SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.16.01.02
Vol.16 (29), No.1, (2020), 13 – 32

THE STEINHAUS-WEIL PROPERTY: I. SUBCONTINUITY AND AMENABILITY

NICHOLAS H. BINGHAM AND ADAM J. OSTASZEWSKI

In memory of Harry I. Miller (1939-2018)

ABSTRACT. The Steinhaus-Weil theorem that concerns us here is the simple, or classical, 'interior-points' property – that in a Polish topological group a non-negligible set B has the identity as an interior point of $B^{-1}B$. There are various converses; the one that mainly concerns us is due to Simmons and Mospan. Here the group is locally compact, so we have a *Haar* reference measure η. The Simmons-Mospan theorem states that a (regular Borel) measure has such a Steinhaus-Weil property if and only if it is absolutely continuous with respect to the Haar measure. This the first of four companion papers (we refer to the others as II [BinO11], III, [BinO12], and IV, [BinO13], below). Here (Propositions 1.1-1.7 and Theorems 11.-1.4) we exploit the connection between the interior-points property and a selective form of infinitesimal invariance afforded by a certain family of *selective* reference measures σ, drawing on Solecki's amenability at 1 (and using Fuller's notion of subcontinuity).

In II, we turn to a converse of the Steinhaus-Weil theorem, the Simmons-Mospan theorem, and related results. In III, we discuss Weil topologies, linking the topological group-theoretic and measure-theoretic aspects. We close in IV with some other interior-point results related to the Steinhaus-Weil theorem.

1. INTRODUCTION

We begin by stating the Steinhaus-Weil Theorem in its simplest form (Steinhaus [Ste] for the line, Weil [Wei, §11, p. 50] for a Polish locally compact group, Grosse-Erdmann [GroE]):

Theorem SW. *In a locally compact Polish group G with (left) Haar measure η_G, for non-null Borel B, $B^{-1}B$ (and likewise BB^{-1}) contains a neighbourhood of the identity.*

The context we work in here and throughout, unless otherwise stated, is that groups and spaces are assumed separable. This both simplifies the exposition

---

2010 *Mathematics Subject Classification.* Primary 22A10, 43A05; Secondary 28C10.

Key words and phrases. Steinhaus-Weil property, amenability at 1, measure subcontinuity, Simmons-Mospan theorem, selective measure, interior-points property, Haar measure, left Haar null.

and emphasizes that we need only the axiom of Dependent Choices (DC – ‘what is needed to make induction work’), rather than the Axiom of Choice (AC); cf. [BinO8]. For comments concerning non-separable settings, see the arXiv version [BinO10, §8.1].

The interior-point property of the measure-theoretically ‘non-negligible’ set B of the theorem is referred to as the *Steinhaus-Weil property*, which encompasses the category variant due to Piccard [Pic] and Pettis [Pet], cf. Cor. $2'$ and Th. 1B of [BinO12] (by reference, when appropriate, to the *quasi-interior* of a set – the largest open set equivalent to it modulo a meagre set). This important result has many ramifications; for example, it is basic to the theory of regular variation – see e.g. [BinGT, Th. 1.1.1].

The results below hinge on work of Solecki [Sol2] on amenability at 1 and on an amendment of Fuller’s concept of subcontinuity (see §2 and below). These are aimed at freeing up the classical dependency on local compactness and the corresponding standard (Haar) reference measure. To the best of our knowledge such aims, in respect of topological groups, were last undertaken by Xia in 1972 in Chapter 3 of [Xia], where the emphasis is on (relative) quasi-invariance (cf. [BinO10, §7.2]), a topic we pursued in the related paper [BinO9] (cf. [Bog1, p.64]) with tools developed here.

For G a topological group with (admissible) metric d (briefly: metric group), denote by $\mathcal{M}(G)$ the family of regular σ-finite Borel measures on G, with $\mathcal{P}(G) \subseteq \mathcal{M}(G)$ the probability measures ([Kec, §17E], [Par]), by $\mathcal{P}_{\text{fin}}(G)$ the larger family of finitely-additive regular probability measures (cf. [Bin], [Myc]), and by $\mathcal{M}_{\text{sub}}(G)$ submeasures (monotone, finitely subadditive set functions μ with $\mu(\emptyset) = 0$). Here *regular* is taken to imply both *inner* regularity (inner approximation by compact subsets, also called the *Radon* property, as in [Bog2, II §7.1] and [Sch]), and *outer* regularity (outer approximation by open sets). We recall that a σ-finite Borel measure on a metric space is necessarily outer regular ([Bog2, II. Th. 7.1.7], [Kal, Lemma 1.34], cf. [Par, Th. II.1.2] albeit for a probability measure) and, when the metric space is complete, inner regular ([Bog2, II. Th. 7.1.7], cf. [Par, Ths. II.3.1 and 3.2]). When G is locally compact we denote Haar measure by η_G or just η (H denoting capital eta in Greek). For X metric, we denote by $\mathcal{K} = \mathcal{K}(X)$ the family of compact subsets of X (the *hyperspace* of X in §1, where we view it as a topological space under the Hausdorff metric, or the Vietoris topology). For $\mu \in \mathcal{M}(G)$ we write ${}_g\mu(\cdot) := \mu(g\cdot)$ and $\mu_g(\cdot) := \mu(\cdot g)$; $\mathcal{M}(\mu)$ denotes the μ-measurable sets of G and $\mathcal{M}_+(\mu)$ those of finite positive measure, and $\mathcal{K}_+(\mu) := \mathcal{K}(G) \cap \mathcal{M}_+(\mu)$. For G a Polish group, recall that $E \subseteq G$ is *universally measurable* ($E \in \mathcal{U}(G)$) if E is measurable with respect to every measure $\mu \in \mathcal{P}(G)$ – for background, see e.g. [Kec, §21D], cf. [Fre, 434D, 432], [Sho]; these form a σ-*algebra*. Examples are analytic subsets (see e.g. [Rog, Part 1 §2.9], or [Kec, Th.

