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SURVEY ON THE KAKUTANI PROBLEM IN P-ADIC ANALYSIS II

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ABSTRACT. Let $\mathbb{K}$ be a complete ultrametric algebraically closed field and let A be the Banach $\mathbb{K}$-algebra of bounded analytic functions in the "open" unit disk D of $\mathbb{K}$ provided with the Gauss norm. Let $Mult(A, \| \,.\, \|)$ be the set of continuous multiplicative semi-norms of A provided with the topology of pointwise convergence, let $Mult_m(A, \| \,.\, \|)$ be the subset of the $\phi \in Mult(A, \| \,.\, \|)$ whose kernel is a maximal ideal and let $Mult_1(A, \| \,.\, \|)$ be the subset of the $\phi \in Mult(A, \| \,.\, \|)$ whose kernel is a maximal ideal of the form $(x-a)A$ with $a \in D$. By analogy with the Archimedean context, one usually calls *ultrametric Corona problem, or ultrametric Kakutani problem* the question whether $Mult_1(A, \| \,.\, \|)$ is dense in $Mult_m(A, \| \,.\, \|)$. In a previous paper, we have recalled the characterization of a large set of continuous multiplicative semi-norms and why the multbijectivity of the algebra A would solve the Corona problem. Here we prove that multbijectivity in the general case. This implies that $Mult_1(A, \| \,.\, \|)$ is dense in $Mult_m(A, \| \,.\, \|)$, beginning by the case when $\mathbb{K}$ is spherically complete and generalizing next.

CONTENTS

I. SPECIAL PRIME CLOSED IDEALS OF A

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As explained in [11], we consider the $\mathbb{K}$-algebra of bounded analytic functions in the "open" disk D of center 0 and diameter 1 and the present paper is aimed at examining whether the set of multiplicative semi-norms whose kernel is a maximal ideal of codimension 1 (which is identified with the disk D) is dense inside the set of of multiplicative semi-norms whose kernel is a maximal ideal, with respect to the topolgy of pointwise convergence: this is the Kakutani problem for the algebra A. That will require to deepen the link between maximal ideals of infinite codimension, ultrafilters on D and multiplicative continuous semi-norms.

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Notations. All notations of [11] are used here. Let $\mathbb{K}$ be an algebraically closed field complete with respect to an ultrametric absolute value $|\ .\ |$. Given $a \in \mathbb{K}$ and $r > 0$, we denote by $d(a,r)$ the disk $\{x \in \mathbb{K} \mid |x-a| \le r\}$, by $d(a,r^-)$ the disk $\{x \in K \mid |x-a| < r\}$, by $C(a,r)$ the circle $\{x \in \mathbb{K} \mid |x-a| = r\}$ and set $D = d(0,1^-)$. Let $a \in D$. Given $r,\ s \in]0,1]$ such that $0 < r < s$ we set $\Gamma(a,r,s) = \{x \in \mathbb{K}\ |r < |x-a| < s\}$.

Let A be the $\mathbb{K}$-algebra of bounded power series converging in D which is complete with respect to the Gauss norm defined as $\left\| \sum_{n=1}^{\infty} a_n x^n \right\| = \sup_{n \in \mathbb{N}} |a_n|$: we know that this norm actually is the norm of uniform convergence on D [5], [13].

Given $a \in D$ and $R \in]0,1[$ we call *circular filter of center a and diameter R on D* the filter $\mathcal{F}$ which admits as a generating system the family of sets $\Gamma(\alpha, r', r'') \cap D$ with $\alpha \in d(a,R), r' < R < r''$, i.e. $\mathcal{F}$ is the filter which admits for basis the family of sets of the form $D \cap \left(\cap_{i=1}^{q} \Gamma(\alpha_i, r_i', r_i'')\right)$ with $\alpha_i \in d(a,R), r_i' < R < r_i''$ $(1 \le i \le q\ ,\ q \in \mathbb{N})$.

We l denote by $\mathcal{W}$ the circular filter on D of center 0 and diameter 1 and by $\mathcal{Y}$ the filter admitting for basis the family of sets of the form $\Gamma(0,r,1) \setminus \left(\bigcup_{n=0}^{\infty} d(a_n, r_n^-)\right)$ with $a_n \in D,\ r_n \le |a_n|$ and $\lim_{n\to\infty} |a_n| = 1$.

We denote by $B(\mathbb{N}, \mathbb{K})$ the $\mathbb{K}$-algebra of bounded sequences of $\mathbb{K}$. Let $S = (a_n)_{n\in N}$ be a sequence in D thinner than $\mathcal{W}$. We will denote by T_S the mapping from A into $B(\mathbb{N}, \mathbb{K})$ which associates to each $f(x) = \sum_{n=0}^{\infty} a_n x^n$ the sequence $(f(a_n)_{n\in\mathbb{N}})$, we will denote by $\Sigma(S)$ the set of ultrafilters thinner than S and by $I(S)$ the ideal of the $f \in A$ such that $f(a_n) = 0\ \forall n \in \mathbb{N}$.

Given $a \in \mathbb{K}$ and $r > 0$, we denote by $\Phi(a,r)$ the set of circular filters secant with $d(a,r)$ i.e. the circular filters of center $b \in d(a,r)$ and radius $s \in [0,r]$.

An ultrafilter $\mathcal{U}$ on D will be called *coroner ultrafilter* if it is thinner than $\mathcal{W}$. Similarly, a sequence (a_n) on D will be called *a coroner sequence* if its filter is a coroner filter, i.e. if $\lim_{n\to+\infty} |a_n| = 1$.

Two coroner ultrafilters $\mathcal{F}$, G are said to be *contiguous* if for every subsets $F \in \mathcal{F}$, $G \in \mathcal{G}$ of D the distance from F to G is null.

As in [11], given a normed $\mathbb{K}$-algebra E, we denote by $Mult(E, \|\ .\ \|)$ the set of continuous multiplicative semi-norms of $\mathbb{K}$-algebra of E [2], [12], [1].

Let $\psi \in Mult(A, \|\ .\ \|)$ be different from $\|\ .\ \|$. Then ψ will be said to be *coroner* if its restriction to $\mathbb{K}[x]$ is equal to $\|\ .\ \|$.

Let $(a_n)_{n\in\mathbb{N}}$ be a coroner sequence in D. The sequence is called *a regular sequence* if $\inf_{j\in\mathbb{N}} \prod_{\substack{n\in\mathbb{N}\\ n\neq j}} |a_n - a_j| > 0$.

An ultrafilter $\mathcal{U}$ is said to be *regular* if it is thinner than a regular sequence. Thus, by definition, a regular ultrafilter is a coroner ultrafilter.

We denote by $B(\mathbb{N}, \mathbb{K})$ the $\mathbb{K}$-algebra of bounded sequences of $\mathbb{K}$.

By Corollary (4.6) in [15], we have Theorem I.1:

Theorem I.1. *Let S be a sequence in D thinner than $\mathcal{W}$. Then T_S is surjective on $B(\mathbb{N}, \mathbb{K})$ if and only if the sequence S is regular.*

Notations. Let S be a regular sequence. Since T_S is surjective, there exists a $\mathbb{K}$-algebra isomorphism Λ_S from $\frac{A}{Ker(T_S)}$ onto $B(N,K)$, where $Ker(T_S) = I(S)$.

For every ultrafilter $\mathcal{G}$ on $\mathbb{N}$ we will denote by $\Theta(\mathcal{G})$ the ideal of $B(\mathbb{N}, \mathbb{K})$ consisting of the sequences $(a_n)_{n\in\mathbb{N}}$ such that $\lim_{\mathcal{G}} a_n = 0$.

The following Theorem I.2 is classical [15]:

Theorem I.2. *Θ is a bijection from the set of ultrafilters on $\mathbb{N}$ onto $Max(B(\mathbb{N}, \mathbb{K}))$. The restriction of Θ to the subset of non-principal ultrafilters on $\mathbb{N}$ is a bijection from this set onto the set of non-principal maximal ideals of $B(\mathbb{N}, \mathbb{K})$. Moreover, a maximal ideal of $B(\mathbb{N}, \mathbb{K})$ is principal if and only it is of codimension 1.*

Theorem I.3. *Let S be a regular sequence and let $\mathcal{M}$ be a maximal ideal of A. The following two statements are equivalent:*

(i) $I(S) \subset \mathcal{M}$,

(ii) There exists an ultrafilter $\mathcal{U}$ thinner than S such that $\mathcal{M} = \mathcal{J}(\mathcal{U})$.

Moreover, the mapping Ψ which associates to each ultrafilter $\mathcal{U}$ thinner than S the ideal $\mathcal{J}(\mathcal{U})$ is a bijection from $\Sigma(S)$ onto the set of maximal ideals of A containing $I(S)$.

