SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.16.01.07
Vol.16 (29), No.1, (2020), 83 – 102

COARSE EXAMPLES OF BAUES COFIBRATION CATEGORY

NADIA E. GHEITH MOHAMAD AND PAUL D. MITCHENER

ABSTRACT. In this article, we set up a framework for homotopy theory in coarse geometry by defining a notion of cofibration for coarse maps. We show that the axioms of a Baues cofibration category hold.

1. INTRODUCTION

Many aspects of topology have an analogue notion in coarse geometry with respect to large scale properties; see [8, 9, 13, 15]. For example, there is a notion of coarse homotopy which is an analogue of the notion of homotopy in topology [13–15].

The usual framework for axiomatic homotopy theory is that of a Quillen model category [3, 12]. Unfortunately, the axioms required for a category to be a Quillen model category are hard to prove in the coarse setting. Baues introduced a weaker notion of cofibration category in [1, 2] as a generalization of a Quillen model category.

Definition 1.1. *A* cofibration category *is a category* $\mathcal{C}$ *with two classes of morphisms,* cof *of cofibrations and* w.e. *of weak equivalences. These are to satisfy:*

- **C1:** *Composition axiom: Isomorphisms are both cofibrations and weak equivalences. If f and g are composable morphisms in $\mathcal{C}$, if two of the three morphisms f, g, and gf are weak equivalences, then so is the third. The composite of cofibrations is a cofibration.*
- **C2:** *Push out axiom: For a cofibration $i : A \hookrightarrow X$ and a morphism $f : A \to Y$ there exists the push out in $\mathcal{C}$*

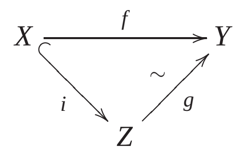
$$\begin{array}{ccc} X & \xrightarrow{f'} & X \cup_A Y \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i'} \\ A & \xrightarrow{f} & Y \end{array}$$

2010 *Mathematics Subject Classification.* 16W20.
Key words and phrases. Coarse map, Coarse space, Coarse category, Baues Cofibration category
.

and i' is a cofibration. Moreover;

- **(a):** *if f is a weak equivalence, so is f',*
- **(b):** *if i is a weak equivalence, so is i'.*

C3: *Factorization axiom: For any map $f\colon X \to Y$ in $\mathcal{C}$ there exists a commutative diagram*



where i is a cofibration and g is a weak equivalence.

Before stating the last axiom, we need to introduce another definition. A morphism in a cofibration category $\mathcal{C}$ is called a trivial cofibration *if it is both a weak equivalence and a cofibration. An object S in $\mathcal{C}$ is called a* fibrant model *if each trivial cofibration $i\colon S \to Q$ in $\mathcal{C}$ admits a map $r\colon Q \to S$, where $ri = 1_S$. We call r a* retraction *of i.*

C4: *Axiom on fibrant models: For each object X in $\mathcal{C}$ there is a trivial cofibration $X \overset{\sim}{\hookrightarrow} SX$ where SX is a model fibrant in $\mathcal{C}$. We call $X \overset{\sim}{\hookrightarrow} SX$ a* fibrant model of X.

In this article, we show that natural notions of weak equivalence and cofibration in coarse geometry give us a Baues cofibration category structure on the category of coarse spaces and coarse maps. For certain axioms involving limits and colimits, we need to use the notion of non-unital coarse spaces (see [6]).

Acknowledgements

The second author would like to thank Thomas Schick for many fruitful conversations on the area covered in this article during his time in Göttingen.

2. Coarse geometry and generalised ray

In the large scale we can define an abstract coarse space in terms of entourages [5, 8, 15, 16].

This definition comes from [8].

Definition 2.1. *Let X be a set. Then X is called a* unital coarse space *if it is equipped with a* coarse structure, *defined to be a collection ε of subsets M of $X \times X$ called* entourages *satisfying the following axioms:*

- **(1):** *If $M \in \varepsilon$ and $M' \subseteq M$, then $M' \in \varepsilon$.*
- **(2):** *Let $M_1, M_2 \in \varepsilon$, then $M_1 \cup M_2 \in \varepsilon$, and $M_1M_2 \in \varepsilon$ where $M_1M_2 = \{(x,z) \mid (x,y) \in M_1, (y,z) \in M_2$ for some $y\}$. We call M_1M_2* the composite *of M_1 and M_2.*
- **(3):** *$\Delta_X \in \varepsilon$ where $\Delta_X = \{(x,x) : x \in X\}$.*

(4): $\bigcup_{M\in\varepsilon} M = X\times X$.
(5): *If* $M\in\varepsilon$, $M^t = \{(y,x) \mid (x,y)\in M\}\in\varepsilon$.

We can use (X,ε) *to refer to a coarse space when we need to emphasise the collection of entourages.*

A subset M *is called* symmetric *if* $M = M^t$.

A *non-unital coarse space* is a coarse space defined as above, but we drop the axiom where Δ_X must be an entourage.

In the nonunital case, we get many more coarse maps. Consequently, using nonunital coarse spaces, it becomes extremely easy to construct (nonzero) categorical products in the coarse category, see [6]. Initially, let us focus on the unital case.

Definition 2.2. *Let* X *and* Y *be unital coarse spaces. Then a map* $f\colon X\to Y$ *is said to be* controlled *or* coarsely uniform *if for every entourage* $M\subseteq X\times X$*, the image*

$$f[M] = \{(f(x), f(y)) : (x,y)\in M\}$$

is an entourage. A controlled map f *is called* coarse *if the inverse image of a bounded set is also bounded.*

Definition 2.3. *We call two coarse maps* $f,g\colon X\to Y$ close*, and write* $f\sim_{Crs} g$*, if the set* $\{(f(x),g(x)) \mid x\in X\}$ *is an entourage,.*

A coarse map $f\colon X\to Y$ is called a *coarse equivalence* if there is a coarse map $g\colon X\to Y$ such that the compositions $f\circ g$ and $g\circ f$ are close to the identity maps 1_X and 1_Y respectively.

We call two coarse spaces X and Y *coarsely equivalent* if a coarse equivalence $f\colon X\to Y$ exists.

A subset $B\subseteq X$ is said to be *bounded* if the inclusion $B\hookrightarrow X$ is close to a constant map, or equivalently if it takes the form $M(x) = \{y : (x,y)\in M\}$ for some entourage $M\subseteq X\times X$ and point $x\in X$.

Definition 2.4. *Let* X *be a set and* ε *a collection of subsets of* $X\times X$*. The* coarse structure generated by ε *is the minimum coarse structure on* X *that contains* ε*. We write this structure* $\langle\varepsilon\rangle$.

Note that here we do *not* assume that the coarse structure generated by a collection is unital.

The following definition comes from [5].

Definition 2.5. *Let* X *be a Hausdorff space. A coarse structure on* X *is said to be* compatible with the topology *if every entourage is contained in an open entourage, and the closure of any bounded set is compact. We call a Hausdorff space equipped with a coarsre structure compatible with the topology a* coarse topological space.

Any coarse topological space is locally compact, and the bounded sets are precisely those which are precompact.

Example 2.1. *Let (X,d) be a proper metric space. Then d induces a coarse structure on X, which is called* metric structure *such that:*

Let $D_r = \{(x,y) \in X \times X \mid d(x,y) < r\}$. Then $E \subseteq X \times X$ is an entourage if $E \subseteq D_r$, for some $r > 0$.

The coarse space X equipped with the metric coarse structure is a coarse topological space if X is locally compact.

The following definition comes from [9].