21.10], [Fre, 434Dc]) and the σ-algebra that they generate. Beyond these are the *provably* Δ_2^1 sets of [FenN] – cf. [BinO8].

Recall that E is *left Haar null*, $E \in \mathcal{HN}$, as in Solecki [Sol1,2,3] (following [Chr1,2]) if there are $B \in \mathcal{U}(G)$ covering E and $\mu \in \mathcal{P}(G)$ with

$$\mu(gB) = 0 \qquad (g \in G).$$

(The terminal brackets here and below indicate *universal quantification* over the free variable.) So if $B \in \mathcal{U}(G)$ is not left Haar null, then for each $\mu \in \mathcal{P}(G)$ there is compact $K = K_\mu \subseteq B$ and $g \in G$ with

$${}_g\mu(K) > 0.$$

The question then arises whether there is also $\delta > 0$ with ${}_g\mu(Kt) > 0$ for *all* $t \in B_\delta$, for $B_\delta = B_\delta(1_G)$ the open δ-ball centered at 1_G : a *right-sided* property *complementing* the earlier *left-sided* property (of nullity, or otherwise). If this is the case for some μ, then (see Corollary $2'$ in §2) $1_G \in \text{int}(K^{-1}K) \subseteq \text{int}(E^{-1}E)$; indeed, one has

$$K \cap Kt \in \mathcal{M}_+({}_g\mu) \qquad (t \in B_\delta), \tag{$*M$}$$

('M for measure', cf $*B$ below , 'B for Baire'), which implies (Lemma 1, §2):

$$B_\delta \subseteq \text{int}(K^{-1}K) \subseteq \text{int}(E^{-1}E);$$

cf. [Kem], [Kuc, Lemma 3.7.2], [BinO1, Th. K], [BinO6, Th. 1(iv)]. As this clearly forces local-compactness of G (see Lemma 1 below), for the more general context we weaken the 'complementing right-sided property' to hold only *selectively*: on a subset (cf. $B_\delta^\Delta(\mu)$ in §2) of B_δ of the form

$$\{z \in B_\delta : |\mu(Kz) - \mu(K)| < \varepsilon\}.$$

We are guided by the close relation between the measure-theoretic *Steinhaus-Weil-like* property (*M) and its category version

$$K \cap Kt \in \mathcal{B}_+(\tau), \tag{$*B$}$$

where the latter term $\mathcal{B}_+(\tau)$ refers to non-meagre *Baire* sets (= with the Baire property) of τ, a refinement of the ambient topology $\mathcal{T}_G = \mathcal{T}_d$ of G, the latter conveniently taken to be generated by a *left*-invariant metric $d = d_L^G$ with associated group-norm (§5), or 'pre-norm' as in [ArhT], $||x|| := d(x, 1_G) = d(tx, t)$ (so that $B_\delta(t) = tB_\delta$ – see Prop. 1). We refer to the (left) invariance of $\mathcal{B}_+(\tau)$ (under translation) as the (left) *Nikodym property* of τ.

Here in Part I, in the context of a metric or Polish group G, we study continuity properties of the maps $m_K : t \mapsto \mu(Kt)$ in the light of theorems of Solecki [Sol2] and of converses to Theorem SW above (see Theorem SM in Part II) and related results. The key here is Fuller's notion of subcontinuity, as applied to the function $m_K(t)$ at $t = 1_G$. This yields a fruitful interpretation of Solecki's notion of *amenability at* 1_G via *selective subcontinuity* and linkage to *shift-compactness* (see Th. 3 below; the term is borrowed from [Par, III.2]). Since commutative Polish groups are amenable at 1 [Sol2, Th. 1(ii)], this widens the field of applicability of shift-compactness to non-Haar-null subsets of these, as in [BinO5], and leads to a conjecture (see remarks preceding Theorem 3) as to whether $\mathcal{HN}$ comprises the negligible sets of some refinement topology of $\mathcal{T}_d$.

We frequently refer for background to the extended commentaries and associated extensive bibliography of [BinO10], the unified arXiv version of this four-part series.

2. Measure under translation – preliminaries

We begin with a form of the 'telescope' or 'tube' lemma (cf. [Mun, Lemma 5.8]), applied in §2. Our usage of upper semicontinuity in relation to set-valued maps follows [Rog], cf. [Bord].

Proposition 1 (cf. [Hey, 1.2.8])**.** *For a metric group G and compact $K \subseteq G$, the map $t \mapsto Kt$ is upper semicontinuous; in particular, for $\mu \in \mathcal{M}(G)$,*

$$m_K : t \mapsto \mu(Kt)$$

is upper semicontinuous, hence μ-measurable. In particular, if $m_K(t) = 0$, then m is continuous at t.