Proof. Obviously, (ii) implies (i). Thus, suppose (i) true. Let $S = (a_n)_{n\in\mathbb{N}}$. By Theorem I.2, the isomorphism Λ_S makes a bijection Ψ from the set of maximal ideals of A containing $I(S)$ to the set of maximal ideals of $B(\mathbb{N}, \mathbb{K})$ and more precisely, it makes a bijection from the set of maximal ideals of A of infinite codimension containing $I(S)$ to the set of maximal ideals of $B(\mathbb{N}, \mathbb{K})$ of infinite codimension which actually are the non-principal maximal ideals of $B(\mathbb{N}, \mathbb{K})$. Let $\mathcal{N} = \Psi(M)$. By Theorem I.1, there exists an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $\mathcal{N}$ is the ideal of the bounded sequences tending to zero along $\mathcal{U}$. Now, let Ξ be the natural bijection from the set of non-principal ultrafilters of $\mathbb{N}$ onto the set of ultrafilters thinner than S and let $V = \Xi(\mathcal{U})$. Then $\mathcal{N} = \Lambda_S(M)$ hence $\mathcal{M} = I(S)$. Moreover, in this way, we can see that $\Psi \circ \Xi^{-1}$ is a bijection from $\Sigma(S)$ onto the set of maximal ideals of A containing $I(S)$. $\square$

Corollaire I.3.1. *If $\mathcal{U}$ is a regular ultrafilter on D, $\mathcal{J}(\mathcal{U})$ is a maximal ideal of A.*

Corollary I.3.2. *If A is multbijective, then for every $\phi \in Mult_m(A, \|\,.\,\|)$ there exists a coroner ultrafilter $\mathcal{U}$ such that $\phi = \varphi_{\mathcal{U}}$.*

In order to prove Theorem I.6, we shall state Theorem I.4:

Theorem I.4. *There exist regular maximal ideals $\mathcal{M}$ of A and $f \in \mathcal{M}$, having a sequence of zeros of order 1 and no other zeros, such that $f' \notin \mathcal{M}$.*

Proof. By Theorem III.4 in [11] there exist bounded sequences $(a_n)_{n\in \mathbb{N}}$ in D such that $a_0 = 1$ and such that the sequence $\left|\frac{a_n}{a_{n+1}}\right|$ is strictly increasing and then the function $f(x) = \sum_{n=0}^{\infty} a_n x^n$ admits a sequence of zeros $(\alpha_n)_{n\in \mathbb{N}^*}$ satisfying $|\alpha_n| = \left|\frac{a_n}{a_{n+1}}\right|$. Thus, particularly, if we set $r_n = \left|\frac{a_n}{a_{n+1}}\right|$, then by Theorem III.4 in [11] f admits exactly a unique zero in each circle $C(0,r_n)$, each of order 1, and has no other zero in D. Consequently, by Lemma I.8 in [11], we can see that $|f'(\alpha_n)| = |f'|(r_n)\ \forall n \in \mathbb{N}^*$. Now, let $\mathcal{U}$ be an ultrafilter thinner than the sequence $(\alpha_n)_{n\in \mathbb{N}^*}$. We can check that the sequence $(\alpha_n)_{n\in \mathbb{N}}$ is regular, hence $\mathcal{U}$ is a regular ultrafilter. Consequently, by Corollary I.3.1 $\mathcal{J}(\mathcal{U})$ is a maximal ideal of A. Now, by construction, f belongs to $\mathcal{J}(\mathcal{U})$. But $\lim_{n\to+\infty} |f'(\alpha_n)| = \lim_{n\to+\infty} |f'|(r_n) = \|f'\| \neq 0$, hence f' does not belong to $\mathcal{J}(\mathcal{U})$. □

Lemma I.5. *Let $(a_n)_{n\in \mathbb{N}}$ be a regular sequence, let $\delta = \inf_{k\in \mathbb{N}} \prod_{n\neq k, n\in \mathbb{N}} |a_n - a_k|$ and let $\rho = \inf_{k\neq n, k,n\in \mathbb{N}} |a_n - a_k|$. Let $f \in A$ admit each a_n as a zero of order 1 and have no other zero. Then $|f'(x)| \geq \|f\|\frac{\delta}{\rho}\ \forall x \in \bigcup_{n=0}^{\infty} d(a_n, (\delta\rho)^-)$.*

Proof. Let us fix $t \in \mathbb{N}$, let $r = |a_t|$. Set $u = x - a_t$, $g(u) = f(x)$ and consider $|g|(\rho)$. Since g has a unique zero in $d(0,\rho^-)$ and admits all the the $a_n - a_t$ as zeros, by Lemma I.5 and I.8 in [11] we can check that $|g|(\rho) \geq \|g\|\delta = \|f\|\delta$. Inside $d(0,\rho^-)$, $g(u)$ is of the form $b_1 u + \sum_{n=2}^{\infty} b_n u^n$ with $|b_1|\rho \geq |b_n|\rho^n\ \forall n \geq 2$. Consequently, $|g'(u)|\rho| \geq \|f\|\delta$ Now $|g'(u)| = |b_1| = \frac{|g(u)|}{\rho}\ \forall u \in d(0,\rho^-)$. Now, of course $f'(x) = g'(u)$ hence

$$|f'(x)| = \frac{|f(x)|}{\rho} \geq \|f\|\frac{\delta}{\rho}\ \forall x \in d(a_t, \rho^-).$$

That holds for every $t \in \mathbb{N}$. □

Theorem I.6. *Suppose $\mathbb{K}$ is spherically complete. Let $\mathcal{M}$ be a regular maximal ideal of A. There exists $f \in \mathcal{M}$, having a sequence of zeros of order 1 and no other zeros, such that $f' \notin \mathcal{M}$.*

Proof. Since $\mathcal{M}$ is a regular maximal ideal, there exists a regular sequence $(a_n)_{n\in \mathbb{N}}$ and a regular ultrafilter $\mathcal{U}$ thinner than the sequence (a_n) such that $\mathcal{M} = \mathcal{J}(\mathcal{U})$. Since the sequence is regular, we have $\delta = \inf_{k\in \mathbb{N}} \prod_{n\neq k, n\in \mathbb{N}} |a_n - a_k| > 0$ and $\rho = \inf_{k\neq n, k,n\in \mathbb{N}} |a_n - a_k| > 0$

Since $\mathbb{K}$ is spherically complete, since $\delta > 0$ we may apply Corollary III.6.1 in [11] showing there exists $f \in A$ admitting each a_n as a zero of order 1 and no other zero. Now by Lemma I.5, we have $|f'(x)| \geq \|f\|\frac{\delta}{\rho}\ \forall x \in \bigcup_{n=0}^{\infty} d(a_n, \delta^-)$ which shows that $\varphi_{\mathcal{U}}(f') > 0$ because the set $E = \bigcup_{n=0}^{\infty} d(a_n, (\delta\rho)^-)$ obviously belongs to $\mathcal{U}$. Consequently, f' does not belong to $\mathcal{M}$. □

By Theorem III.5 in [11] we may notice the following proposition I.7:

Proposition I.7. *Let $\mathbb{K}$ be spherically complete. Let $(a_j)_{j\in\mathbb{N}}$ be a coroner sequence such that $\prod_{n=0}^{\infty}|a_n|>0$. There exists $f\in A$ admitting each a_n as a zero of order* 1 *and having no other zeros.*

Proof. Let E be the divisor on D $(a_n,1)_{n\in\mathbb{N}}$ By Theorem III.16 in [11], there exists $f\in B$ such that $\mathcal{T}(f)\geq E$ and such that $|f|(r)\leq 2|E|(r)\ \forall r\in]0,1[$. But since $\prod_{n\in\mathbb{N}}|a_n|>0$, by Corollary II.8.1 in [11] $|E|(r)$ is bounded in $]0,1[$ and hence f belongs to A. Consequently, I is not null. □

Notation. Recall that we denote by U the disk $d(0,1)$. Considering the ring $U[x]$ of polynomials with coefficients in U, we denote by $\mathcal{H}$ be the family of ideals J of $U[x]$ such that $J\cap U\neq\{0\}$ and, given an integer $s\in\mathbb{N}^*$, let $\mathcal{H}_s$ be the set of $J\in\mathcal{H}$ generated by s elements. For every ideal $J\in\mathcal{H}$ we put $t(J)=\sup\{|x|\mid x\in J\cap U\}$ and $\ell(J)=\inf\{\sup_{f\in J}|f(x)|\mid x\in V\}$ and we denote by $u(J)$ the number such that $t(J)=\ell(J)^{u(J)}$. Finally, we put $m(s)=\sup\{u(J)\mid J\in\mathcal{J}_s\}$. Henceforth, given $f_1,...,f_s\in H(U)$ such that $\|f_i\|<1\ \forall i=1,...,s$, we set $w(f_1,...,f_s)=\inf\{\max_{1\leq i\leq s}|f_i(x)|\mid x\in U\}$. Moreover, given $f_1,...,f_s\in A$ we set $\lambda(f_1,...,f_s)=\inf\{\max_{1\leq i\leq s}|f_i(x)|\mid x\in D\}$.

