{m(i)}) = \sum\nolimits_{i \le n} \eta(B_{\delta(i)} d_{m(i)}) < \varepsilon.$$

Then, as η is left-invariant,

$$\sum\nolimits_{i \le n} \eta(d_{m(i)}^{-1} B_{\delta(i)}) = \sum\nolimits_{i \le n} \eta(B_{\delta(i)}) \le \varepsilon/M.$$

But $\{d_{m(i)}^{-1} B_{\delta(i)} : i \le n\}$ covers K^{-1} by the symmetry of the balls B_δ (by the symmetry of the norm); so, as $\varepsilon > 0$ is arbitrary, $\eta(K^{-1}) = 0$.

As for the final assertion, if $\eta(E^{-1}) > 0$ for some measurable E, then $\eta(K^{-1}) > 0$ for some compact $K^{-1} \subseteq E^{-1}$, by regularity; then $\eta(K) > 0$, and so $\eta(E) > 0$. □

Proposition 1 and Lemma H immediately give:

Theorem SakM (cf. [Sak, III.11], [Mos], [BarFF, Th. 7])**.** *For G a locally compact group with left Haar measure η and ν a Borel measure on G, if the set S is η-null, then for η-almost all t*

$$\nu(tS) = 0.$$

In particular, this is so for S the support of a measure ν singular with respect to η.

This in turn allows us to prove the locally compact (separable) case of the Simmons-Mospan Theorem, Theorem SM ([Sim, Th. 1], [Mos, Th. 7], recently rediscovered in the abelian case [BarFF, Th. 10]). Then in Theorem 2 below we pursue a non-locally compact variant.

Theorem SM. *In a locally compact Polish group, a Borel measure has the Steinhaus-Weil property if and only if it is absolutely continuous with respect to Haar measure.*

Proof. For K compact and μ absolutely continuous w.r.t. Haar measure η, if $\mu(K) > 0$ then $\eta(K) > 0$ and so as η, being invariant, is subcontinuous, Lemma 1 of Part I §2 gives the Steinhaus-Weil property. Otherwise, decomposing μ into its singular and absolutely continuous parts w.r.t. η, choose K a compact subset of the support of the singular part of μ; then $\mu(K) > \mu_-(K) = 0$, by Theorem SakM above, and so Prop 6 (ii) (the converse part – see I §2) on the Mospan property applies, giving a non-null compact set C without the interior-point property. □

Proposition 2 (after Simmons, cf. [Sim, Lemma] and [BarFF, Th. 8]). *For G a Polish group, $\mu, \nu \in \mathcal{M}(G)$ and $\nu \perp^{\text{L-inv}} \mu$ concentrated on a compact left invertibly μ-null set K, there is a Borel $B \subseteq K$ such that $K \backslash B$ is ν-null and both BB^{-1} and $B^{-1}B$ have empty interior.*

Proof. As we are concerned only with the subspace $KK^{-1} \cup K^{-1}K$, w.l.o.g. the group G is separable. By Prop. 1 above, $Z := \{x : \nu(xK) = 0\}$ is dense and so also is

$$Z_1 := \{x : \nu(K \cap xK) = 0\},$$

since $\nu(K \cap xK) \leq \nu(xK) = 0$, so that $Z \subseteq Z_1$. Take a denumerable dense set $D \subseteq Z_1$ and put

$$S := \bigcup_{d \in D} K \cap dK.$$

Then $\nu(S) = 0$. Take $B := K \backslash S$. If $\emptyset \neq V \subseteq BB^{-1}$ and $d \in D \cap V$, then for some $b_1, b_2 \in B \subseteq K$

$$d = b_1 b_2^{-1}: \qquad b_1 = db_2 \in K \cap dK \subseteq S,$$

a contradiction, since $B \cap S = \emptyset$. So $(K \backslash S)(K \backslash S)^{-1}$ has empty interior. A similar argument based on

$$T := \bigcup_{d \in D} Kd \cap K$$

ensures that also $(K \backslash S \backslash T)^{-1}(K \backslash S \backslash T)$ has empty interior. □

In order to generalize the Simmons Theorem from its locally compact context we will need to cite the following result. Here $\mathbb{Q}_+ := \mathbb{Q} \cap (0,\infty)$ denotes the positive rationals, and $B_\delta^{K,\Delta}(\sigma) := \{z \in B_\delta : \sigma(Kz) > \Delta\}$ as in Part I §2.

Theorem 1 (Disaggregation Theorem, [BinO2, Th. 7.1, Prop. 7.1]**).** *Let G be a Polish group that is strongly amenable at 1, and let $\mathbf{t}$ be a regular null sequence. For $\sigma = \sigma(\mathbf{t})$ there are a countable family $\mathcal{H}$ with $\mathcal{H} \subseteq \mathcal{K}_+(\sigma)$, a countable set $D = D(\mathcal{H}) \subseteq G$ dense in G, and a dense subset $G(\sigma)$ of G on which the sets below are the sub-basic sets of a metrizable topology:*

$$B_\delta^{K,\Delta}(\sigma) \qquad (K \in \mathcal{H}, \delta, \Delta \in \mathbb{Q}_+,\ \Delta < \sigma(K)).$$

In particular, the space $G(\sigma)$ is continuously and compactly embedded in G. Moreover, each such sub-basic open set contains a cofinal subsequence of $\mathbf{t}$.