Proof. For K compact and $V \supseteq K$ open, pick for each $k \in K$ a $\delta(k) > 0$ with $kB_{2\delta(k)} \subseteq V$. By compactness, there are $k_1, ..., k_n$ with $K \subseteq \bigcup_j k_j B_{\delta(k_k)} \subseteq V$; then for $\delta := \min_j r(k_j) > 0$

$$Kt \subseteq \bigcup_j k_j B_{\delta(k_k)} t \subseteq \bigcup_j k_j B_{2\delta(k_k)} \subseteq V \qquad (||t|| < \delta).$$

To prove upper semicontinuity of m_K, fix $t \in G$. For $\varepsilon > 0$, as Kt is compact, choose by outer regularity an open $U \supseteq Kt$ with $\mu(U) < \mu(Kt) + \varepsilon$; as before, there is an open ball B_δ at 1_G with $KtB_\delta \subseteq U$, and then $\mu(KtB_\delta) \leq \mu(U) < \mu(Kt) + \varepsilon$. The final assertion follows from positivity of m_K. □

We continue with an analogue. The result is folklore, cf. [BeeV, Th. 3.2(i)]; it comes close to matters touched on in [Ost, §3]. Here and below the vertical section of a set A is denoted $A_x := \{y : (x, y) \in A\}$.

Proposition 2 (Sectional upper semicontinuity)**.** *For a metric group G, compact $F \subseteq G$ and compact $K \subseteq G^2$, the map*

$$x \mapsto K_x \qquad (x \in F)$$

is upper semicontinuous.

Proof. For $V \subseteq G$ open with $K_x \subseteq V$, suppose for $x_n \in F$ with $x_n \to x$ that $(x_n, y_n) \in K \backslash (G \times V)$. By compactness of K, we may suppose w.l.o.g. that $y_n \to y$. Then $(x, y) \in K \backslash (G \times V)$, and so $(x, y) \in \{x\} \times K_x$ and $y \notin V$; but $y \in K_x \subseteq V$, a contradiction. □

From Prop. 2 on *upper* semicontinuity, we obtain information about $m_K : t \mapsto \mu(Kt)$ below. This links with *lower* semicontinuity. By a theorem of Fort, the ε-continuity points (defined in terms of the Hausdorff metric: see [For]) of an upper semicontinuous compact-valued mapping of a metric space into a totally bounded metric space form a dense open set, implying in a real-valued context such as here *continuity* on a co-meagre set. We return to this shortly in Theorem LB below.

Proposition 3 (Sectional upper semicontinuity under a measure). *For a metric group G, compact $F \subseteq G$ and compact $K \subseteq G^2$, and $\mu \in \mathcal{M}(G)$, the map*

$$m : x \mapsto \mu(K_x) \qquad (x \in F = \mathrm{proj}_1 K)$$

is upper semicontinuous, and so Borel.

Proof. Fix $x \in F$. Let $\varepsilon > 0$. By outer regularity, take V open in G with $K_x \subseteq V$ and $\mu(V) < \mu(K_x) + \varepsilon$. By Prop. 2 $x \mapsto \{x\} \times K_x$ is upper semicontinuous on F; so for some open neighbourhood U of x

$$\bigcup\nolimits_{y \in U \cap F} \{y\} \times K_y = K \cap (U \times G) \subseteq K \cap (U \times V).$$

So, for $y \in F \cap U$, $\mu(K_y) \leq \mu(V) < \mu(K_x) + \varepsilon$, proving the first assertion. The second assertion follows since

$$m^{-1}(a,b) = \bigcap\nolimits_{n \in \mathbb{N}} m^{-1}[0,b) \backslash m^{-1}[0,a+1/n). \qquad \square$$

For further results on Borel-measurability of regular Borel measures see [BeeV, Th. 2.2] (there termed 'Radon measures').

We will need the following result in [BinO12,13] (see Lemma 2, §4, and [BinO12, Th. 1, §2]), preferable to the usual Fubini Theorem as using qualitative rather than quantitative measure theory (like the Kuratowski-Ulam Theorem [FreNR]). Interestingly, it may be proved by mimicking the proof of Prop. 1 above, yielding a simplification to that by Eric van Dowen [vDo], itself a simplification of that in [Oxt2, Ch. 14]: for the proof (omitted here), see [BinO10, §8.12].

Theorem FN (Fubini theorem for null sets). *For a metric group G and $A \subseteq G^2$ measurable under $\mu \times \nu$, with $\mu, \nu \in \mathcal{M}(G)$: if the 'exceptional set' of points x for which the vertical section A_x is ν-non-null is itself μ-null, then A is $\mu \times \nu$-null.*

We close this section with a study of the continuity properties of the map $m_K : t \mapsto \mu(Kt)$ for compact K, extending Prop. 3.