Remark. Characterizing the coroner ultrafilters $\mathcal{U}$ such that $\mathcal{J}(\mathcal{U})$ is a maximal ideal appears very hard. For instance, consider an ultrafilter $\mathcal{U}$ thinner than $\mathcal{Y}$. It is a coroner ultrafilter. But $\mathcal{J}(\mathcal{U})=\{0\}$. Indeed, suppose a non-identically zero function f lies in $\mathcal{J}(\mathcal{U})$. Let (a_n) be its sequence of zeros, set $r_n=|a_n|$, $n\in\mathbb{N}$, and let $E=D\setminus\bigcup_{n=0}^{\infty}d(a_n,r_n^-)$. Clearly $|f(x)|=|f|(|x|)\ \forall x\in E$. However, E belongs to $\mathcal{Y}$ and therefore, $\mathcal{U}$ is secant with E, a contradiction with the hypothesis $f\in\mathcal{J}(\mathcal{U})$.

On the other hand, the mapping $\mathcal{J}$ from the set of coroner ultrafilters to the set of ideals of A is not injective: as noticed in [11], two contiguous coroner ultrafilters define the same ideal.

Thus, by Theorem III.3 in [11], if an element $\psi\in Mult(A,\|\ .\ \|)$ is neither the Gauss norm nor of the form $\varphi_{\mathcal{F}}$ on the whole set A, with $\mathcal{F}$ a circular filter on D of diameter $r<1$, then, its restriction to $\mathbb{K}[x]$ must be the Gauss norm on $\mathbb{K}[x]$. So its kernel is a prime closed ideal included in a maximal ideal of the form $\mathcal{J}(\mathcal{U})$, with $\mathcal{U}$ a coroner ultrafilter.

Here we shall first examine the problem of the continuation of $\varphi_{\mathcal{W}}$ to A through multiplicative norms, what was not done in [11].

Theorem I.8. *Let $\mathcal{U}$ be a regular ultrafilter. Then the ideal $\mathcal{M}=\mathcal{J}(\mathcal{U})$ is a maximal ideal of A and there exists $f\in\mathcal{M}$, having a sequence of zeros of order 1 and no other zeros, such that $f'\notin\mathcal{M}$.*

Proof. By Theorem III.4 in [11] we know that there exist sequences $(a_n)_{n\in\mathbb{N}}$ in D such that the sequence $\left|\frac{a_n}{a_{n+1}}\right|$ is strictly increasing and then the function $f(x)=\sum_{n=0}^{\infty}a_nx^n$ admits a sequence of zeros $(\alpha_n)_{n\in\mathbb{N}^*}$ satisfying $|\alpha_n|=\left|\frac{a_n}{a_{n+1}}\right|$.

Thus, particularly, if we set $r_n = \left|\frac{a_n}{a_{n+1}}\right|$ then f admits exactly a unique zero in each circle $C(0, r_n)$, each of order 1, and has no other zero in D. Consequently, by Lemma I.8 in [11], we can see that $|f'(\alpha_n)| = |f'|(r_n)\ \forall n \in \mathbb{N}^*$. Now, let $\mathcal{U}$ be an ultrafilter thinner than the sequence $(\alpha_n)_{n\in\mathbb{N}^*}$. We can check that the sequence (α_n) is regular, hence $\mathcal{U}$ is a regular ultrafilter. Consequently, by Corollary I.3.1 the ideal $\mathcal{M} = \mathcal{I}(\mathcal{U})$ is a maximal ideal of A. On the other hand, by lemma I.5, f' does not belong to $\mathcal{M}$. □

By Theorem I.8 and Corollary 7.2, we now have this corollary:

Corollary I.8.1. *A admits maximal ideals of infinite codimension.*

Following Corollary III.3.1 in [11], we can now complete the characterization of continuous multiplicative norms on A.

Theorem I.9. *Let $\psi \in Mult(A, \|\ .\ \|)$ be coroner. Then ψ is not a norm. Moreover, for every $f \in A$ such that $\psi(f) < \|f\|$, there exists $g \in Ker(\psi)$ admitting each zero of f as a zero of order superior or equal to its order as a zero of f.*

Proof. Suppose that ψ is a norm different from the Gauss norm $\|\ .\ \|$ on A. So, there exists a circular filter $\mathcal{F}$ on D, of diameter $r \le 1$ such that $\psi(P) = \varphi_{\mathcal{F}}(P)\ \forall P \in \mathbb{K}[x]$. But by Corollary III.13.1 in [11], we know that $r = 1$ and hence, the restriction of ψ to $\mathbb{K}[x]$ is the Gauss norm. Now, since ψ is not the Gauss norm on A, there exists $f \in A$ such that $\psi(f) < \|f\|$. Actually, without loss of generality, we can choose $f \in A$ such that $\psi(f) < 1 \le \|f\|$. Let $\rho = \psi(f)$. And, up to a change of origin, we can also assume that $f(0) \neq 0$. By Proposition II.4 in [11], f is not quasi-invertible, hence f has a sequence of zeros $(a_n)_{n\in\mathbb{N}}$ in D, with $|a_n| \le |a_{n+1}|$. For each $n \in \mathbb{N}$, let q_n be the multiplicity order of a_n. By Theorem II.8 in [11] we know that $\sum_{n=0}^{\infty} -q_n \log|a_n| < +\infty$. Now, there clearly exists a sequence t_n of strictly positive integers satisfying

$t_n \le t_{n+1},\ n \in \mathbb{N}$,
$\lim_{n\to\infty} t_n = +\infty$,
$\sum_{n=0}^{\infty} t_n q_n \log(|a_n|) < +\infty$.

By Theorem III.5 in [11] there exists a function $g \in A$ admitting each a_n as a zero of order $s_n \ge t_n q_n$, such that $|g|(|a_n|) \le 2\left|\prod_{k=0}^{n}\left(\frac{a_n}{a_k}\right)^{t_n q_n}\right|\ \forall n \in \mathbb{N}$ and consequently, g belongs to A.

Now, for each $n \in \mathbb{N}$ and for each $k = 0, \dots, n$, let $u_{n,k} = \max(0, t_n q_k - s_k)$ and let $P_n(x) = \prod_{k=0}^{n}(x - a_k)^{u_{n,k}}$. Clearly, all coefficients of P_n lie in D except the leading coefficient that is 1. Consequently, $\|P_n\| = 1\ \forall n \in \mathbb{N}$ and therefore

$$\|P_n g\| = \|g\|. \tag{1}$$

On the other hand, since the sequence t_n is increasing, we can check that for each fixed $n \in \mathbb{N}$, each zero a_k of f^{t_n} is a zero of $P_n g$ of order $\ge t_k q_k$. Consequently, by Lemma I.5 in [11] in the ring A we can write $P_n g$ in the form $f^{t_n}\sigma_n$, with $\sigma_n \in A$.

By (1), we have $\|\sigma_n\|\|f^{t_n}\| = \|g\|$ hence, since $\|f\| \geq 1$, we can see that $\|\sigma_n\| \leq \|g\|$. But now, since the restriction of ψ to $\mathbb{K}[x]$ is $\| \, . \, \|$, we have $\psi(P_n) = 1$, hence $\psi(P_n g) = \psi(P_n)\psi(g) = \psi(g)$ and therefore

$$\psi(g) = \psi(f^{t_n}\sigma_n) = \psi(f)^{t_n}\psi(\sigma_n) \leq \rho^{t_n}\|g\|. \tag{2}$$

Relation (2) holds for every $n \in \mathbb{N}$ hence $\lim_{n\to+\infty} \rho^{t_n}\|g\| = 0$. Consequently, $\psi(g) = 0$. This, ψ is not a norm. Moreover the function g we have constructed admits each zero of f as a zero of order superior or equal to its order as a zero of f. □

Notation. We will denote by $Mult_n(A, \| \, . \, \|)$ the set of continuous multiplicative norms of A.

By Theorems III.3 in [11], we now can also state Corollary I.9.1:

Corollary I.9.1. *Let $\psi \in Mult(A, \| \, . \, \|)$ be a norm. If ψ is not $\| \, . \, \|$, there exists a circular filter $\mathcal{F}$ on D, of diameter $r < 1$, such that $\psi = \varphi_{\mathcal{F}}$.*

On the other hand, each coroner maximal ideal is the kernel of some coroner continuous multiplicative semi-norm of A.