Definition 2.6. *Let R be the topological space $[0,\infty)$ equipped with a coarse structure compatible with the topology. We call the space R a* generalised ray *if the following conditions hold.*

- *Let $M,N \subseteq R \times R$ be entourages. Then the sum*
$$M+N = \{(u+x, v+y) \mid (u,v) \in M, (x,y) \in N\}$$
is an entourage.
- *Let $M \subseteq R \times R$ be an entourage . Then the set*
$$M^s = \{(u,v) \in R \times R \mid x \leq u, v \leq y, (x,y) \in M\}$$
is an entourage.
- *Let $N \subseteq R \times R$ be an entourage , and $a \in R$. Then the set*
$$a+N = \{(a+x, a+y) \mid (x,y) \in N\}$$
is an entourage.

For example, the space $\mathbb{R}_+$ (with the metric coarse structure) is a generalised ray.

Proposition 2.1. *Let X be a coarse space, and let R be a generalised ray. Let $p,q\colon X \to R$ be controlled maps, then $p+q$ is a controlled map.*

Proof. Let $M \subseteq X \times X$ be an entourage. Then the images $p[M]$, $q[M]$ are entourages. Now
$$(p+q)[M] = \{((p+q)(x),\ (p+q)(y)) \ :\ (x,y) \in M\} \subseteq p[M] + q[M]$$
which implies that $(p+q)[M]$ is an entourage. Hence $p+q$ is controlled. □

Definition 2.7. *Let X and Y be coarse spaces, equipped with collections of entourages ε_X and ε_Y respectively. Then we define the* product *of X and Y to be the Cartesian product $X \times Y$ equipped with the coarse structure defined by forming finite compositions, unions of entourages, and all subsets of entourages in the set*
$$\{M \times N : M \in \varepsilon_X, N \in \varepsilon_Y\}.$$

Unfortunately, the above product is not a product in the category-theoretic sense since the projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are not in general coarse maps. We shall see that there is a solution to this issue in the world of non-unital coarse spaces.

Now we define two different coarse versions of disjoint union of coarse spaces.

The following definition comes from [7].

Definition 2.8. *Let X and Y be coarse spaces. Then we define the* disjoint union *to be the set $X \sqcup Y$ equipped with the coarse structure given by defining the entourages to be subsets of unions of the form*

$$M \cup N \cup (B_X \times B_Y) \cup (B_Y' \times B_X')$$

where $M \subseteq X \times X$ and $N \subseteq Y \times Y$ are entourages, and $B_X, B_X' \subseteq X$ and $B_Y, B_Y' \subseteq Y$ are bounded subsets. We denote this disjoint union by $X \sqcup Y$.

The following result is easy to check.

Proposition 2.2. *Let X and Y be coarse spaces, R be a generalised ray. Let $p_X : X \to R$ and $p_Y : Y \to R$ be controlled maps. Then $X \sqcup Y$ the map $p_{X \sqcup Y} : X \sqcup Y \to R$ defined by the formula*

$$p_{X \sqcup Y}(x) = \begin{cases} p_X(x) & x \in X \\ p_Y(x) & x \in Y \end{cases}$$

is a controlled map. □

Definition 2.9. *Let X and Y be coarse spaces. Then we define another type of* disjoint union *to be the set $X \sqcup Y$ equipped with the coarse structure given by defining the entourages to be subsets of unions of the form $M \cup N$ where $M \subseteq X \times X$ and $N \subseteq Y \times Y$ are entourages. We denote this disjoint union by $X \sqcup_\infty Y$.*

The space $X \sqcup_\infty Y$ is a non-unital coarse space even when X and Y are unital coarse spaces.

The following is also easy to check.

Proposition 2.3. *Let X and Y be coarse spaces, R be a generalised ray. Let $p_X : X \to R$ and $p_Y : Y \to R$ be controlled maps. Then $X \sqcup_\infty Y$ is a coarse space, and the map $p_{X \sqcup_\infty Y} : X \sqcup_\infty Y \to R$ defined by the formula*

$$p_{X \sqcup_\infty Y}(x) = \begin{cases} p_X(x) & x \in X \\ p_Y(x) & x \in Y \end{cases}$$

is a controlled map. □

3. Non-Unital Coarse Maps and Coarse Homotopy

In this section, we define a notion of locally proper, coarse maps between non-unital coarse spaces, and a coarse version of abstract homotopy groups.

The following definitions are prompted from [6].

Definition 3.1. *Let X, Y be coarse spaces and $f\colon X \to Y$ a map.*

- *We call f a* locally proper map *if $f|_{X'}$ is proper whenever $X' \subseteq X$ is a unital coarse subspace, that is, the inverse image of a bounded set $B \subseteq Y$ under the map $f|_{X'}$ is bounded.*
- *We call f a* coarse map between non-unital coarse spaces *if it is a controlled and locally proper map.*

Any proper map is locally proper, but the converse in not always true.
We define two maps between non-unital coarse spaces being close as follows.

Definition 3.2. *Let $f, g\colon X \to Y$ be two coarse maps between non-unital coarse spaces. We say that f is* close *to g if for any unital subspace $X' \subseteq X$, we have $f|_{X'}$ is close to $g|_{X'}$ in sense of definition (2.3).*

We call f a coarse equivalence between non-unital coarse spaces *if $f|_{X'}$ is a coarse equivalence in sense of definition (2.3) whenever $X' \subseteq X$ is a unital coarse subspace.*

Let X be a topological space. The product $X \times [0,1]$ is called a cylinder on X. We need to define a coarse version of the topological cylinder in order to define a coarse version of homotopy.

The following definition comes from [10].

Definition 3.3. *Let X be a coarse space, R be a generalised ray, and $p\colon X \to R$ be some controlled map. Then we define the* p-cylinder *of X:*

$$I_pX = \{(x,t) \in X \times R \mid t \le p(x)+1\}$$

The p-cylinder is a coarse space. We define the projection $p'\colon I_pX \to R$ by the formula $p'(x,t) = p(x)+t$ and we define coarse maps $i_0, i_1\colon X \to I_pX$ by the formula $i_0(x) = (x,0)$ and $i_1(x) = (x, p(x)+1)$ respectively.

Our aim in this work is to define a Baues cofibration category on the category of non-unital coarse spaces. The above definition yields ideas of homotopy and mapping cylinder which are vital to the axioms.

Definition 3.4. *Let $f_0, f_1\colon X \to Y$ be unital coarse maps. A* coarse homotopy *between f_0, f_1 is a coarse map $H\colon I_pX \to Y$ for some controlled map $p\colon X \to R$ such that $f_0 = H \circ i_0$ and $f_1 = H \circ i_1$ respectively.*

We say the maps $f_0, f_1\colon X \to Y$ are coarsely homotopic between non-unital coarse spaces *if $f_0|_{X'}$ is coarsely homotopic to $f_1|_{X'}$ whenever $X' \subseteq X$ is a unital coarse subspace.*

A coarse map $f\colon X \to Y$ is termed coarse homotopy equivalence *if there is a coarse map $g\colon Y \to X$ such that the compositions $g \circ f$ and $f \circ g$ are coarsely homotopic to the identities 1_X and 1_Y respectively.*

The proof of the following lemma is found in [4] in the unital case. The non-unital case follows easily.

Lemma 3.1. *Let $f_0, f_1 \colon X \to Y$ be close maps between coarse spaces. If f_0 is a controlled map, then f_1 is a controlled map. If f_0 is a coarse map, then f_1 is a coarse map. If f_0 is a coarse equivalence, then f_1 is a coarse equivalence. If f_0 is a coarse homotopy equivalence, then f_1 is a coarse homotopy equivalence.* □

Example 3.1. *Let X and Y be coarse spaces, and let $p \colon X \to R$ be a controlled map. Consider two close coarse maps $f_0, f_1 \colon X \to Y$. Then we can define a coarse homotopy $H \colon I_pX \to Y$ between the maps f_0 and f_1 by the formula*

$$H(x,t) = \begin{cases} f_0(x) & t<1 \\ f_1(x) & t \geq 1 \end{cases}$$

Thus, close coarse maps are also coarse homotopic. In particular, any coarse equivalence is a coarse homotopy equivalence.