Corollary 1 (Fort [For]). *In Proposition 2, $t \mapsto \mu(Kt)$ is lower semi-continuous (so also continuous) on a co-meagre set.*

We can improve on the preceding result by recourse to a natural generalization, for our compact sectional context, of the classical continuity theorems of Luzin [Hal, §55] and Baire [Oxt2, Th. 8.1] – see also [Sch, Ch. 1, §5] and [Bog2, Th. 2.2,10, Th. 7.1.13]. Below for a *compact* metric space X, recall that $\mathcal{K}(X)$ the *hyperspace of* X, the space of compact subsets of X, is equipped with the Hausdorff metric, or Vietoris topology; here this is also a compact space ([Eng, 2.7.28], [Kec, Th. 4.25], [Mic]). Then (LB for 'Luzin-Baire'):

Theorem LB. *For G a metric group and compact $K \subseteq G^2$, the map $\kappa : G \to \mathcal{K}(G) : x \mapsto K_x$ is Borel-measurable, and so*

i) *κ is continuous relative to a co-meagre set.*
For $\mu \in \mathcal{P}(G)$:

ii) *for each $\varepsilon > 0$ there is a Borel set S_ε with $\mu(G \backslash S_\varepsilon) < \varepsilon$ such that $x \mapsto K_x$ is continuous on S_ε; equivalently:*

ii)′ *there is an increasing sequence of Borel sets S_n with union μ-almost all of G such that $x \mapsto K_x$ is continuous on each S_n.*

Proof. See the arXiv version [BinO10, Th. LB], based on [Zak]. □

A first corollary is the following result on the continuity of the map

$$x \mapsto ||x||_E^\mu = \mu(xE \triangle E),$$

for *measurable* E, by compact approximation (cf. Part III). Below, the sets C_x associated with points x should be interpreted as neighbourhoods of x in the spirit of a Hashimoto ideal topology for the ideal of μ-null sets, for which see [LukMZ], or [BinO6]. This mimicks Weil's *proof* of the 'fragmentation lemma' ([BinO12, Part III, §1 Lemma 2]) in [Hal, Ch. XII §62 Th. A] (cf. [Wei, Ch. VII, §31]).

Proposition 4 (Almost everywhere continuity)**.** *For a metric group G, $\delta > 0$, $\mu \in \mathcal{P}(G)$, $E \in \mathcal{M}_+(\mu)$, and $F \in \mathcal{K}_+(\mu)$:*

there is a compact $C \subseteq F$ with $\mu(F \backslash C) < \delta$ such that for any $\varepsilon > 0$ and each $x \in C$ there is a μ-non-null measurable $C_x \subseteq C$ containing x with

$$|\mu(xE \triangle E) - \mu(yE \triangle E)| < \varepsilon \qquad (y \in C_x).$$

In particular, there is an increasing family of compact sets C_n with union μ-almost all of G satisfying the above with C_n for C.

Proof. See [BinO10, Prop. 4]. □

A proof similar to but simpler than that above (omitted here – see [BinO10, §8.13]) improves Prop. refprop1:

Proposition 5 (Almost everywhere upper semicontinuity)**.** *For a metric group G, $\delta > 0$, $\mu \in \mathcal{P}(G)$, $E \in \mathcal{M}_+(\mu)$, and $F \in \mathcal{K}_+(\mu)$:*

there is a compact $C \subseteq F$ with $\mu(F \backslash C) < \delta$ such that for any $\varepsilon > 0$ each $x \in C$ has a neighbourhood U_x with

$$\mu(yE) < \mu(xE) + \varepsilon \qquad (y \in C \cap U_x).$$

In particular, there are disjoint compact sets C with union μ-almost all of G for which this holds.

3. SUBCONTINUITY OF MEASURES

Proposition refprop1 above, on upper semicontinuity, motivates the following definitions, the key one being an adaptation of *subcontinuity* (of functions) due to Fuller [Ful] (for which see Remark 4 below) to the context of measures. We focus on the *right-sided* version of the concept. Subcontinuity is a natural auxiliary in the quest for fuller forms of continuity: as one instance, see [Bou] for the step from separate to joint continuity; as another (classic) instance, note that a subcontinuous set-valued map with closed graph (yet another relative of upper semicontinuity) is continuous – see [HolN] for an extensive bibliography. Here its relevance to the Steinhaus-Weil Theorem (which seems to be new here) yields Theorems 1 and 3, linking *amenability at 1* with *shift-compactness*, for which see Theorem 3 below (the latter term is borrowed from [Par, III.2]). Thus subcontinuity passes between local compactness and the pathology of invariance associated with non-local compactness: see [Oxt1] and [DieS, Ch. 10].

Definition ([BinO6])**.** For $\mu \in \mathcal{P}_{\text{fin}}(G)$, and (compact) $K \in \mathcal{K}(G)$, noting that $\mu_\delta(K) := \inf\{\mu(Kt) : t \in B_\delta\}$ is weakly decreasing in δ, put

$$\mu_-(K) := \sup_{\delta>0} \inf\{\mu(Kt) : t \in B_\delta\},$$

and, for $\mathbf{t} = \{t_n\}$ a *null sequence*, i.e. with $t_n \to 1_G$,

$$\mu_-^{\mathbf{t}}(K) := \liminf_{n\to\infty} \mu(Kt_n).$$

Then

$$0 \le \mu_-(K) \le \mu(K) = \inf_{\delta>0} \sup\{\mu(Kt) : t \in B_\delta\},$$

by Proposition refprop1. We say that a null sequence $\mathbf{t}$ is *non-trivial* if $t_n \neq 1_G$ infinitely often. Define as follows:

i) μ is *translation-continuous* ('*continuous*' or '*mobile*') if $\mu(K) = \mu_-(K)$ $(K \in \mathcal{K}(G))$;
ii) μ is *maximally discontinuous* at $K \in \mathcal{K}(G)$ if $0 = \mu_-(K) < \mu(K)$;
iii) μ is *subcontinuous* if $0 < \mu_-(K) \le \mu(K)$ $(K \in \mathcal{K}_+(\mu))$;
iv) μ is *(selectively) subcontinuous at* $K \in \mathcal{K}_+(\mu)$ *along* $\mathbf{t}$ if $\mu_-^{\mathbf{t}}(K) > 0$.