Concerning the Corona Problem, we may notice this:

Corollary I.9.2. *The set of norms in $Mult(A, \| \, . \, \|)$ is included in the closure of $Mult_1(A, \| \, . \, \|)$.*

Theorem I.10. *Let G be the set of circular filters on D of diameter $r \in]0,1[$. For every circular filter $\mathcal{F} \in G$, $\varphi_{\mathcal{F}}$ is a multiplicative norm on A and the mapping from G into $Mult(A, \| \, . \, \|)$ that associates to each $\mathcal{F} \in \Phi(0,1)$ the norm $\varphi_{\mathcal{F}}$ is a bijection from G onto the set of norms of $Mult(A, \| \, . \, \|) \setminus \{\| \, . \, \|\}$.*

Proof. By Corollary III.13.1 in [11], we know that for each $\mathcal{F} \in G$, if $0 < \mathrm{diam}(\mathcal{F}) < 1$, $\varphi_{\mathcal{F}}$ has extension to a multiplicative norm on A and the mapping G into $Mult(A, \| \, . \, \|)$ that associates to each $\mathcal{F} \in G$ the norm $\varphi_{\mathcal{F}}$ is obviously injective because its restriction to $H(D)$ is already injective.

Now, consider a multiplicative semi-norm ϕ on A which is not of the form $\varphi_{\mathcal{F}}$ with $\mathcal{F}$ a circular filter of diameter $r < 1$. Then, if ϕ is not $\| \, . \, \|$, ϕ is coroner, therefore by Corollary I.19.1, it is not a norm. Consequently, the mapping from G into $Mult(A, \| \, . \, \|)$ that associates to each $\mathcal{F} \in G$ the norm $\varphi_{\mathcal{F}}$ is a surjection from G onto the set of multiplicative norms of A other than $\| \, . \, \|$. □

Thus, Theorem I.10 lets us characterize the continuous multiplicative norms of A, which we can summarize in this way:

Corollary I.10.1. *Let $\phi \in Mult(A, \| \, . \, \|)$ be different from $\| \, . \, \|$. Then ϕ is a norm if and only if it is of the form $\varphi_{\mathcal{F}}$ with $\mathcal{F}$ a circular filter of diameter $r \in]0,1[$.*

II. Maximal ideal of infinite codimension

Now we will consider the question whether $Mult_1(A, \| \, . \, \|)$ is dense in $Mult_m(A, \| \, . \, \|)$. We have to prove many intermediate results.

The following Proposition II.2 is also called Corona Statement in dimension 1. However, on a non-archimedean field, it is not at all proven to be equivalent to a property of density for maximal ideals defined by points of D inside the set $Mult(A, \| \,.\, \|)$. We will prove Proposition II.2 in the same way as in [15]. Proposition II.1 is proven in [15] and is indispensable for further results.

Notation. Recall that we denote by U the disk $d(0,1)$. Let $\mathcal{H}$ be the family of ideals J of $U[x]$ such that $J \cap U \neq \{0\}$ and, given an integer $s \in \mathbb{N}^*$, let $\mathcal{H}_s$ be the set of $J \in \mathcal{H}$ generated by s elements. For every ideal $J \in \mathcal{H}$ we put $t(J) = \sup\{|x| \mid x \in J \cap U\}$ and $\ell(J) = \inf\{\sup_{f \in J} |f(x)| \mid x \in V\}$ and we denote by $u(J)$ the number such that $t(J) = \ell(J)^{u(J)}$. Finally, we put $m(s) = \sup\{u(J) \mid J \in \mathcal{J}_s\}$. Henceforth, given $f_1, \dots, f_s \in H(U)$ such that $\|f_i\| < 1 \ \forall i = 1, \dots, s$, we set $w(f_1, \dots, f_s) = \inf\{\max_{1 \le i \le s} |f_i(x)| \mid x \in U\}$. Moreover, given $f_1, \dots, f_s \in A$ we set $\lambda(f_1, \dots, f_s) = \inf\{\max_{1 \le i \le s} |f_i(x)| \mid x \in D\}$.

Proposition II.1. *Let J be a finitely generated ideal of $U[x]$ such that $J \cap U \neq \{0\}$. Then $m(s) = 2 \ \forall s \ge 2$.*

In the proof of Proposition II.2, we will follow a similar way as in [15].

Proposition II.2. *Let $s \in \mathbb{N}^*$. For any $f_1, \dots, f_s \in A$ satisfying $\|f_i\| < 1$ $(1 \le i \le s)$ and $\lambda(f_1, \dots, f_s) > 0$. There exist $g_1, \dots, g_s \in A$ satisfying $\sum_{i=1}^s f_i g_i = 1$ and $\|g_i\| < \lambda(f_1, \dots, f_s))^{-2}$.*

Proof. As a first step, we will prove that for any $f_1, \dots, f_s \in H(U)$ satisfying $\|f_i\| < 1 \ \forall i = 1, \dots, s$ and $w(f_1, \dots, f_s) > 0$, there exist $g_1, \dots, g_s \in H(d(0,R))$ satisfying $\sum_{i=1}^s f_i g_i = 1$ and $\|g_i\| < (w(f_1, \dots, f_s))^{-2} \ \forall i = 1, \dots, s$.

Since $\mathbb{K}[x]$ is dense in $H(U)$, we can find polynomials $P_1, \dots, P_s \in \mathbb{K}[x]$ such that $\|P_i - f_i\| \le (w(f_1, \dots, f_s))^2 \ \forall i = 1, \dots, s$. Since $w(f_1, \dots, f_s) > 0$, by Corollary III.3.1 in [11], there exists no maximal ideal $\mathcal{M}$ of $H(U)$ containing the ideal generated by $f_1, \dots, f_s$. Consequently, there exist $g_1, \dots, g_s \in H(U)$ such that $\sum_{i=1}^s g_i f_i = 1$. Therefore, we can define $h_1, \dots, h_s \in H(U)$ such that $\|h_i\| \le 1 \ \forall i = 1, \dots, s$ and such that $\sum_{i=1}^s h_i f_i$ is an element $P_0 \in U$. Let I be the ideal of $U[x]$ generated by $P_0, P_1, \dots, P_s$. Since $I \cap U \neq \{0\}$, we have $0 < t(I) < \ell(I)$. Consequently, by Proposition II.1 we can find $Q_0, Q_1, \dots, Q_s \in U[x]$ such that $\sum_{i=0}^s Q_i P_i$ be an element $a \in U$ satisfying $|a| \ge (w(f_1, \dots, f_s))^{m(s)} = (w(f_1, \dots, f_s))^2$. Then

$$\sum_{i=1}^{s} a^{-1} f_i + a^{-1} Q_0 \sum_{i=1}^{s} h_i f_i = 1 + \sum_{i=1}^{s} a^{-1} Q_i (f_i - P_i).$$

By construction, we have $\|\sum_{i=1}^s a^{-1} Q_i (f_i - P_i)\| < 1$ and hence $1 + \sum_{i=1}^s a^{-1} Q_i (f_i - P_i)$ is an invertible element u in $H(U)$. So, $\sum_{i=1}^s u^{-1}\big(a^{-1}(Q_i + Q_0 h_i)\big) f_i = 1$ and hence for every $i = 1, \dots, s$, we have $\|u^{-1} a^{-1}(Q_i + Q_0 h_i)\| \le (w(f_1, \dots, f_s))^{-2}$ $\forall i = 1, \dots, s$. Thus, putting $g_i = u^{-1}\big(a^{-1}(Q_i + Q_0 h_i)\big)$, $i = 1, \dots, s$ we have proven that for any $f_1, \dots, f_s \in H(U)$ satisfying $\|f_i\| < 1 \ \forall i = 1, \dots, s$ and $w(f_1, \dots, f_s) > 0$, there exist $g_1,\dots,g_s \in H(U)$ satisfying $\sum_{i=1}^{s} f_i g_i = 1$ and $\|g_i\| < (w(f_1,\dots,f_s))^{-2}$ $\forall i = 1,\dots,s$.

Now, let us prove the conclusion of Proposition II.2. So, we take $f_1,\dots,f_s \in A$.

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{K}$ such that $0 < |u_n| < |u_{n+1}| < \dots < 1$ and $\lim_{n\to+\infty} |u_n|^n = 1$. For each $i = 1,\dots,s$, set $f_{i,n}(x) = f_i(u_n x)$. Since each u_n belongs to D, each $f_{i,n}$ is a power series of radius $r > 1$ and hence belongs to $H(U)$. Then by the claim we have just proven, for each $n \in \mathbb{N}$, there are $g_{i,n} \in H(U)$ such that $\sum_{i=1}^{s} g_{i,n} f_{i,n} = 1$ and $\|g_{i,n}\| < (w(f_{1,n},\dots,f_{s,n}))^{-2}$. Now, each $g_{i,n}$ is a power series $\sum_{k=0}^{+\infty} g_{i,n,k} x^k$. Let $h_{i,n}$ be the power series $\sum_{k=0}^{2n} g_{i,n,k}(u_n)^{-k} x^k$. Then we have $\|h_{i,n}\| < |u_n|^{-2n}(w(f_1,\dots,f_s))^{-2}$ and hence $\sum_{i=1}^{s} h_{i,n} f_i$ is of the form $1 + x^n t_n(x)$ with $t_n \in H(U)$.