Theorem 3.2. *Let X, Y be coarse spaces. Coarse homotopy defines an equivalence relation on the set of all coarse maps from X to Y.*

Recall that if X is a coarse space, $M \subseteq X \times X$ is an entourage, and $A \subset X$, then we write

$$M[A] = \{(a,b) \in M \mid a \in A\}.$$

The following definition comes from [9].

Definition 3.5. *We call a union $X = A \cup B$* coarsely excisive decomposition *if for any entourage $m \subseteq X \times X$, there is an entourage $M \subseteq X \times X$ such that $m[A] \cap m[B] \subseteq M[A \cap B]$.*

Lemma 3.3. *Let $f \colon X \to Y$ be a map. Let $X = A \cup B$ be a coarsely excisive decomposition such that the restrictions $f|_A$ and $f|_B$ are coarse maps. Then f is a coarse map.*

Proof. For any entourage M, we have that $M \cup M^t$ is a symmetric entourage, so without loss of generality, let $m \subseteq X \times X$ be a symmetric entourage containing the diagonal. We need to show that the image $f[m]$ is an entourage.

The sets $f[m \cap (A \times A)]$ and $f[m \cap (B \times B)]$ are entourages as $f|_A$ and $f|_B$ are controlled maps. It is therefore enough to show that $f[m \cap (A \times B)]$ and $f[m \cap (B \times A)]$ are entourages.

Pick a symmetric entourage M containing m such that $m[A] \cap m[B] \subseteq M[A \cap B]$. Let $(x,y) \in m \cap (A \times B)$. Then $x \in A \cap m[B]$ and $y \in B \cap m[A]$ so

$$x \in A \cap M[A \cap B] \qquad y \in B \cap M[A \cap B]$$

so $(x,y) \in M \cap (M[A \cap B] \times M[A \cap B])$.

By hypothesis, the image $f[M\cap(M[A\times B]\times M[A\cap B])]$ is an entourage, so $f[m\cap(A\times B)]$ is also an entourage. Similarly, the image $f[m\cap(B\times A)]$ is an entourage, so the map f is controlled.

Now let $C\subseteq Y$ be a bounded subset, then $(f|A)^{-1}[C]$ and $(f|B)^{-1}[C]$ are bounded subsets since $f|A$ and $f|B$ are coarse maps, and the inverse $f^{-1}(C)\subseteq(f|A)^{-1}[C]\cup(f|B)^{-1}[C]$, so $f^{-1}(C)$ is bounded. Hence f is a coarse map. □

Corollary 3.1. *Let $p\colon X\to R$ be a controlled map, $A=\{(x,t)\in X\times R \mid t\leq p(x)\}$, and $B=\{(x,t)\in X\times R \mid t\geq p(x)\}$, $p\colon X\to R$, so $X\times R=A\cup B$.*

Suppose that $f\colon X\times R\to Y$ is a map such that the restrictions $f|_A$ and $f|_B$ are coarse maps. Then f is a coarse map.

Proof. It is enough to show that $X\times R=A\cup B$ is a coarsely excisive decomposition.

To prove that $X\times R=A\cup B$ is a coarsely excisive decomposition, let $m\subseteq X\times R\times X\times R$ be an entourage. Without loss of generality, suppose that m is symmetric, and $m=m_1\times m_2$ where $m_1\subseteq X\times X$, $m_2\subseteq R\times R$ are symmetric entourages containing the diagonal.

Let $(z,w)\in X\times R$ be such that $(z,w)\in m[A]\cap m[B]$, so there exists $(x,s)\in A$ and $(y,t)\in B$ such that $((x,s),(z,w))\in m$, that is, $(x,z)\in m_1$, $(s,w)\in m_2$ and $((z,w),(y,t))\in m$, that is $(z,y)\in m_1$, $(w,t)\in m_2$. By definition of A and B, we have

$$s\leq p(x),\ t\geq p(y),\ \text{ and either } w\geq p(z),\ \text{ or } w\leq p(z).$$

We prove the case when $s\leq p(x)$, $t\geq p(y)$, and $w\geq p(z)$, the other cases are identical.

So we have

$$s\leq p(z)\ \text{ or } s\geq p(z)$$

and

$$t\leq p(z)\ \text{ or } t\geq p(z)$$

By symmetery of the entourages m_1 and m_2, we have $(p(z),w)\in[p\times p(m_1)]^s m_2$.

Now let $M=m_1\times[p\times p(m_1)]^s m_2$ where M depends on m and the controlled map p. Then $((z,p(z)),(z,w))\in M$, and this means $(z,w)\in M(A\cap B)$, and we are done. □

Proof of Theorem 3.2: Let $f\colon X\to Y$ be a coarse map, then f is coarsely homotopic to itself using a constant coarse homotopy $F\colon I_pX\to Y$ such that $F(x,t)=f(x)$, $x\in X$ where $p\colon X\to R$ is some controlled map. Then $F(x,0)=f(x)=F(x,p(x)+1)$ and since f is a coarse map, then F is also.

Let $f,g\colon X\to Y$ be coarse maps such that f is coarsely homotopic to g, so there exists a coarse homotopy $F\colon I_pX\to Y$, where $p\colon X\to R$ is some controlled map such that $F(x,0)=f(x)$, and $F(x,p(x)+1)=g(x)$.

Define a new coarse map $G\colon I_pX\to Y$ by $G(x,t)=F(x,p(x)+1-t)$, then G is a coarse homotopy from g to f.

To show this, we first see that $(p+1)$ is a controlled map since for an entourage $M \subseteq X \times X$, then $p+1(M)$ is an entourage as the map p is controlled. Then by definition of a generalised ray, the set

$$\{(p(x)+1-s,\ p(y)+1-t) \mid\ (x,y) \in M,\ 0 \leq s \leq p(x)+1,\ 0 \leq t \leq p(y)+1\}$$

is an entourage.

Now since F is a coarse map, it is easy to show that the map G is coarse. This proves that the equivalence relation is symmetric.

Now, we prove that the equivalence relation is transitive. Let $f\colon X \to Y$, $g\colon X \to Y$ and $h\colon X \to Y$ be coarse maps such that f is coarsely homotopic to g and g is coarsely homotopic to h.

So we can define two coarse homotopies $H\colon I_pX \to Y$ and $G\colon I_qX \to Y$ such that $H(x,0) = f(x)$, $H(x,p(x)+1) = g(x) = G(x,0)$, and $G(x,q(x)+1) = h(x)$ for all $x \in X$ where $p\colon X \to R$ and $q\colon X \to R$ are some controlled maps, and $p+q$ is controlled by proposition 2.1. Define the map $H+G\colon I_{p+q}X \to Y$ as follow

$$(H+G)(x,t) = \begin{cases} H(x,2t) & 0 \leq t \leq (p(x)+1)/2 \\ G(x,2t-(p(x)+1)) & (p(x)+1)/2 \leq t \leq ((p+q)(x)/2)+1 \\ G(x,q(x)+1) & ((p+q)(x)/2)+1 \leq t \leq (p+q)(x)+1 \end{cases}$$

Note that the maps $K,L\colon I_pX \to I_pX$ defined by $K(x,t) = (x,2t)$ and $L(x,t) = (x,2t-(p(x)+1))$ are coarse maps, and the maps $((p+q)/2)+1\colon X \to R$ and $(p+1)/2\colon X \to R$ are controlled. Now the map $H+G$ is a coarse homotopy by corollary 3.1. □

Lemma 3.4. *Let $f\colon X \to Y$ be a map, and $i : A \hookrightarrow X$ be a coarse equivalence. Suppose that the restriction $f|_A$ is a coarse map, then f is coarse.*

Proof. Since i is a coarse equivalence, there exists a coarse map $g\colon X \to A$ such that $g \circ i$ and $i \circ g$ are close to 1_A and 1_X.