Remarks. 1. $m_K(.)$ is *continuous* if μ is continuous, since $m_K(st) = m_{Ks}(t)$ and Ks is compact whenever K is compact; for directional continuity of measures in linear spaces see [Bog3, §3.1]. In [LiuR] (cf. [LiuRW], [Gow1,2]) a Radon measure μ on a space X, on which a group G acts homeomorphically, is called *mobile* if $t \mapsto \mu(Kt)$ is continuous for all $K \in \mathcal{K}(X)$.
2. For G locally compact (i) holds for μ the left Haar measure η_G, and also for $\mu \ll \eta_G$ (absolutely continuous w.r.t. to η_G).
3. A measure μ singular w.r.t. Haar measure is maximally discontinuous for its support: this is at the heart of the analysis offered by Simmons (and independently, much later by Mospan) – see Corollary 2′ below.

4. *Subcontinuity*, in the sense of [Ful], of a map $f : G \to (0,\infty)$ requires that, for every $t_n \to t \in G$, there is a subsequence $t_{m(n)}$ with $f(t_{m(n)})$ convergent in the range (i.e. to a positive value). The distinguished role of null sequences emerges below in the *Subcontinuity Theorem* (Theorem 1). Null sequences should be viewed here as selecting stepwise (or even pathwise, under local connectedness, as suggested by Tomasz Natkaniec) 'asymptotic directions' justifying the phrase '*along* **t**' in (iv) above, and allowing (iv) to be interpreted as a *selective subcontinuity* in 'direction' **t**. The analogous selective concept in a linear space is 'along a vector' as in [Bog3, §3.1].
5. *Selective versus uniform subcontinuity.* Definition (iii) is equivalent to demanding for $K \in \mathcal{K}_+(\mu)$ that *any* null sequence $\mathbf{t} = \{t_n\}$ have a subsequence $\mu(Kt_{m(n)})$ bounded away from 0; then (iii) may be viewed as demanding 'uniform subcontinuity': selective subcontinuity along *each* **t** for all $K \in \mathcal{K}_+(\mu)$.
6. *Left- versus right-sided versions.* Writing $\tilde{\mu}(E) := \mu(E^{-1})$ with E Borel in G for the *inverse measure* captures versions associated with right-sided translation such as $\tilde{\mu}_-$ and

$$\tilde{\mu}^{\mathbf{t}}_-(K) := \liminf_{n\to\infty} \mu(t_n K).$$

Definition. We will say that μ is *symmetric* if $\mu = \tilde{\mu}$; then B is null iff B^{-1} is null for B a Borel set, or $B \in \mathcal{U}(G)$.

In Lemma 1 below it suffices for μ to be a bounded, regular submeasure which is *supermodular:*

$$\mu(E \cup F) \geq \mu(E) + \mu(F) - \mu(E \cap F) \qquad (E, F \in \mathcal{U}(G));$$

recall, however, from [Bog2, 1.12.37] the opportunity to replace, for any $K \in \mathcal{K}(G)$, a supermodular submeasure μ by a dominating $\mu' \in \mathcal{M}_{\text{fin}}(G)$, i.e. with $\mu'(K) \geq \mu(K)$.

For $K \in \mathcal{K}_+(\mu)$ and $\delta, \Delta > 0$, put

$$B^{\Delta}_{\delta} = B^{K,\Delta}_{\delta}(\mu) := \{z \in B_\delta : \mu(Kz) > \Delta\},$$

which is monotonic in $\Delta : B^{\Delta}_{\delta} \subseteq B^{\Delta'}_{\delta}$ for $0 < \Delta' \leq \Delta$. Note that $1_G \in B^{\Delta}_{\delta}$ for $0 < \Delta < \mu(K)$.

The specialization below to a mobile measure (see above) may be found in [Gow1,2].

Lemma 1 (cf. [BinO6, Th. 2.5])**.** *Let $\mu \in \mathcal{P}_{\text{fin}}(G)$ for G a metric group. For $K \in \mathcal{K}_+(\mu)$, if $\mu^{\mathbf{t}}_-(K) > 0$ for some non-trivial null sequence* **t***, then for $\Delta \geq \mu^{\mathbf{t}}_-(K)/4 > 0$ there is $\delta > 0$ with $t_n \in B^{\Delta}_{\delta}$ for all large enough n and*

$$\Delta \leq \mu(K \cap Kt) \qquad (t \in B^{\Delta}_{\delta}),$$

so that

$$K \cap Kt \in \mathcal{M}_+(\mu) \qquad (t \in B^{\Delta}_{\delta}). \tag{$*$}$$

In particular,

$$K \cap Kt \neq \emptyset \qquad (t \in B_\delta^\Delta),$$

or, equivalently,

$$B_\delta^\Delta \subseteq K^{-1}K, \tag{$**$}$$

so that B_δ^Δ has compact closure.

A fortiori, if $\mu_-(K) > 0$, then $\delta, \Delta > 0$ may be chosen with $\Delta < \mu_-(K)$ and $B_\delta \subseteq B_\delta^\Delta$ so that (*) and (**) hold with B_δ replacing B_δ^Δ, and in particular G is locally compact.