We will get to a conclusion thanks to a Banach process. Let $\mathcal{E}$ be the Banach space of bounded sequences in $\mathbb{K}$ provided with the classic norm $\|(a_n)_{n\in\mathbb{N}}\|_0 = \sup_{n\in\mathbb{N}} |a_n|$ and let $\mathcal{C}$ be the closed sub-space of converging sequences. For every sequence $(a_n)_{n\in\mathbb{N}} \in \mathcal{C}$, we put $\mathcal{L}((a_n)) = \lim_{n\to+\infty} a_n$.

Suppose first that $\mathbb{K}$ is spherically complete. There exists a linear map $\widetilde{\mathcal{L}}$ of norm 1 from $\mathcal{E}$ to $\mathbb{K}$ expanding $\mathcal{L}$ to $\mathcal{E}$. Set $h_{i,n} = \sum_{k=0}^{+\infty} x^k$ and let $l_{i,k}$ be the sequence $(h_{i,n,k})_{1\le n}$. For each pair (i,k), we can now define $g_{i,k} = \widetilde{\mathcal{L}}(l_{i,k})$ and put $g_i = \sum_{k=0}^{+\infty} g_{i,k} x^k$. Since $\widetilde{\mathcal{L}}$ is of norm 1, each sequence $g_{i,k}$ is bounded and hence g_i belongs to A. Moreover, by construction, g_i satisfies $\|g_i\| < (\lambda(f_1,\dots,f_s))^{-2}$ $\forall i = 1,\dots,s$. Now, we have

$$\sum_{i=1}^{s} g_i f_i = \sum_{k=0}^{+\infty} \Big(\sum_{i=1}^{s} (\sum_{j=0}^{k} \widetilde{\mathcal{L}}(l_{i,j}) f_{i,k-j} \Big) x^k \tag{1}$$

and for each fixed $k \in \mathbb{N}$,

$$\sum_{i=1}^{s} \Big(\sum_{j=0}^{k} \widetilde{\mathcal{L}}(l_{i,j}) f_{i,k-j} \Big) = \widetilde{\mathcal{L}} \Big(\sum_{i=1}^{s} (\sum_{j=0}^{k} f_{i,j} h_{i,n,k-j}) \Big).$$

Consequently, since $\sum_{i=1}^{s} h_{i,n} f_i = 1 + x^n t_n(x)$, we can check that

$$\lim_{n\to+\infty} \Big(\textstyle\sum_{i=1}^{s} (\sum_{j=0}^{k} f_{i,j} h_{i,n,k-j}) \Big) = 1 \text{ whenever } k = 0$$

and

$$\lim_{n\to+\infty} \Big(\textstyle\sum_{i=1}^{s} (\sum_{j=0}^{k} f_{i,j} h_{i,n,k-j}) \Big) = 0 \text{ whenever } k \neq 0.$$

Consequently, since $\widetilde{\mathcal{L}}$ extends to $\mathcal{E}$ the limit on $\mathcal{C}$, we have

$$\widetilde{\mathcal{L}} \Big(\textstyle\sum_{i=1}^{s} (\sum_{j=0}^{k} f_{i,j} h_{i,n,k-j}) \Big) = 1 \text{ whenever } k = 0$$

and

$$\widetilde{\mathcal{L}} \Big(\textstyle\sum_{i=1}^{s} (\sum_{j=0}^{k} f_{i,j} h_{i,n,k-j}) \Big) = 0 \text{ whenever } k \neq 0.$$

Therefore by (1) we obtain $\sum_{i=1}^{s} g_i f_i = 1$.

Consider now the general case when $\mathbb{K}$ is no longer supposed to be spherically complete. Consider a spherically complete algebraically closed extension $\widehat{\mathbb{K}}$ of

$\mathbb{K}$. Let $\widehat{D}$ be the disk of $\widehat{K}$: $\{x \in \widehat{K} \mid |x| < 1\}$ and let $\widehat{A}$ be the algebra of bounded analytic functions from $\widehat{D}$ to $\widehat{K}$. Set $B = \lambda(f_1, ..., f_s))^{-2}$. Then by what forgoes, there exist $h_1, \dots, h_s \in \widehat{A}$ such that $\sum_{j=1}^{s} h_j f_i = 1$ and $\max_{1 \le j \le s} \|h_j\| < B$.

Inside $\widehat{A}$, let F be the closed subspace of the $\mathbb{K}$-Banach space generated by 1 and all coefficients of all the h_j. Let us take $\varepsilon > 0$ such that $(1+\varepsilon)\max_{1\le j\le s}\|h_j\| \le B$. Since F is a $\mathbb{K}$-Banach space of countable type, there exists a $\mathbb{K}$-linear map ℓ from F to $\mathbb{K}$ satisfying $\ell(1) = 1$ with a norm $\| \,.\, \|^*$ satisfying $\|\ell\|^* \le 1+\varepsilon$. Let T be the closed subspace of $\widehat{A}$ consisting of the power series with coefficients in F. Then T is a A-module and then we have an extension $\mathcal{L}$ of ℓ from T to A defined as $\mathcal{L}(\sum_{k=0}^{\infty} e_k x^k) = \sum_{k=0}^{\infty} \ell(e_k) x^k$ which is A-linear and its norm satisfies $\|\mathcal{L}\|^* \le 1+\varepsilon$. Now, putting $g_j = \mathcal{L}(h_j),\ 1 \le j \le s$, we have $\sum_{i=1}^{s} f_i g_i = 1$ and $\max_{1\le j\le s}\|g_j\| < B$.

Notation. Let I be an ideal of A. For each $f \in A$ and for each $\varepsilon > 0$, we set $E(f,\varepsilon) = \{x \in D \mid |f(x)| \le \varepsilon\}$.

Corollary II.2.1. *Let I be a proper ideal of A. The family $(E(f,\varepsilon),\ f \in I,\ \varepsilon > 0)$ generates a filter $\mathcal{F}$ on D such that $I \subset \mathcal{J}(\mathcal{F})$.*

Proof. Let $(E(f_j,\varepsilon_j),\ 1 \le j \le n$ be such that $\bigcap_{j=1}^{n} E(f,\varepsilon_j) = \emptyset$. Let $\varepsilon = \min_{1\le j\le n}(\varepsilon_j)$ and let $F_j = D \setminus E(f_j,\varepsilon),\ 1 \le j \le n$.

Then $\bigcup_{j=1}^{n} F_j = D$, hence by Theorem II.2, $1 \in I$, a contradiction. □

By Lemma I.5 and Corollary II.2.1, we can derive Corollary II.2.2:

Corollary II.2.2. *Let $\mathcal{M}$ be a maximal ideal of A. Then there exists an ultrafilter $\mathcal{U}$ on D such that $\mathcal{M} = \mathcal{J}(\mathcal{U})$. Moreover, if $\mathcal{U}$ converges, $\mathcal{M}$ is of codimension 1 and of the form $(x-a)A$, $a \in D$. If $\mathcal{U}$ does not converge, it is coroner, $\mathcal{M}$ is of infinite codimension and for every $f \in \mathcal{M}$, f is not quasi-invertible.*

III. Density of $Mult_1(A, \| \,.\, \|)$ in $Mult_m(A, \| \,.\, \|)$

Let us recall the definition of an increasing pierced filter [5].

Definition. *Let $a \in \mathbb{K}$, let $R > 0$ and consider the filter $\mathcal{G}$ admitting for basis the annuli $\Gamma(a,r,R)$ $r \in]0,R[$. This filter is called increasing filter of center a and diameter R. Moreover, if all F is an infraconnected subset of $\mathbb{K}$ and if every annulus $\Gamma(a,r,R)$ contain holes of F, then the filter on F induced by $\mathcal{G}$ is said to be pierced. Next, a pierced filter is called a T-filter if its holes satisfy a certain relation [5].*

By using properties of T-filters and particularly idempotent T-sequences [14], by Lemma 35.1 and Proposition 37.1 in [5], Proposition 1.6 and [15]), we have the following proposition:

Proposition III.1. *Let $(r_n)_{n\in\mathbb{N}}$ be a sequence in $|\mathbb{K}|$ such that $0 < r_n < r_{n+1}$, $\lim_{n\to+\infty} r_n = 1$, let $(q_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{N}$ such that $q_n \le q_{n+1}$ and $\lim_{n\to+\infty}\left(\frac{r_n}{r_{n+1}}\right)^{q_n} = 0$. Let $l \in]0,1[$ and for each $n \in \mathbb{N}$, let $b_n \in C(0,(r_n)^{q_n})$, let*

$a_{n,1},\dots,a_{n,q_n}$ *be the q_n-th roots of b_n and let* $F = D \setminus \left(\bigcup_{n\in\mathbb{N}}(\bigcup_{j=1}^{q_n} d(a_{n,j},\, l^-)\right)$. *Set* $f_n(x) = \prod_{k=1}^{n}\prod_{j=1}^{q_k}\left(\frac{1}{1-\frac{x}{a_{k,j}}}\right)$. *Then each f_n belongs to $R(F)$ and the sequence $(f_n)_{n\in\mathbb{N}}$ converges in $H(F)$ to an element f strictly vanishing along the pierced increasing filter of center* 0 *and diameter* 1.