We have $f \circ i = f|_A$. Then $f \circ i \circ g = f|_A \circ g$, but $i \circ g$ is close to 1_X and $f \circ 1_X$ is close to f, so then f is close to $f|_A \circ g$, and $f|_A \circ g$ is a coarse map.

Now we need to show that f is controlled.

Let $M = \{(f|_A \circ g(x), f(x)) : x \in X\}$. By closeness M is an entourage.

Let E be an entourage in $X \times X$, then $f \times f(E) \subseteq M^t\{f|_A \circ g(x), f|_A \circ g(y)) : (x,y) \in E\}M$. Since $f|_A \circ g$ is coarse, $\{f|_A \circ g(x), f|_A \circ g(y)) : x,y \in E\}$ is an entourage, and also M^t so the composite is an entourage. This shows that f is a controlled map.

Now let $B \subseteq Y$ be bounded, and M the entourage defined above then $M[B]$ is bounded. Since $f|_A \circ g$ is coarse, we have $(f|_A \circ g)^{-1}(M[B])$ is bounded. Let $x \in f^{-1}(B)$, then $f(x) \in B$ and so $f|_A \circ g(x) \in M[B]$ which means $f^{-1}[B]$ is contained in $(f|_A \circ g)^{-1}(M[B])$, so $f^{-1}[B]$ is bounded. Hence f is a coarse map, and we are done. □

Theorem 3.5. *Let $f_i\colon X \to Y$ and $g_i\colon Y \to Z$ be coarse maps where $i = 0,1$. If f_0 is coarsely homotopic to f_1 and g_0 is coarsely homotopic to g_1, then $g_0 \circ f_0$ is coarsely homotopic to $g_0 \circ f_1$ and $g_0 \circ f_1$ is coarsely homotopic to $g_1 \circ f_1$. Further, then $g_0 \circ f_0$ is coarsely homotopic to $g_1 \circ f_1$.*

Proof. Let $F : I_pX \to Y$ define a coarse homotopy between the coarse maps f_0 and f_1 such that $F(x,0) = f_0(x)$ and $F(x, p(x)+1) = f_1(x)$ for all $x \in X$ where $p_X : X \to R$ is some controlled map, and $G\colon I_qY \to Z$ define a coarse homotopy between the coarse maps g_0 and g_1 such that $G(y,0) = g_0(y)$ and $G(y,q(x)+1) = g_1(y)$ for all $y \in Y$ where $p_Y : Y \to R$ is some controlled map.

Define the map $K\colon I_pX \to Z$ by $K(x,t) = g_0 \circ F(x,t)$. Then K defines a coarse homotopy between $g_0 \circ f_0$ and $g_0 \circ f_1$. Now we define the map $H\colon I_{q\circ f_1}X \to Z$ by $H(x,t) = G(f_1(x),t)$, then again H defines a coarse homotopy between $g_0 \circ f_1$ and $g_1 \circ f_1$. By theorem (3.2) we have $g_0 \circ f_0$ is coarsely homotopic to $g_1 \circ f_1$. □

Lemma 3.6. *The projection map $\pi\colon I_pX \to X$ defined by $\pi(x,t) = x$ where $x \in X$ and $t \in R$ is a coarse homotopy equivalence where $p\colon X \to R$ is some controlled map.*

Proof. Define the inclusion map $i : X \hookrightarrow I_pX$ by the formula $i(x) = (x,0)$. Then $\pi \circ i = 1_X$, and we can define a coarse homotopy $H\colon I_{p\circ\pi}(I_pX) \to I_pX$ (which is also a controlled homotopy) between the map $i\circ\pi\colon I_pX \to I_pX$ and the identity on I_pX where $i \circ \pi(x,t) = (x,0)$ as follows;

$$H((x,t),s) = \begin{cases} (x,s+t) & s \leq p(x) \\ (x,0) & s > p(x) \end{cases}$$

The maps π, i and H are coarse maps, so H defines a coarse homotopy between $i\circ\pi$ and the identity on I_pX. □

Lemma 3.7. *Let $f\colon X \to Y$ be a map, and $i : A \hookrightarrow X$ be a coarse homotopy equivalence. Suppose that the restriction $f|_A$ is a coarse map, then f is coarse.*

Proof. Since i is a coarse homotopy equivalence, there exists a coarse map $g\colon X \to A$ such that $g \circ i$ and $i \circ g$ are coarse homotopic to 1_A and 1_X .

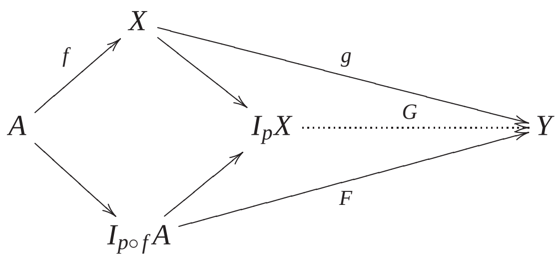
We have $f \circ i = f|_A$. Then $f \circ i \circ g = f|_A \circ g$, but $i \circ g$ is coarse homotopic to 1_X and by theorem (3.5) the map $f \circ 1_X$ is coarse homotopic to f. Since f is coarse homotopic to $f|_A \circ g$, and since $f|_A \circ g$ is a coarse map, we must have that the map f is a coarse map. □

4. The Quotient Coarse Cofibration Category

Let QCrs be the *quotient coarse category* consisting of all non-unital coarse spaces and closeness equivalence classes of coarse maps. Denote the class of a coarse map f by $[f]$. In [6], Luu proved that the quotient coarse category has all non-zero limits and colimits. This is good as we can prove that the push out diagram exists in this category easily.

Definition 4.1. *Let A, X be non-unital coarse spaces. A coarse map $f\colon A \to X$ is called a* coarse cofibration *if given a coarse map $g\colon X \to Y$, some controlled map $p\colon X \to R$ and a coarse homotopy $F\colon I_{p\circ f}A \to Y$ such that $g(f(a)) = F(a,0)$ for all $a \in A$, $t \in R$, we can find a coarse homotopy $G\colon I_pX \to Y$ such that $g(x) = G(x,0)$ for all $x \in X$, $G(f(a),t) = F(a,t)$ for all $a \in A$, $t \in R$.*

This definition is illustrated by the following commutative diagram:



Lemma 4.1. *Let $f, g\colon X \to Y$ be close maps between non-unital coarse spaces. If f is a coarse cofibration, then g is a coarse cofibration. If f is a coarse homotopy equivalence, then g is a coarse homotopy equivalence.*

Proof. First, suppose that f is close to g and g is a coarse cofibration, we need to show that f is a coarse cofibration. Let $p\colon Y \to R$ be some controlled map, $h\colon Y \to Z$ be a coarse map, and $F\colon I_{p\circ f}X \to Z$ be a coarse homotopy such that $F(x,0) = h(f(x))$ for all $x \in X$.

Since f is close to g, so $p\circ f$ is close to $p\circ g$ which implies that the cylinders $I_{p\circ f}X$ and $I_{p\circ g}X$ are coarsely equivalent.

Let $K\colon I_{p\circ g}X \to I_{p\circ f}X$ be a coarse equivalence such that $K(x,0) = (x,0)$ for all $x \in X$.

Define a map $F^{'}\colon I_{p\circ g}X \to Z$ by

$$F^{'}(x,t) = \begin{cases} h\circ g(x) & t = 0 \\ F\circ K(x,t) & otherwise \end{cases}$$

But $F^{'}$ is close to F, and since F is a coarse map, by lemma (3.1) we have $F^{'}$ is a coarse map, and $F^{'}(x,0) = h\circ g(x)$.