Proof. For the first part fix a null sequence $\mathbf{t}$ and $K \in \mathcal{K}_+(\mu)$ with $\mu_-^{\mathbf{t}}(K) > 0$; take any $\Delta \geq \mu_-^{\mathbf{t}}(K)/4 > 0$, and, as above, write B_δ^Δ for $B_\delta^{K,\Delta}$. Then, for $\mu(Kt) > 2\Delta \geq \mu_-^{\mathbf{t}}(K)/2$ and $\delta > 0$ arbitrary, $t \in B_\delta^\Delta$; and so $t_n \in B_\delta^\Delta$ for all large enough n (since also $t_n \in B_\delta$ for all large enough n). So $B_\delta^\Delta(K)\backslash\{1_G\}$ is non-empty for $\mathbf{t}$ non-trivial.

Put $H_t := K \cap Kt \subseteq K$. By outer regularity of μ, choose $U = U(\Delta, K)$ open with $K \subseteq U$ and $\mu(U) < \mu(K) + \Delta$. By upper semicontinuity of $t \mapsto Kt$, we may now fix $\delta = \delta(\Delta, K) > 0$ so that $KB_\delta \subseteq U$. For $t \in B_\delta^\Delta$, by finite additivity of μ, since $2\Delta < \mu(Kt)$

$$\begin{aligned} 2\Delta + \mu(K) - \mu(H_t) &\leq \mu(Kt) + \mu(K) - \mu(H_t) = \mu(Kt \cup K) \\ &\leq \mu(U) \leq \mu(K) + \Delta. \end{aligned}$$

Comparing the ends gives

$$0 < \Delta \leq \mu(H_t) \qquad (t \in B_\delta^\Delta).$$

For $t \in B_\delta^\Delta$, as $K \cap Kt \in \mathcal{M}_+(\mu)$, take $s \in K \cap Kt \neq \emptyset$; then $s = kt$ for some $k \in K$, so $t = k^{-1}s \in K^{-1}K$. Conversely, $t \in B_\delta^\Delta \subseteq K^{-1}K$ yields $t = k^{-1}k'$ for some $k, k' \in K$; then $k' = kt \in K \cap Kt$.

By the compactness of $K^{-1}K$, B_δ^Δ has compact closure.

As for the final assertions, if $\mu_-(K) > 0$, now take $\Delta := \mu_-(K)/2$. Then $\inf\{\mu(Kt) : t \in B_\delta\} > \Delta$ for all small enough $\delta > 0$, and so in particular $\mu(Kt) > \Delta$ for $t \in B_\delta$, i.e. $B_\delta \subseteq B_\delta^\Delta$. So the argument above applies for small enough $\delta > 0$ with B_δ in lieu of B_δ^Δ, just as before. Here the compactness of $K^{-1}K$ now implies local compactness of G itself. □

As an immediate and useful corollary, we have

Lemma 1′. *For $\mu \in \mathcal{P}_{\text{fin}}(G)$, with G a metric group, any null sequence $\mathbf{t}$ and any $K \in \mathcal{K}(G)$: if $\mu_-^{\mathbf{t}}(K) > 0$, then there is $m \in \mathbb{N}$ with*

$$0 < \mu_-^{\mathbf{t}}(K)/4 < \mu(K \cap Kt_n) \qquad (n > m). \tag{$*'$}$$

In particular,

$$t_n \in K^{-1}K \qquad (n > m). \tag{$**'$}$$

Proof. Apply Lemma 1 to obtain $\Delta, \delta > 0$; for $t \in B_\delta^\Delta$, $\mu(Kt) > \Delta$, so as above $t_n \in B_\delta^\Delta$ for all large enough n. □

This permits a connection with left Haar null sets; recall that a group G is *amenable at* 1 [Sol2] (see below for the origin of this term) if, given $\mu :=: \{\mu_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(G)$ with $1_G \in \mathrm{supp}(\mu_n)$ (the support of μ_n), for $n \in \mathbb{N}$ there are σ and σ_n in $\mathcal{P}(G)$ with $\sigma_n \ll \mu_n$ satisfying:

$$\sigma_n * \sigma(K) \to \sigma(K) \qquad (K \in \mathcal{K}(G)).$$

In view of Theorem 1 below, we term σ (or $\sigma(\mu)$ if context requires) a *selective measure* and to the measures σ_n, if needed, as *associated measures* (corresponding to the sequence $\{\mu_n\}_{n\in\mathbb{N}}$).

Solecki explains ([Sol2, end of §2]) the use of the term 'amenability at 1' as a localization (via the restriction that supports contain 1_G) of a *Reiter-like condition* [Pat, Prop. 0.4] which characterizes amenability: for $\mu \in \mathcal{P}(G)$ and $\varepsilon > 0$, there is $\nu \in \mathcal{P}(G)$ with

$$|\nu * \mu(K) - \nu(K)| < \varepsilon \qquad (K \in \mathcal{K}(G)).$$

Lemma 1 and the next several results disaggregate Solecki's Interior-point Theorem [Sol2, Th 1(ii)] (Corollary 2 below), shedding more light on it and in particular connecting it to shift-compactness (Theorem 3 below). Indeed, we see that interior-point theorem itself as an 'aggregation' phenomenon. Theorem 1 of Part II identifies subgroups with a 'disaggregation' topology, refining $\mathcal{T}_G$ by using sets of the form $B_\delta^{K,\Delta}(\sigma)$, the measures σ being provided in our first result:

Theorem 1 (Subcontinuity Theorem, after Solecki [Sol2, Th. 1(ii)]). *For G Polish and amenable at 1_G and $\mathbf{t}$ a null sequence, there is $\sigma = \sigma(\mathbf{t}) \in \mathcal{P}(G)$ such that for each $K \in \mathcal{K}_+(\sigma)$ there is a subsequence $\mathbf{s} = \mathbf{s}(K) := \{t_{m(n)}\}$ with*

$$\lim\nolimits_n \sigma(Kt_{m(n)}) = \sigma(K) \qquad (n \in \mathbb{N}), \text{ so} \qquad \sigma_-^{\mathbf{s}}(K) > 0.$$

Proof. For $\mathbf{t} = \{t_n\}$ null, put $\mu_n := 2^{n-1}\sum_{m\geq n} 2^{-m}\delta_{t_m^{-1}} \in \mathcal{P}(G)$; then $1_G \in \mathrm{supp}(\mu_n) \supseteq \{t_m^{-1} : m > n\}$. By definition of amenability at 1_G, in $\mathcal{P}(G)$ there are σ and $\sigma_n \ll \mu_n$, with $\sigma_n * \sigma(K) \to \sigma(K)$ for all $K \in \mathcal{K}(G)$. For $n \in \mathbb{N}$ choose $\alpha_{mn} \geq 0$ with $\sum_{m\geq n} \alpha_{mn} = 1$ $(n \in \mathbb{N})$ and with $\sigma_n := \sum_{m\geq n} \alpha_{mn}\delta_{t_m^{-1}}$.

Fix $K \in \mathcal{K}_+(\sigma)$ and θ with $0 < \theta < 1$. As K is compact, $\sigma_n * \sigma(K) \to \sigma(K)$; then w.l.o.g.

$$\sigma_n * \sigma(K) > \theta\sigma(K) \qquad (n \in \mathbb{N}).$$

Then, for each $n \in \mathbb{N}$,

$$\sup\{\sigma(Kt_m) : m \geq n\} \cdot \sum\nolimits_{m\geq n} \alpha_{mn} \geq \sum\nolimits_{m\geq n} \alpha_{mn}\sigma(Kt_m) > \theta\sigma(K).$$

So for each n there is $m = m(\theta) \geq n$ with

$$\sigma(Kt_m) > \theta\sigma(K).$$

Now choose $m(n) \geq n$ inductively so that $\sigma(Kt_{m(n)}) > (1-2^{-n})\sigma(K)$; then, by Proposition 1, $\lim_n \sigma(Kt_{m(n)}) = \sigma(K)$: σ is subcontinuous along $\mathbf{s} := \{t_{m(n)}\}$ on K. □

Remark. The selection above of the subsequence $\mathbf{s}$ mirrors the role of 'admissible directions' which we encounter subsequently in Cameron-Martin theory ([BinO13, §2] and [BinO10, §8.2]).

We are now able to deduce Solecki's interior-point theorem in a slightly stronger form, which asserts that the sets B^{Δ}_{δ} reconstruct the open sets of G using the compact subsets of a 'non-negligible set', as follows. We recall that $\mathcal{K}(X)$ denotes the family of compact subsets of X; below we use the notation $\delta(\Delta, K)$ established in the proof of Lemma 1.

Theorem 2 (Aggregation Theorem). *For G Polish and amenable at* 1_G, *if* $E \in \mathcal{U}(G)$ *is not left Haar null – then, setting*

$$\hat{E} := \bigcup\nolimits_{\Delta>0, g\in G, \mathbf{t}} \{B^{gK,\Delta}_{\delta(gK,\Delta)}(\sigma(\mathbf{t})) : K \in \mathcal{K}(E), 0 < \sigma(\mathbf{t})(gK)/4 \le \Delta < \sigma(\mathbf{t})(gK)\},$$

$$1_G \in \mathrm{int}(\hat{E}) \subseteq \hat{E} \subseteq E^{-1}E.$$

In particular, for E open, $1_G \in \mathrm{int}(\hat{E})$.

Proof. Suppose otherwise; then, as in Lemma 1, for $g \in G$, any null sequence $\mathbf{z}$, compact $K \subseteq E$ with $0 < \sigma(\mathbf{z})(gK)/4 \le \Delta$ and $\delta = \delta(gK, \Delta)$,

$$B^{gK,\Delta}_{\delta}(\sigma(\mathbf{z})) \subseteq (gK)^{-1}gK = K^{-1}K \subseteq E^{-1}E,$$

so that $\hat{E} \subseteq E^{-1}E$. Next suppose there is for each n

$$t_n \in B_{1/n} \backslash \hat{E}.$$

Consider $\sigma = \sigma(\mathbf{t})$. As E is not left Haar null, there is g with $\sigma(gE) > 0$. Choose compact $K \subseteq gE$ with $\sigma(K) > 0$. Then with $h := g^{-1}$ and $H := hK \subseteq E$, $\sigma(K) = \sigma(gH) = \sigma^{\mathbf{s}}(gH) > 0$ for some subsequence $\mathbf{s} = \{t_{m(k)}\}$. So, again as above and as in Lemma 1, with $\Delta := \sigma(gH)/4$ for some $\delta = \delta(K, \Delta) > 0$

$$B^{gH,\Delta}_{\delta}(\sigma(\mathbf{t})) \subseteq (gH)^{-1}gH = H^{-1}H \subseteq E^{-1}E.$$