Notation. Given a unital commutative ultrametric $\mathbb{K}$-algebra B and $f \in B$, we denote by $sp(f)$ the spectrum of f i.e. the set of $x \in \mathbb{K}$ such that $f - x$ is not invertible in B.

Proposition III.2. *Let $(B, \|\ .\ \|)$ be a unital commutative ultrametric Banach $\mathbb{K}$-algebra. Suppose there exist $\ell \in B$, $\phi,\ \psi \in Mult(B, \|\ .\ \|)$ such that $\psi(\ell) < \phi(\ell)$, $sp(\ell) \cap \Gamma(0, \psi(\ell), \phi(\ell)) = \emptyset$ and there exists $\varepsilon \in]0, \phi(\ell) - \psi(\ell)[$ satisfying further $\|(\ell - a)^{-1}\| \le M\ \forall a \in \Gamma(0, \psi(\ell), \phi(\ell) - \varepsilon)$. Then there exists $f \in B$ such that $\psi(f) = 1$, $\phi(f) = 0$.*

Proof. Let $s = \psi(\ell)$, $t = \phi(\ell)$, $Q = \|\ell\|$, $R = t - \varepsilon$ and $l = \frac{1}{M}$. Let $r_0 \in]s, t - \varepsilon[$. Consider the sequence $(a_{n,j})_{n\in\mathbb{N}, 1\le j\le q_n}$ defined in Proposition III.1 and the set $E = d(0, Q^-) \setminus \left(\bigcup_{n\in\mathbb{N}}\left(\bigcup_{j=1}^{q_n} d(a_{n,j},\, l^-)\right)\right)$. Then in $H(E)$ we have

$$\Big\|\frac{1}{x-b}\Big\|_E \le l\ \forall b \in \bigcup_{n\in\mathbb{N}}\big(\bigcup_{j=1}^{q_n} d(a_{n,j},\, l^-)\big). \tag{1}$$

There exists a natural homomorphism θ from $R(E)$ into B such that $\theta(x) = \ell$. Since $Q = \|\ell\|$ and $\|(\ell - b)^{-1}\| \le M\ \forall b \in \Gamma(0, s, t)$, by Theorem I.18 in [11] and by (1) θ is clearly continuous with respect to the norms $\|\ .\ \|_E$ of $R(E)$ and $\|\ .\ \|$ of B. Consequently, θ has continuation to a continuous homomorphism from $H(E)$ to B.

Now, let $\psi' = \psi \circ \theta$, $\phi' = \phi \circ \theta$. Then both ϕ', ψ' belong to $Mult(H(E), \|\ .\ \|)$ and satisfy $\psi'(x) = s$, $\phi'(x) = t - \varepsilon$. So, ψ' is of the form $\varphi_{\mathcal{F}}$ with $\mathcal{F}$ a circular filter on E secant with $C(0, s)$ and ϕ' is of the form $\varphi_{\mathcal{G}}$ with $\mathcal{G}$ a circular filter on E secant with $C(0, t)$.

Consider now the function f constructed in Proposition III.1 which, by construction, belongs to $H(E)$ and has no zero and no pole in $d(0, s^-)$. Consequently, $|f(x)| = |f(0)| = 1\ \forall x \in d(0, s^-)$. Moreover, we have $\lim_{\mathcal{G}} f(x) = 0$, hence $\phi'(f) = 0$. Let $g = \theta(f)$. Then $\psi(g) = \psi'(f) = 1$ and $\phi(g) = \phi'(f) = 0$, which ends the proof. □

Proposition III.3. *Let $\mathcal{M}$ be a non-principal maximal ideal of A and let $\mathcal{U}$ be an ultrafilter thinner than $\mathcal{G}_{\mathcal{M}}$. Then $\varphi_{\mathcal{U}}$ belongs to the closure of $Mult_1(A, \|.\|)$ in $Mult_m(A, \|.\|)$.*

Proof. Let V be neighborhood of $\varphi_{\mathcal{U}}$ in $Mult(A, \|.\|)$. It contains some set of the form $\mathcal{V}(\varphi_{\mathcal{U}}, f_1, \dots, f_q, \varepsilon)$, where $f_1, \dots, f_q \in A$ and $\varepsilon > 0$, with respect to the topology of pointwise convergence i.e.

$\mathcal{V}(\varphi_r, f_1, ..., f_q, \varepsilon) = \{\phi \in Mult(A, \|\,.\,\|) \mid \big|\varphi_r(f_j) - \phi x f_j)\big|_\infty \le \varepsilon,\ j = 1, ..., q,\ q \in \mathbb{N}^*\}$. For each $j = 1, ..., q$, there exists $E_j \in \mathcal{U}$ such that $\big|\,|f_j(x)| - \varphi_{\mathcal{U}}(f_j)\big|_\infty \le \varepsilon$ $\forall x \in E_j$. Let $E = \bigcap_{j=1}^{q} E_j$. Then

$$\big|\ |f_j(x)| - \varphi_{\mathcal{U}}(f_j)\ \big|_\infty \le \varepsilon\ \forall x \in E, \forall j = 1, ..., q.$$

Consequently, φ_a belongs to $\mathcal{V}(\varphi_{\mathcal{U}}, f_1, ..., f_q, \varepsilon)$ for all $a \in E$. □

Corollary III.3.1. *Let $\mathcal{M}$ be a univalent non-principal maximal ideal of A and let $\phi \in Mult_m(A, \|\,.\,\|)$ satisfy $Ker(\phi) = \mathcal{M}$. Then ϕ is of the form $\phi(f) = \lim_{\mathcal{U}} |f(x)|$ with $\mathcal{U}$ a coroner ultrafilter such that $\mathcal{I}(\mathcal{U}) = \mathcal{M}$. Moreover, ϕ belongs to the closure of $Mult_1(A, \|\,.\,\|)$ in $Mult_m(A, \|\,.\,\|)$.*

Proposition III.4. *Let $\mathcal{U}$ be a coroner ultrafilter on D, let $f \in A \setminus \mathcal{I}(\mathcal{U})$ be non-invertible in A, such that $\|f\| \le 1$ and let $g \in A$, $h \in \mathcal{I}(\mathcal{U})$ be such that $fg = 1 + h$. Let $\tau = \varphi_{\mathcal{U}}(f)$, let $\varepsilon \in]0, \tau[$ and let $\Lambda = \{x \in D \ \Big|\ |f(x)g(x)| - 1\Big|_\infty < \varepsilon,\ \Big|\ |f(x)| - \tau\Big|_\infty < \varepsilon\}$.*

Suppose that there exist a function $\widetilde{h} \in A$ admitting for zeros in D the zeros of h in $D \setminus \Lambda$ and a function $\overline{h} \in A$ admitting for zeros the zeros of h in Λ, each counting multiplicities, so that $h = \overline{h}\widetilde{h}$. Then $|\widetilde{h}(x)|$ has a strictly positive lower bound in Λ and $\overline{h}$ belongs to $\mathcal{I}(\mathcal{U})$.

Moreover, there exists $\omega \in]0, \tau[$ such that $\omega \le \inf\{\max(|f(x)|, |\overline{h}(x)|) \mid x \in D\}$. Further, for every $a \in d(0, (\tau - \varepsilon))$, we have $\omega \le \inf\{\max(|f(x) - a|, |\overline{h}(x)|) \mid x \in D\}$.