Now since g is a coarse cofibration, there is a coarse homotopy $G\colon I_pY \to Z$ such that $G(y,0) = h(y)$ for all $y \in Y$, and $G(g(x),t) = F^{'}(x,t)$ for all $x \in X$, $t \in R$. Define a map $G^{'}\colon I_pY \to Z$ by

$$G^{'}(y,t) = \begin{cases} h(y) & t = 0 \\ F(x,t) & y = f(x) \\ G(y,t) & otherwise \end{cases}$$

But $G^{'}$ is close to G, and since G is a coarse map, by lemma (3.1) we have that $G^{'}$ is a coarse map. Therefore f is a coarse cofibration.

The second statement is straightforward. □

Considering the above lemma, we have the following definition.

Definition 4.2. *Let X, Y be non-unital coarse spaces. We call a closeness class $[f]\colon A \to X$ a* coarse cofibration class *if the representative coarse map f is a coarse cofibration. A closeness class $[f]\colon A \to X$ is called a* coarse homotopy equivalence class *if the representative coarse map f is a coarse homotopy equivalence as defined in definition (3.4).*

Lemma 4.2. *Let X be a coarse space, $p\colon X \to R$ be some controlled map. Let $[i] : A \hookrightarrow X$ be an inclusion class in the quotient coarse category, then the inclusion class $[j]\colon X \times \{0\} \to I_{p_A}A \cup (X \times \{0\})$ is a coarse homotopy equivalence class, where $p_A = p\mid_A$.*

Proof. Define a closeness equivalence class $[\pi]\colon I_{p_A}A \cup (X \times \{0\}) \to X \times \{0\}$ by $\pi(x,t) = (x,0)$ for all $x \in A$ or $t = 0$. Then π is a representative coarse map, and $[\pi \circ j] = [1_{X\times\{0\}}]$. Define a controlled map $p_0\colon X \times \{0\} \to R$ by $p_0(x,0) = p(x)$ for all $x \in X$, so we can define a coarse homotopy class $[H]\colon I_{p_0\circ\pi}(I_{p_A}A \cup (X \times \{0\})) \to I_{p_A}A \cup (X \times \{0\})$ between the class $[j \circ \pi]$ and the identity on $[I_{p_A}A \cup (X \times \{0\})]$ where $j \circ \pi(x,t) = (x,0)$ for all $x \in X$ as follows;

$$H((x,t),s) = \begin{cases} (x, s+t) & s \leq p_0 \circ \pi(x,t) \\ (x,0) & s > p_0 \circ \pi(x,t) \end{cases}$$

The classes π, j, and H are closeness equivalence classes so $[H]$ defines a coarse homotopy class between $[j \circ \pi]$ and the identity class that defined on the space $I_{p_A}A \cup (X \times \{0\})$. □

Lemma 4.3. *Let X be a non-unital coarse space, $p\colon X \to R$ be some controlled map. Let $i : A \hookrightarrow X$ be an inclusion, where $p_A = p\mid_A$. Write*

$$(I_{p_A}A) \cup (X \times \{0\}) = \{(x,t) \in I_pX : x \in A \text{ or } t = 0\}$$

Let $j : (I_{p_A}A) \cup (X \times \{0\}) \hookrightarrow I_pX$ be the obvious inclusion. Then the following are equivalent;

- **(1):** *$[i] : A \hookrightarrow X$ is a coarse cofibration class.*
- **(2):** *Suppose we have an equivalence class $[f]\colon (I_{p_A}A) \cup (X \times \{0\}) \to Y$, then there exists a coarse homotopy $G\colon I_pX \to Y$ such that $G \circ j = f$.*
- **(3):** *There is a coarse homotopy class $[r]\colon I_pX \to (I_{p_A}A) \cup (X \times \{0\})$ such that $r(x,t) = (x,t)$ for all $(x,t) \in (I_{p_A}A) \cup (X \times \{0\})$.*

Proof. First note that since $p_A = p|_X$ and i is the inclusion, then $p_A(a) = p(i(a))$ and p_A is a controlled map.

$((1) \Rightarrow (2))$ Let $[f]\colon (I_{p_A}A) \cup (X \times \{0\}) \to Y$ be a closeness equivalence class, then f is a representative coarse map. Define a map $f_0\colon X \times \{0\} \to Y$ by $f_0 = f\mid_{X\times\{0\}}$, and $H\colon I_{p_A}A \to Y$ such that $H = f\mid_{I_{p_A}A}$. Then f_0, H are coarse maps, and since $[i]$ is a coarse cofibration class, so i is a coarse cofibration. This means that there exists a coarse homotopy $G\colon I_pX \to Y$ such that $G(i(a),t) = H(a,t)$ for all $a \in A$, $t \in R$ and so then $G \circ j(x,t) = f(x,t)$.

((2) $\Rightarrow$ (3)) Suppose that $[I]\colon (I_{p_A}A) \cup (X \times \{0\}) \to (I_{p_A}A) \cup (X \times \{0\})$ is the identity class, so I is the representative identity map. By (2) there exists a coarse homotopy class $[r]\colon I_pX \to (I_{p_A}A) \cup (X \times \{0\})$ such that $r \circ j = I$, that is, $r(j(x,t)) = r(x,t) = (x,t)$ for all $(x,t) \in (I_{p_A}A) \cup (X \times \{0\})$.

((3) $\Rightarrow$ (2)) Suppose we have a coarse homotopy class $[r]\colon I_pX \to (I_{p_A}A) \cup (X \times \{0\})$ such that $r(x,t) = (x,t)$ for all $(x,t) \in (I_{p_A}A) \cup (X \times \{0\})$.

Let $[f]\colon (I_{p_A}A) \cup (X \times \{0\}) \to Y$ be a closeness equivalence class, then f is the representative coarse map. So we can define a coarse homotopy $G\colon I_pX \to Y$ by writing $G(x,t) = f(r(x,t))$. Then $G \circ j(x,t) = G(x,t) = f(r(x,t)) = f(x,t)$, so $G \circ j = f$.

((2) *and* (3) $\Rightarrow$ (1)) Let $[i]\colon A \to X$ be a closeness equivalence class. We need to prove that the representative coarse map i is a coarse cofibration. Let $F\colon I_{p_A}A \to Y$ be a coarse homotopy and $g\colon X \to Y$ be a coarse map such that $F(a,0) = g(i(a))$ for all $a \in A$. By (3) we have a coarse homotopy $r\colon I_pX \to (I_{p_A}A) \cup (X \times \{0\})$ such that $r(x,t) = (x,t)$ for all $(x,t) \in (I_{p_A}A) \cup (X \times \{0\})$.

Define a map $f_0\colon X \times \{0\} \to Y$ by $f_0(x,0) = g(x)$. Then f_0 is a coarse map. Let $f\colon (I_{p_A}A) \cup (X \times \{0\}) \to Y$ be a map such that $f|_{X \times \{0\}} = f_0$ and $f|_{I_{p_A}A} = F$, we need to show that f is a coarse map. Since the inclusion $X \times \{0\} \hookrightarrow (I_{p_A}A) \cup (X \times \{0\})$ is a coarse homotopy equivalence by lemma (4.2) and since the map f_0 is a coarse map, so by lemma 3.7, the map f is coarse as required.

Therefore by (2) there exists a coarse homotopy $G\colon I_pX \to Y$ defined by $G(x,t) = f \circ r(x,t)$, and then $G(x,0) = f_0(x,0) = g(x)$, for all $x \in X$, and $G(i(a),t)) = fr(i(a),t) = f(a,t) = F(a,t)$ for all $a \in A$, $t \in R$. Therefore $[i]$ is a coarse cofibration class. □

Theorem 4.4. *The quotient coarse category Qcrs is a Baues cofibration category. The weak equivalences are the coarse homotopy equivalence classes, and the cofibrations are the coarse cofibration classes.*

Proving this theorem requires us to prove the following results in order to satisfy the axioms required of a Baues cofibration category.