For a proof we refer the reader to [BinO2]; the result relies on I Cor. 4, the closing result in I, establishing the sub-basic property referred to here. The subspace $G(\sigma)$ above is a topological analogue of the *Cameron-Martin* subspace $H(\gamma)$ of a locally convex topological vector space equipped with a Radon Gaussian measure γ – see [BinO3, §8.2-3].

We are now ready for the promised generalization. This requires equivalence of the selective measure $\sigma(\mathbf{t})$ (Part I §2) and its inverse – which is valid at least in Polish abelian groups (see Part I, Th. 4 §2 on strong amenability at 1).

Theorem 2 (**Generalized Simmons Theorem**, cf. [Sim, Th. 2])**.** *Let G be a Polish group that is strongly amenable at 1 (which holds e.g. if G is abelian), let $\sigma = \sigma(\mathbf{t})$ be a selective measure corresponding to a regular null sequence $\mathbf{t}$, which we assume is equivalent to its inverse $\tilde{\sigma}$ (e.g. if G is abelian), and let $G(\sigma)$ be the dense subspace endowed with the refinement topology as in the preceding theorem. Then:*
$\nu \in \mathcal{M}(G)$ is left invertibly-singular w.r.t. σ iff ν has a support that is a σ-compact union of compact sets K_n with each of the compact sets $K_nK_n^{-1}$ and $K_n^{-1}K_n$ nowhere dense (equivalently: having empty interior) in the topology of $G(\sigma)$.

Proof. If $\nu \in \mathcal{M}(G)$, by Theorem LD write

$$\nu = \nu_a + \nu_s \text{ with } \nu_a <^{\text{L-inv}} \sigma \text{ and } \nu_s \perp^{\text{L-inv}} \sigma.$$

If ν is concentrated as in the statement of the theorem on a σ-compact set B with $B^{-1}B$ having empty interior in $G(\sigma)$, then $\nu_a = 0$, and so ν is left invertibly-singular w.r.t. μ. Indeed, as ν is concentrated on B, so is ν_a. We claim that $\nu_a(B) = 0$. Otherwise, $\nu_a(K_n) > 0$ for one of the sequence of compact sets K_n with union B. So $K := K_n \notin \mathcal{M}_0^{\text{L-inv}}(\sigma)$, as $\nu_a <^{\text{L-inv}} \sigma$. The argument now splits into two cases, according as $K \notin \mathcal{M}_0^{\text{L}}(\sigma)$ or $K^{-1} \notin \mathcal{M}_0^{\text{L}}(\sigma)$

First, suppose that $\sigma(gK) > 0$ for some $g \in G$; then, by Part I, Lemma 1, there are $\delta > 0$ and $0 < \Delta < \sigma_-^{\mathbf{t}}(gK)$ with

$$B_\delta^{gK,\Delta}(\sigma) \subseteq (gK)^{-1}gK = K^{-1}K \subseteq B^{-1}B,$$

contradicting the above property of B.

Next, suppose that $\sigma(Kg) = \sigma^{-1}(g^{-1}K^{-1}) > 0$ for some $g \in G$; so $\sigma(g^{-1}K^{-1}) > 0$, as $\tilde{\sigma}$ is equivalent to σ. Then, again by Part I Lemma 1, there are $\delta > 0$ and $0 < \Delta < \sigma_-^{\mathbf{t}}(g^{-1}K^{-1})$ with

$$B_\delta^{g^{-1}K^{-1},\Delta}(\sigma) \subseteq (g^{-1}K^{-1})^{-1}g^{-1}K^{-1} = KK^{-1} \subseteq BB^{-1},$$

again contradicting the above property of B. So $\nu = \nu_{\mathrm{a}}$ is invertibly singular w.r.t. σ.

The rest of the proof is as in Simmons [Sim, Th. 2], using I Prop. 6 (Mospan property): the Baire-category argument still holds, since compactness implies closure under the $G(\sigma(\mathbf{t}))$-topology, the latter being a finer topology; avoidance of interior-points requires second countability, assured by Th. 1 above. □

Corollary 5 (Simmons Theorem: [Sim, Th. 2]**).** *For G separable and locally compact and η left Haar measure:*
$\nu \in \mathcal{M}(G)$ is singular w.r.t. η iff ν has a support that is a σ-compact union of compact sets K_n with each of the compact sets $K_nK_n^{-1}$ nowhere dense (equivalently: having empty interior).

For the non-separable version of the above, see [BinO3, §8.1].

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(Received: November 17, 2019)
(Revised: July 07, 2020)

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