Proof. Let $u \in \Lambda$ and let s be the distance of u from $\mathbb{K} \setminus \Lambda$. So, the disk $d(u, s^-)$ is included in Λ, hence fg has no zero inside this disk. Consequently, $|f(x)g(x)|$ is a constant b in $d(u, s^-)$. Consider the family F_u of radii of circles $C(u, r)$, containing at least one zero of fg. By Theorem I.4 in [11], F_u has no cluster point different from 1. Consequently, there exists $\rho \ge s$ such that fg admits at least one zero in $C(u, \rho)$ and admits no zero in $d(u, \rho^-)$. And then $|f(x)g(x)|$ is a constant c in $d(u, \rho^-)$. But then, at u we see that $b = c$ and therefore $d(u, \rho^-)$ is included in Λ. Hence $\rho = s$ and therefore fg admits at least one zero α in $C(u, s)$. Thus, at α we have $h(\alpha) = -1$. Therefore, in the disk $d(\alpha, s^-)$ we can check that $\varphi_{\alpha,s}(h) \ge 1$. But by Theorem I.9 in [11], we have $\varphi_{\alpha,s}(h) = \varphi_{u,s}(h)$, hence $\varphi_{u,s}(h) \ge 1$.

Now,

$$\frac{\|h\|}{\varphi_{u,s}(h)} = \frac{\|\widetilde{h}\|}{\varphi_{u,s}(\widetilde{h})} \frac{\|\overline{h}\|}{\varphi_{u,s}(\overline{h})} \ge \frac{\|\widetilde{h}\|}{\varphi_{u,s}(\widetilde{h})}.$$

Therefore, since $\varphi_{u,s}(h) \ge 1$, we obtain

$$\frac{\|\widetilde{h}\|}{\varphi_{u,s}(\widetilde{h})} \le \|h\|. \qquad (1)$$

But since by definition $d(u,s^-)$ is included in Λ, $\widetilde{h}$ has no zero in this disk, hence $|\widetilde{h}(x)|$ is constant and equal to $\varphi_{u,s}(\widetilde{h})$. Consequently, by (1) we obtain $\frac{\|\widetilde{h}\|}{|\widetilde{h}(u)|} \leq \|h\|$ and therefore

$$|\widetilde{h}(u)| \geq \frac{\|\widetilde{h}\|}{\|h\|} \ \forall u \in \Lambda.$$

This shows that $\widetilde{h}$ does not belong to $\mathcal{J}(\mathcal{U})$, hence, $\varphi_{\mathcal{U}}(\widetilde{h}) \neq 0$. Consequently, $\varphi_{\mathcal{U}}(\overline{h}) = 0$.

Now, by hypothesis, we have $fg - \overline{h}\widetilde{h} = 1$. Since both g, $\widetilde{h}$ belong to A and therefore are bounded in D, it is obvious that $\inf\{\max(|f(x)|, |\overline{h}(x)|) \mid x \in D\} > 0$. So, we may obviously choose $\omega \in]0, \tau - \varepsilon[$ such that

$$\omega \leq \inf\{\max(|f(x)|, |\overline{h}(x)|) \mid x \in D\}. \qquad (2)$$

Let us now show that for every $a \in d(0, (\tau - \varepsilon))$, we have $\omega \leq \inf\{\max(|f(x) - a|, |\overline{h}(x)|) \mid x \in D\}$.

Let $\Lambda' = \{x \in D \mid \ |f(x)| \geq \tau - \varepsilon\}$ and let $a \in d(0, (\tau - \varepsilon)^-)$. When β lies in Λ', we have $|f(\beta)| > |a|$, hence by (2), $\max(|f(\beta) - a|, |\overline{h}(\beta)|) \geq \omega$ because by(2), either $\omega \leq |\overline{h}(\beta)|$, or $\omega \leq |f(\beta)| = |f(\beta) - a|$.

Now, let β lie in $D \setminus \Lambda'$ and let t be the distance from β to Λ'. Since $D \setminus \Lambda'$ is open, t is > 0. Consider $\varphi_{\beta,t}(f)$. Either there exists $\mu \in \Lambda'$ such that $|\beta - \mu| = t$ and then $\varphi_{\beta,t}(f) \geq |f(\mu)| \geq \tau - \varepsilon$, or there exists a sequence $(x_n)_{n \in \mathbb{N}} \in \Lambda'$ such that $\lim_{n \to +\infty} |\beta - x_n| = t$ and $|x_n - \beta| > t$. Suppose that we are in the second case: there exists a sequence $(x_n)_{n \in \mathbb{N}} \in \Lambda'$ such that $\lim_{n \to +\infty} |\beta - x_n| = t$ and $|x_n - \beta| > t$. Then the sequence is thinner than the circular filter of center β and diameter t, hence

$$\lim_{n \to +\infty} |f(x_n)| = \varphi_{\beta,t}(f)$$

hence $\varphi_{\beta,t}(f) \geq \tau - \varepsilon$ again. If f has no zero in $d(\beta, t^-)$, then $|f(x)|$ is a constant in that disk, hence of course $\varphi_{\beta,t}(f) < \tau - \varepsilon$, a contradiction. Consequently, f must have a zero γ in $d(\beta, t^-)$. Therefore, due to (2), we have $|\overline{h}(\gamma)| \geq \omega$. But since by definition, $\Lambda \subset \Lambda'$, the zeros of $\overline{h}$ belong to Λ'. And since $d(\beta, t^-) \cap \Lambda' = \emptyset$ actually $\overline{h}$ has no zero in $d(\beta, t^-)$. Consequently $|\overline{h}(x)|$ is constant in $d(\beta, t^-)$ and hence $|\overline{h}(\beta)| \geq \omega$, which completes the proof. $\square$

The following basic Proposition is easily checked and is an application of Proposition 10 in [3]:

Proposition III.5. *Let S be a set and let E be a subset. Let $\mathcal{F}$ be an ultrafilter on E. Then the filter $\widetilde{\mathcal{F}}$ on S with basis $\mathcal{F}$ is an ultrafiter inducing on E the ultrafilter $\mathcal{F}$.*

Corollary III.5.1. *Let S be a set and let E be subset of S. Let $\mathcal{F}$ be an ultrafilter on E and let $\widehat{\mathcal{F}} = \mathcal{G}$ be the ultrafilter on S having $\mathcal{F}$ as a basis of filter. Let f be a function defined on S with values in a compact topological space T.*

Then $\lim_{\mathcal{G}} f(x) = \lim_{\mathcal{F}} f(x)$.

Proof. Suppose that f admits distinct limits on $\mathcal{F}$ and $\mathcal{G}$. Then $\mathcal{F}$ is a basis of a filter on S that is not secant with $\mathcal{G}$, a contradiction since $\mathcal{F}$ is the ultrafilter induced by $\mathcal{G}$ on E. □

We can now prove Proposition III.6 in the general context of a field that is not supposed to be spherically complete:

Proposition III.6. *Let $\mathcal{M}$ be a non-principal maximal ideal of A and let $\mathcal{U}$ be an ultrafilter on D such that $\mathcal{M} = \mathcal{J}(\mathcal{U})$. Let $f \in A \setminus \mathcal{M}$ satisfy $\|f\| < 1$, let $\tau = \varphi_{\mathcal{U}}(f)$ and let $\varepsilon \in]0,\tau[$. There exists $c > 0$ such that, for every $a \in d(0,\tau-\varepsilon)$, there exists $g_a \in A$ satisfying $(f-a)g_a - 1 \in \mathcal{M}$ and $\|g_a\| \leq c$.*

Proof. Suppose first that f is invertible in A. By Theorem I.4 in [11], $|f(x)|$ is a constant and hence is equal to τ. Therefore, $|f(x)-a| = \tau \ \forall a \in d(0,\tau-\varepsilon)$. Consequently, $f-a$ is invertible and its inverse g_a satisfies $\|g_a\| = \tau^{-1}$. Thus, we only have to show the claim when f is not invertible. Since f does not belong to $\mathcal{M}$, we can find $g \in A$ and $h \in \mathcal{M}$ such that $fg = 1+h$ with $h \in \mathcal{M}$.