Proposition 4.1. *Let $[f]\colon X \to Y$ be an isomorphism in the quotient coarse category. Then $[f]$ is both a coarse cofibration class and coarse homotopy equivalence class.*

Proof. Let $[f]$ be an isomorphism, then f is the representative coarse map, so we have a coarse map $g\colon Y \to X$ such that $g \circ f$ is close to 1_X, $f \circ g$ is close to 1_Y. Therefore by example (3.1) the map f is a coarse homotopy equivalence. Hence $[f]$ is a coarse homotopy equivalence class.

Now let $F: I_{p_Y \circ f} X \to Z$ be a coarse homotopy where $p_Y: Y \to R$ is some controlled map. Let $h: Y \to Z$ be a coarse map such that $F(x,0) = h(f(x))$ for all $x \in X$.

Define a map $G: I_{p_Y} Y \to Z$ by $G(y,t) = F(g(y),t)$, then $G(f(x),t) = F(g(f(x)),t)$ for all $x \in X$, $t \in R$, and $G(y,0) = F(g(y),0) = h(f \circ g(y))$ for all $y \in Y$.

But by assumption we have $g \circ f$ is close to 1_X, $f \circ g$ is close to 1_Y which implies that $h \circ f \circ g$ is close to h. Therefore $G|_{Y \times \{0\}}$ is close to h and $G|_{I_{p_Y} f(X)}$ is close to F.

Now define another map $G': I_{p_Y} Y \to Z$ by

$$G'(y,t) = \begin{cases} h(y) & t = 0 \\ F(x,t) & y = f(x) \\ G(y,t) & otherwise \end{cases}$$

But G' is close to G, and since G is a coarse map so by lemma (3.1) we have G' is a coarse map. Therefore f is a coarse cofibration, and hence $[f]$ is a coarse cofibration class. □

The next two axioms follow immediately from the definitions.

Proposition 4.2. *Consider two equivalence classes $[f]: X \to Y$ and $[g]: Y \to Z$. If any two of the morphisms $[f]$, $[g]$ and $[gf]$ are coarse homotopy equivalence classes, then so is the third.* □

Proposition 4.3. *Composition of coarse cofibration classes is a coarse cofibration class.* □

Proposition 4.4. *Let X, Y be non-unital coarse spaces, and let $[i]: X \to Y$ be an equivalence class that is both a coarse cofibration class and a coarse homotopy equivalence class. Then there is a coarse map $[r]: Y \to X$ such that $[r \circ i] = [1_X]$.*

Proof. Since $[i]$ is a coarse homotopy equivalence class, the map i is a coarse homotopy equivalence which means that there exists a coarse map $h: Y \to X$ such that $h \circ i$ and $i \circ h$ are coarsely homotopic to 1_X, 1_Y respectively.

So we have a coarse homotopy $F: I_p X \to X$ such that $F(x,0) = h(i(x))$, $F(x, p(x) + 1) = x$ for all $x \in X$. Without loss of generality, we can assume $p = qi$ for a controlled map $q: Y \to R$. Since i is a coarse cofibration, there exists a coarse homotopy $G: I_q Y \to X$ such that $G(y,0) = h(y)$ for all $y \in Y$, and $G(i(x),t) = F(x,t)$ for all $x \in X$, $t \in R$.

Define a coarse map $r: Y \to X$ by the formula $r(y) = G(y, q(y) + 1)$. By construction we have $r \circ i(x) = G(i(x), p_Y(i(x)) + 1) = F(x, p_Y(i(x)) + 1) = x$ for all $x \in X$. Hence $[r \circ i] = [1_X]$. □

The following definition comes from [6].

Definition 4.3. *Suppose that (X, ε_X) is a non-unital coarse space, Y is any set, and $f: X \to Y$ is any map.*

The push-forward *coarse structure of ε_X along f is defined by;*

$$f_*\varepsilon_X =< \{(f\times f)(F) : F\in \varepsilon_X\} >.$$

The following obvious proposition also comes from [6].

Proposition 4.5. *Suppose that (X,ε_X) is a non-unital coarse space, and $f\colon X\to Y$ is any locally proper map. Then $f_*\varepsilon_X$ is the minimum coarse structure on Y which makes f into a coarse map.* □

Definition 4.4. *Let $f,g\colon A\to X$ be coarse maps between non-unital coarse spaces. The* Coequalizer *of $[f]$, $[g]$ is defined by writing $Coeq([f],[g]) = X$, equipped with the coarse structure*

$$\varepsilon_{Coeq([f],[g])} =< \varepsilon_X, f_*\varepsilon_A, g_*\varepsilon_A, \{(f\times g)(F) : F\in\varepsilon_A\} >_X,$$

together with the identity map in level of sets $\theta\colon X\to Coeq([f],[g])$ (which is a coarse map).

The proof of the following lemma is found in [6].

Lemma 4.5. *Let $f,g\colon A\to X$ be coarse maps between non-unital coarse spaces. The coarse space $Coeq([f],[g])$ is a coequalizer of $[f]$ and $[g]$ (in the category-theoretic sense) in the category Qcrs.* □

Definition 4.5. *Let A, X, and Y be coarse spaces between non-unital coarse spaces. Suppose that we have coarse maps $i\colon A\to X$ and $f\colon A\to Y$. Then we define*

$$X\vee_A Y = Coeq([\tilde{i}],[\tilde{f}])$$

where

$$\tilde{i} : A \xrightarrow{i} X \xrightarrow{i_X} X\sqcup_\infty Y, \qquad \tilde{f} : A \xrightarrow{f} Y \xrightarrow{i_Y} X\sqcup_\infty Y$$

are the representative coarse maps and $X\sqcup_\infty Y$ is the disjoint union in definition (2.9).

Theorem 4.6. *Let X, Y, and A be non-unital coarse spaces, suppose we have equivalence classes of two coarse maps $[i]\colon A\to X$, $[f]\colon A\to Y$. Then we have a push out diagram in the quotient coarse category.*

$$\begin{array}{ccc} X & \xrightarrow{[f']} & X\vee_A Y \\ {\scriptstyle [i]}\uparrow & & \uparrow{\scriptstyle [i']} \\ A & \xrightarrow{[f]} & Y \end{array}$$

Proof. Define the classes $[\tilde{i}]\colon A\to X\sqcup_\infty Y$, $[\tilde{f}]\colon A\to X\sqcup_\infty Y$ by $[\tilde{i}] = [i_X\circ i]$ and $[\tilde{f}] = [i_Y\circ f]$ where i_X and i_Y are the representative inclusion maps, so the maps $\tilde{i}$ and $\tilde{f}$ are representative coarse maps.

Now the map $\theta\colon X\sqcup_\infty Y\to Coeq([\tilde{i}],[\tilde{f}])$ is the identity map at the level of sets, and clearly it is a coarse map. So we can factor the representative maps i', f' as follows;

$$Y \xrightarrow{i_Y} X\sqcup_\infty Y \xrightarrow{\theta} X\vee_A Y \qquad X \xrightarrow{i_X} X\sqcup_\infty Y \xrightarrow{\theta} X\vee_A Y$$

respectively, where i_Y, i_X are the inclusions. Since θ, i_Y, and i_X are coarse maps, then $i^{'}$, $f^{'}$ are also coarse maps.

Let $g\colon X\sqcup_\infty Y\to Z$ be a map defined by writing

$$g(x)=\begin{cases} g_1(x) & x\in X\\ g_2(x) & x\in Y\end{cases}$$

then g is coarse such that $g\circ\tilde{i}$ is close to $g\circ\tilde{f}$. Let $h\colon X\vee_A Y\to Z$ be the same map (as a set map) as g, then clearly $g=h\circ\theta$. Hence $[g]=[h]\circ[\theta]$. We need to show that $[h]$ is a unique coarse map.

First since $g\circ\tilde{i}$ is close to $g\circ\tilde{f}$, then h is controlled. Second, suppose that $B\subseteq Z$ is a bounded subset, and $W\subseteq X\vee_A Y$ is a unital subspace, then we can write $W=\theta(W^{'})$ for some unital subspace $W^{'}\subseteq X\sqcup_\infty Y$.

Now since g is a locally proper map, then $(g\,|_{W'})^{-1}(B)$ is bounded, but $g=h\circ\theta$, and $(g\,|_{W'})^{-1}(B)=(\theta\,|_{W'})^{-1}((h\,|_{\theta(W')})^{-1})(B))$. Since θ is surjective, $h\,|_W^{-1}(B)$ is bounded. Therefore h is locally proper. Hence h is a coarse map.

Now $h\circ\tilde{i}(a)=h(\theta(i_y(i(a))))=h(\theta(i(a)))$, and
$h\circ\tilde{f}(a)=h(\theta(i_y(f(a))))=h(\theta(f(a)))$ for all $a\in A$.

To check uniqueness, consider another coarse map $l\colon X\vee_A Y\to Z$ such that g is close to $l\circ\theta$. We need to show that l is close to h, so let $E\subseteq X\vee_A Y\times X\vee_A Y$ be an entourage.

Clearly if $E\subseteq X\sqcup_\infty Y\times X\sqcup_\infty Y$, then $(l\times h)(E)$ is an entourage. Now let $E=(\tilde{i}\times\tilde{f})(F)$ for some F an entourage in A. Since the map g is close to $l\circ\theta$, $g\circ\tilde{i}$ is close to $l\circ\theta\circ\tilde{f}$. Therefore

$$h\times l((\tilde{i}\times\tilde{f})(F))=((g\circ\tilde{i})\times(l\circ\theta\circ\tilde{f}))(F)$$

which is an entourage in Z as needed. □

Proposition 4.6. *Let $[i]\colon A\to X$ be a coarse cofibration class, and let $[f]\colon A\to Y$ be an equivalence class of a coarse map f. Then in the push-out diagram:*

$$\begin{array}{ccc} X & \xrightarrow{[f^{'}]} & X\vee_A Y\\ \uparrow{\scriptstyle [i]} & & \uparrow{\scriptstyle [i^{'}]}\\ A & \xrightarrow{[f]} & Y\end{array}$$

the map $[i^{'}]$ is a coarse cofibration class.

Proof. The above theorem proves the first part. We need to show that $[i^{'}]$ is a coarse cofibration class, so it is enough to show that $i^{'}$ is a coarse cofibration.

Suppose that $q\colon X\sqcup_\infty Y\to R$ is some controlled map, $F: I_{q\circ i'}Y\to Z$ is a coarse homotopy, and $g\colon X\sqcup_\infty Y\to Z$ is a coarse map such that $F(y,0)=g(i^{'}(y))$ for all $y\in Y$.

Define a map $G\colon I_{q\circ i'\circ f}A \to I_{q\circ i'}Y$ by $G(a,t) = (f(a),t)$ for all $a \in A$, then G is a coarse map. Since $q \circ i' \circ f = q \circ f' \circ i$ by the above push out diagram, then the cylinders $I_{q\circ i'\circ f}A$ and $I_{q\circ f'\circ i}A$ are the same. The map $F \circ G\colon I_{q\circ f'\circ i}A \to Z$ is a coarse homotopy such that $F \circ G(a,0) = F(f(a),0) = g(i'(f(a))) = g(f'(i(a)))$ for all $a \in A$ by the above push out diagram.

By the universal property we have $g'\colon X \to Z$ defined by $g'(x) = g(f'(x))$ for all $x \in X$, then g' is a coarse map and $F \circ G(a,0) = g'(i(a))$ for all $a \in A$.

Since the class $[i]$ is a coarse cofibration class, i is a coarse cofibration which implies that there is a coarse homotopy $H\colon I_{q\circ f'}X \to Z$ such that $H(x,0) = g'(x)$ for all $x \in X$, $H(i(a),t) = F \circ G(a,t)$ for all $a \in A$, $t \in R$. We can define a new coarse homotopy $H'\colon I_q(X \sqcup_\infty Y) \to Z$ by writing

$$H'(f'(x),t) = H(x,t),\ H'(i'(y),t) = F(y,t).$$

Let $w \in X \sqcup_\infty Y$ such that $w = f'(x)$ or $w = i'(y)$ $x \in X, y \in Y$, then

$$H'(w,0) = H'(f'(x),0) \text{ when } w = f'(x), \text{ and } H'(i'(x),0)) \text{ when } w = i'(x).$$

This is equivalent to saying

$$H(x,0) = g(f'(x)), \text{ and } F(y,0) = g(i'(y)).$$

Then by the above $H'(w,0) = g(w)$ for all $w \in X \sqcup_\infty Y$, and $H'(i'(y),t) = F(y,t)$ for all $y \in Y$. Hence i' is a coarse cofibration which implies that $[i']$ is a coarse cofibration class. □

Proposition 4.7. *Let $[i]\colon A \to X$ be a coarse homotopy equivalence class, and let $[f]\colon A \to Y$ be an equivalence class of a coarse map f. Then in the push-out diagram:*

$$\begin{array}{ccc} X & \xrightarrow{[f']} & X \vee_A Y \\ {\scriptstyle [i]}\uparrow & & \uparrow{\scriptstyle [i']} \\ A & \xrightarrow{[f]} & Y \end{array}$$

the map $[i']$ is a coarse cofibration class.

Proof. Suppose that $[i]$ is a coarse homotopy equivalence class, then i is a coarse homotopy equivalence, and there exists a coarse map $h\colon X \to A$ such that $i \circ h$ is coarsely homotopy to id_X, and $h \circ i$ is coarsely homotopy to id_A. The map i' is defined by $i'(y) = \pi(y)$ for all $y \in Y$.

Now define a map $r\colon X \sqcup_\infty Y \to Y$ by

$$r(\pi(y)) = y,\ y \in Y, \text{ and, } r(\pi(x)) = f \circ h$$

A similar argument in theorem (4.6) shows that the map r is a coarse map.

The composites $r\circ i^{'}(y) = r(\pi(y)) = y$ for all $y \in Y$, $i^{'}\circ r(\pi(y)) = i^{'}(y) = \pi(y)$, and $i^{'}\circ r(\pi(x)) = i^{'}(f\circ h(x))$, by the push out diagram this implies that $i^{'}\circ f\circ h = f^{'}\circ i\circ h$, but $i\circ h$ is coarsely homotopic to the identity id_X.

By theorem (3.5), we have $f^{'}\circ i\circ h$ is coarsely homotopic to $f^{'}$, and therefore $i^{'}\circ r$ is coarsely homotopic to $id_{X\sqcup_\infty Y}$. Hence $i^{'}$ is a coarsely homotopy equivalence which implies that $i^{'}$ is a coarse homotopy equivalence. □

Definition 4.6. *Let* $[f]\colon X\to Y$ *be the closeness equivalence class of a coarse map, and* $p\colon X\to R$ *be a controlled map. Then we define* the mapping cylinder *of* $[f]$, C_f *to be the push out* $I_pX\vee_X Y$ *which is defined to be* $Coeq([\tilde{f}],[\tilde{i}])$ *where* $\tilde{f}\colon X\to I_pX\sqcup_\infty Y$ *and* $\tilde{i}\colon X\to I_pX\sqcup_\infty Y$ *are coarse maps.*

Proposition 4.8. *We have a coarse cofibration class* $[i]\colon X\to C_f$ *and a coarse homotopy equivalence class* $[r]\colon C_f\to Y$ *such that* $[f]=[r\circ i]$.

Proof. Let $\theta\colon I_{p_X}X\sqcup_\infty Y\to C_f$ be the coequalizer coarse map. Then we define the maps $i\colon X\to C_f$, $r\colon C_f\to Y$ by

$$i(x) = \theta(x,0)$$

$$r(\theta(y)) = y,\ y\in Y \text{ and } r(\theta(x,t)) = f(x), x\in X,\ t\in R$$

Since θ is a coarse map, then i is also, and by a similar argument to that in theorem (4.6) we show that r is a coarse map.