We know that there exists be an algebraically closed spherically complete extension $\widehat{\mathbb{K}}$ of $\mathbb{K}$ (Theorems 7.4 and 7.6 in [11]). Let $\widehat{D}$ be the disk $\{x \in \widehat{\mathbb{K}} \mid |x| < 1\}$. Let $\widehat{A}$ be the algebra of bounded power series converging in $\widehat{D}$ with coefficients in $\widehat{\mathbb{K}}$.$\mathcal{U}$ makes a basis of a filter $\widehat{\mathcal{U}}$ on $\widehat{D}$ and by definition, $\mathcal{U}$ is the the filter induced by $\widehat{\mathcal{U}}$ on D. By Corollary III.5.1 $\widehat{\mathcal{U}}$ is an ultrafilter on $\widehat{D}$. Consider now f as an element of $\widehat{A}$. Then $\widehat{\mathcal{U}}$ defines an element ψ of $Mult(\widehat{A},\|\,.\,\|)$ as $\psi(\ell) = \lim_{\widehat{\mathcal{U}}} |\ell(x)|, \forall \ell \in \widehat{A}$. Consequently, by Corollary III.5.1 τ is equal to $\lim_{\mathcal{U}} |f(x)|$. Let

$$\Lambda = \{x \in \widehat{D} \;\big|\; |\,|f(x)g(x)|-1|_\infty < \varepsilon,\ |\,|f(x)|-\tau|_\infty < \varepsilon\}.$$

Since $\widehat{\mathbb{K}}$ is spherically complete, by Proposition I.7. we can factorize h in the form $\widetilde{h}\overline{h}$ where $\widetilde{h} \in \widehat{A}$ is a function admitting for zeros in $\widehat{D}$ the zeros of h in $\widehat{D} \setminus \Lambda$ and $\overline{h} \in \widehat{A}$ is a function admitting for zeros the zeros of h in Λ, each counting multiplicities. Moreover, we can choose $\overline{h}$ so that $\|\overline{h}\| < 1$. Now, in the field $\widehat{\mathbb{K}}$, by Proposition III.4, there exists $\omega > 0$ such that for every $a \in \widehat{d}(0,(\tau-\varepsilon))$, we have $\omega \leq \inf\{\max(|f(x)-a|, |\overline{h}(x)|) \;\big|\; x \in \widehat{D}\}$. This implies that $\inf\{\max(|f(x)-a|, |\overline{h}(x)|) \ |x \in D\} \geq \omega \ \forall a \in \widehat{d}(0,\tau-\varepsilon)$. We notice that $\|f-a\| < 1$ for every $a \in \widehat{d}(0,\tau-\varepsilon)$, so we may apply Proposition II.2 and obtain a bound b only depending on f and h and functions $\ell_a,\ h_a \in \widehat{A}$ such that $(f-a)\ell_a + \overline{h}h_a = 1$, with

$$\|\ell_a\| < b,\ \|h_a\| < b \ \forall a \in \widehat{d}(0,\tau-\varepsilon). \tag{1}$$

By hypothesis we have $\lim_{\mathcal{U}} h(x) = 0$. Hence by Corollary III.5.1, on $\widehat{D}$ we have $\lim_{\widehat{\mathcal{U}}} h(x) = 0$. Then, by Corollary III.5.1 we have $\lim_{\widehat{\mathcal{U}}} \overline{h}(x) = 0$ hence, on D,

$$\lim_{\mathcal{U}} \overline{h}h_a(x) = 0 \ \forall a \in d(0,\tau-\varepsilon). \tag{2}$$

Now, let us fix $a \in d(0,\tau-\varepsilon)$. Let G be the closed $\mathbb{K}$-vector subspace of $\widehat{\mathbb{K}}$ (considered as a $\mathbb{K}$-Banach space), linearly generated over $\mathbb{K}$ by 1 and all coefficients of ℓ_a. Take $\eta > 0$ such that $(1+\eta)\max(\|\ell_a\|, \|h_a\|) \leq b$. We notice that G

is a $\mathbb{K}$-Banach space of countable type, hence there exists a $\mathbb{K}$-linear mapping Ξ from G to $\mathbb{K}$ of norm $\leq 1+\eta$, such that $\Xi(1)=1$ [15]. Let F be the closed $\mathbb{K}$-vector subspace of $\widehat{A}$ consisting of all power series with coefficients in E. Then F is a A-module and Ξ has continuation to a A-linear mapping $\widehat{\Xi}$ from F to A defined as $\widehat{\Xi}(\sum_{n=0}^{\infty} b_n x^n) = \sum_{n=0}^{\infty} \Xi(b_n)x^n$. This mapping $\widehat{\Xi}$ has a norm bounded by $1+\eta$. Set $g_a = \widehat{\Xi}(\ell_a)$. Then by (1) we have

$$\|g_a\| \leq b(1+\eta)\ \forall a \in d(0,\tau-\varepsilon). \tag{3}$$

On the other hand, by construction, for every $z \in G$, we have $|\widehat{\Xi}(z)| \leq |z|(1+\eta)$: that holds particularly for elements of $G \cap D$. Now, since $(f-a)(l_a) - \overline{h}h_a = 1$, for all $x \in D$, we have $l_a(x) \in G$, $f(x)-a \in K$ and hence $\overline{h}h_a(x)$ belongs to G. Therefore the inequality applies and shows that $|\widehat{\Xi}(\overline{h}h_a)(x)| \leq |(\overline{h}h_a)(x)|(1+\eta)$, hence by (2) we can derive $\lim_{\mathcal{U}} \widehat{\Xi}(\overline{h}h_a)(x) = 0\ \forall a \in d(0,\tau-\varepsilon)$. But since $\widehat{\Xi}$ is a A-module linear mapping, we have $\widehat{\Xi}((f-a)h_a - 1) = (f-a)g_a - 1$. Consequently, $\lim_{\mathcal{U}} |(f(x)-a)g_a(x) - 1| = 0\ \forall a \in d(0,\tau-\varepsilon)$ and hence $(f-a)g_a - 1$ belongs to $\mathcal{J}(\mathcal{U})$. Putting $c = b(1+\eta)$, by (3) we are done. $\square$

Theorem III.7. *A is multbijective.*

Proof. Suppose A is not multbijective and let $\mathcal{M}$ be a maximal ideal which is not univalent. Let F be the quotient field $\frac{A}{\mathcal{M}}$, let θ be the canonical surjection from A onto F and let $\|\ .\ \|_q$ be the $\mathbb{K}$-Banach algebra quotient norm of F. By Corollary II.2.2 there exists an ultrafilter $\mathcal{U}$ on D such that $\mathcal{M} = \mathcal{J}(\mathcal{U})$. Thus, there exists $\psi \in Mult(A, \|\ .\ \|)$ such that $Ker(\psi) = \mathcal{M}$ and $\psi \neq \varphi_{\mathcal{U}}$. Consequently, there exists $f \in A$ such that $\psi(f) \neq \varphi_{\mathcal{U}}(f)$, with $\psi(f) \neq 0$, $\varphi_{\mathcal{U}}(f) \neq 0$. We shall check that we may also assume $\psi(f) < \varphi_{\mathcal{U}}(f)$. Indeed, suppose $\psi(f) > \varphi_{\mathcal{U}}(f)$. Let $g \in A$ be such that $\theta(g) = \theta(f)^{-1}$. Then we can see that $\psi(g) = \psi(f)^{-1}$, $\varphi_{\mathcal{U}}(g) = (\varphi_{\mathcal{U}}(f))^{-1}$, therefore $\psi(g) < \varphi_{\mathcal{U}}(g)$. Thus, we may assume $\psi(f) < \varphi_{\mathcal{U}}(f)$ without loss of generality. Similarly, we may obviously assume that $\|f\| < 1$.

By construction, $\varphi_{\mathcal{U}}$ factorizes in the form $\phi_1 \circ \theta$ and similarly, ψ factorizes in the form $\phi_2 \circ \theta$ with $\phi_1,\ \phi_2 \in Mult(F, \|\ .\ \|_q)$. So, on F we have $\phi_1(\theta(f)) > \phi_2(\theta(f))$.

Let $\sigma = \varphi_{\mathcal{U}}(f)$ and let $\varepsilon \in]0,\sigma[$. By Proposition III.6, there exists $c > 0$ such that, for every $a \in d(0,\sigma-\varepsilon)$, there exists $g_a \in A$ satisfying $(f-a)g_a - 1 \in \mathcal{M}$ and $\|g_a\| \leq c$. Now, $\theta(g_a) = (\theta(f-a))^{-1}$. Thus, $\|(\theta(f-a))^{-1}\|_q \leq c\ \forall a \in d(0,\sigma-\varepsilon)$. Therefore, by applying Proposition III.2 to the Banach $\mathbb{K}$-algebra F, we can see that there exists $y \in F$ such that $\phi_1(y) = 1, \phi_2(y) = 0$. Therefore, taking $g \in A$ such that $\theta(g) = y$, we get $\varphi_{\mathcal{U}}(g) = 0$, $\psi(g) = 1$, a contradiction to the hypothesis $Ker(\varphi_{\mathcal{U}}) = Ker(\psi)$. This finishes showing that A is multbijective. $\square$

Corollary III.7.1. *For every* $\phi \in Mult_m(A, \|\ .\ \|) \setminus Mult_1(A, \|\ .\ \|)$, *there exists a coroner ultrafilter* $\mathcal{U}$ *such that* $\phi(f) = \lim_{\mathcal{U}} |f(x)|\ \forall f \in A$.

Corollary III.7.2. $Mult_1(A, \|\ .\ \|)$ *is dense in* $Mult_m(A, \|\ .\ \|)$.

Corollary III.7.2 together with Theorem III.7 in [11] obviously suggests the following conjecture:

Conjecture. *$Mult_1(A, \|\,.\,\|)$ is dense in $Mult(A, \|\,.\,\|)$.*

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