Define a map $s\colon Y\to C_f$ by $s(y) = \theta\circ i_Y(y)$, for all $y\in Y$, then s is a coarse map, and $r\circ s = 1_Y$ and by proposition (3.6), the map $X\hookrightarrow I_{p_X}X$ is a coarse homotopy equivalence. Therefore r is a coarse homotopy equivalence by the push out diagram and proposition (4.7). This implies that any coarse map close to r is a coarse homotopy equivalence. Hence $[r]$ is a coarse homotopy equivalence class, and that $f = r\circ i$.

Now we need to prove that $[i]\colon X\to C_f$ is a coarse cofibration class. First, i is a coarse map since it is a composite of two coarse maps, so it is enough to show that i is a coarse cofibration. Let $q\colon C_f\to R$ be a controlled map. Suppose we are given a coarse homotopy $F\colon I_{q\circ g}X\to Z$ and a coarse map $h\colon C_f\to Z$ such that $F(x,0) = h(i(x))$ for all $x\in X$.

We can define a map $G\colon I_qC_f\to Z$ by writing $G(\theta(y),t) = h(\theta(y))$ for all $y\in Y$, and

$$G(\theta(x,s),t) = \begin{cases} h(\theta(x,s-\frac{t}{2})) & 0\le s\le (q(i(x))+1)/2,\ t\le 2s \\ F(x,t-2s) & 0\le s\le (q(i(x))+1)/2,\ t\ge 2s \\ h(\theta(x,q(i(x))+1-s-\frac{t}{2})) & (q(i(x))+1)/2\le s\le q(i(x))+1, \\ & t\le 2(q(i(x))+1)-2s \\ F(x,s-q(i(x))+1+\frac{t}{2}) & (q(i(x))+1)/2\le s\le q(i(x))+1, \\ & t\ge 2(q(i(x))+1)-2s \end{cases}$$

The maps $(x,s,t)\mapsto g(\pi(x,s-\frac{t}{2}))$, $(x,s,t)\mapsto g(\pi(x,q(i(x))+1-s-\frac{t}{2}))$, $(x,s,t)\mapsto F(x,t-2s)$, $(x,s,t)\mapsto F(x,s-q(i(x))+1+\frac{t}{2})$ are all controlled. Using the same argument as in lemma (3.3 (2)), the set

$$\{(x,s,t) : 0 \leq s \leq (q(i(x))+1)/2\} \cup \{(x,s,t) : (q(i(x))+1)/2 \leq s \leq q(i(x))+1\}$$

is coarsely excisive decomposition.

Now since the maps $g \circ \pi$ and F are both controlled maps on the sets $\{(x,s,t) : 0 \leq s \leq (q(i(x))+1)/2\}$ and $\{(x,s,t) : (q(i(x))+1)/2 \leq s \leq q(i(x))+1\}$, by corollary 3.1 (2), the map G is controlled on the set

$$\{(x,s,t) : 0 \leq s \leq (q(i(x))+1)/2\} \cup \{(x,s,t) : (q(i(x))+1)/2 \leq s \leq q(i(x))+1\}$$

It is clear that $F(x,0) = g(\pi(x,0))$ when $t = 2s$ and $t = 2(q(i(x))+1) - 2s$. Hence we have that G is a coarse homotopy as required. □

Using the above results, we can prove theorem (4.4) as follows;

Proof of Theorem (4.4): Axiom (C1) is proposition (4.1), proposition (4.2), and proposition (4.3). Axiom (C2) is proposition (4.4). Axiom (C3) is proposition (4.6), and the last axiom (C4) is just proposition (4.8). □

Now since we have proved an important theorem that shows our quotient coarse category Qcrs is a Baues cofibration category [(4.4)], it is not hard to prove the following result (see [11]).

Proposition 4.9. *Let $[f]\colon A \to Y$ be a coarse homotopy equivalence class in the quotient coarse cofibration category, Then in the push out diagram*

$$\begin{array}{ccc} X & \xrightarrow{[f']} & X \vee_A Y \\ \uparrow{\scriptstyle [i]} & & \uparrow{\scriptstyle [i']} \\ A & \xrightarrow{[f]} & Y \end{array}$$

the map $[f']\colon X \to X \vee_A Y$ is a coarse homotopy equivalence class. □

Corollary 4.1. *The inclusions $i_0, i_1 : X \hookrightarrow I_pX$ are coarse cofibrations where $p\colon X \to R$ is some controlled map.*

Proof. The cylinder I_pX is the mapping cylinder of the identity $X \to X$ where $p\colon X \to R$ is some controlled map. It follows by proposition (4.8) that the inclusion map $[i_0]$ is a coarse cofibration. The map $\kappa\colon I_pX \to I_pX$ defined by the formula

$$\kappa(x,t) = (x, p(x)+1-t)$$

is an isomorphism, and the map i_1 is the composition $\kappa \circ i_0$.. □

REFERENCES

[1] H. J. Baues. *Algebraic homotopy*, volume 15 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1989.

[2] H. J. Baues. *Combinatorial foundation of homology and homotopy*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1999. Applications to spaces, diagrams, transformation groups, compactifcations, differential algebras, algebraic theories, simplicial objects, and resolutions.

[3] W. G. Dwyer and J. Spaliński. *Homotopy theories and model categories*. In Handbook of algebraic topology, pages 73–126. North-Holland, Amsterdam, 1995.

[4] B. Grave. *Coarse Geometry and Asymptotic Dimension*. PhD thesis, Georg-August Universit *ä*t Gottingen, 2006.

[5] N. Higson, E. K. Pedersen, and J. Roe. *$C^\star$-algebras and controlled topology*. *K*- Theory, 11(3):209–239, 1997.

[6] V-T. Luu. *Coarse categories i: Foundations*. unpublished, Found in: https://arxiv.org/abs/0708.3901, 2006.

[7] P. D. Mitchener, B. Norouzizadeh, and T. Schick. *Coarse homotopy groups*. Unpublished Preprint: http://www.mitchener.staff.shef.ac.uk/.

[8] P. D. Mitchener. *Coarse homology theories*. Algebr. Geom. Topol., 1:271–297 (electronic), 2001.

[9] P. D. Mitchener. *Addendum to: "Coarse homology theories"* .[Algebr. Geom. Topol. 1 (2001), 271–297 (electronic); mr1834777]. Algebr. Geom. Topol., 3:1089–1101 (electronic), 2003.

[10] P. D. Mitchener. *The general notion of descent in coarse geometry*. Algebr. Geom. Topol., 10(4):2419–2450, 2010.

[11] N. M. Gheith. *Coarse Version of Homotopy Theory (Axiomatic Structure)*. PhD thesis, University of Sheffield, 2013.

[12] D. G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin, 1967.

[13] J. Roe. *Coarse Cohomology and Index Theory on Complete Riemannian Manifolds*, volume 497 of Memoirs of American Mathematical Society. American Mathematical Society, 1993.

[14] J. Roe. *Index theory, coarse geometry, and topology of manifolds*, volume 90 of CBMS. Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996.

[15] J. Roe. *Lectures on coarse geometry*, volume 31 of University Lecture Series. American Mathematical Society, Providence, RI, 2003.

[16] G. Skandalis, J.L. Yu, and G. Yu. *The coarse baum-connes conjucture and groupoids*. Topology, 41:807–834, 2002.

(Received: January 13, 2019)
(Revised: February 12, 2019)

Nadia E. Gheith Mohamad
Gharian University
Mathematics Department, Faculty of science
Gharian, 64418, Libya
e-mail: *nadiamg74@yahoo.com*
and
Paul D. Mitchener
University of Sheffield
School of Mathematics and Statistics
Hicks Building
Sheffield, S3 7RH,UK
e-mail: *P.Mitchener@sheffield.ac.uk*