Choose n with $n > 1/\delta$. Then $t_n \in B_\delta$ for all $m > n$; so for infinitely many k

$$t_{m(k)} \in B^{gH,\Delta}_{\delta}(\sigma(\mathbf{t})) \subseteq \hat{E},$$

a contradiction. As for the last assertion, for E open, D countable and dense, $G \subseteq \bigcup_{d\in D} dE$, so for any $\mu \in \mathcal{P}(G)$ (in particular for σ) $\mu(dE) > 0$ for some $d \in D$, and so E is not left Haar null. □

The immediate consequence is

Corollary 2 (Solecki's Interior-Point Theorem [Sol2, Th 1(ii)]). *For G Polish and amenable at* 1_G, *if* $E \in \mathcal{U}(G)$ *is not left Haar null, then* $1_G \in \mathrm{int}(E^{-1}E)$.

Corollary 2′. *For G a Polish group, if* $E \in \mathcal{U}(G)$ *is not left Haar null and is in* $\mathcal{M}_+(\mu)$ *for some subcontinuous* $\mu \in \mathcal{P}_{\mathrm{fin}}(G)$, *then for some* $\delta > 0$

$$B_\delta \subseteq \mathrm{int}(E^{-1}E).$$

In particular, this inclusion holds for some $\delta > 0$ *in a locally compact group* G, *for any Baire non-meagre set* E.

Proof. The first assertion is immediate from Lemma 1. As for the second, for a non-meagre Baire set E, if $\tilde{E}$ is the quasi-interior and $K \subseteq \tilde{E}$ is compact with non-empty interior, then $\eta_G(K) > 0$. Since η is subcontinuous, there is $\delta > 0$ with

$$Kt \cap K \neq \emptyset \qquad (||t|| < \delta),$$

and so

$$\tilde{E}t \cap \tilde{E} \neq \emptyset \qquad (||t|| < \delta);$$

then $U := (Et)\tilde{} \cap \tilde{E} \neq \emptyset$, since $(Et)\tilde{} = \tilde{E}t$ (the Nikodym property of the *usual* topology of G). So since U is open and non-meagre, also $Et \cap E \neq \emptyset$, and so again (**). □

The next result establishes the embeddability by (left-sided) translation of an appropriate subsequence of a given *null* sequence into a given target set that (like-sidedly) is non-left-Haar null. This property of embedding into a *non-negligible* set, first studied in respect of category and measure negligibility on $\mathbb{R}$ by Kestelman and much later independently by Borwein and Ditor and thereafter also by other authors, mostly for combinatorial challenges, has emerged as an important general unifying principle, termed *shift-compactness*. This is applicable in a much wider context embracing metric groups G under various topologies refining $\mathcal{T}_G$ and so defining various notions of negligibility; for the background here see [BinO2,3,7], [MilO]. Its consequences include various uniform-boundedness theorems as well as the Effros and the Open Mapping Theorems. Here we establish the said property, announced in [MilO], in relation to the ideal $\mathcal{HN}$ of left Haar null sets. (It is a σ-ideal for Polish G in the presence of amenability at 1 [Sol2, Th 1(i)].) This leaves open the 'converse question' (this is the 'conjecture' of §1) of the existence of a a refinement topology for which $\mathcal{HN}$ is the associated notion of negligibility; this seems plausible under the continuum hypothesis, CH, if one restricts attention only to Borel sets in $\mathcal{HN}$ and their subsets by lifting a result concerning $\mathbb{R}$ in [CieJ, Cor. 4.2] to G – see also the Remark 1 following our next result.

Theorem 3 (Shift-compactness Theorem for $\mathcal{HN}$). *For G Polish and amenable at* 1_G, *if* $E \in \mathcal{U}(G)$ *is not left Haar null and* z_n *is null, then there are* $s \in E$ *and an infinite* $\mathbb{M} \subseteq \mathbb{N}$ *with*

$$\{sz_m : m \in \mathbb{M}\} \subseteq E.$$

Indeed, this holds for quasi all $s \in E$, *i.e. off a left Haar null set.*

Proof. Put $t_n := z_n^{-1}$, which is null. With $\sigma = \sigma(\mathbf{t})$ as in the Subcontinuity Theorem, since E is not left Haar null, there is g with $\sigma(gE) > 0$. For this g, put $\mu := {}_g\sigma$. Fix a compact $K_0 \subseteq E$ with $\mu(K_0) > 0$ and then, passing to a subsequence of $\mathbf{t}$ as necessary (by Th.1), we may assume that $\mu_-^{\mathbf{t}}(K_0) > 0$. Choose inductively a sequence $m(n) \in \mathbb{N}$, and decreasing compact sets $K_n \subseteq K_0 \subseteq E$ with $\mu(K_n) > 0$ such that

$$\mu(K_n \cap K_n t_{m(n)}) > 0.$$

To check the inductive step, suppose K_n already defined. As $\mu(K_n) > 0$, by the Subcontinuity Theorem, there is a subsequence $\mathbf{s} = \mathbf{s}(K_n)$ of $\mathbf{t}$ with $\mu_-^{\mathbf{s}}(K_n) > 0$. By Lemma 1′, there is $k(n) > n$ such that $\mu(K_n \cap K_n s_{k(n)}) > 0$. Putting $t_{m(n)} = s_{k(n)}$ and $K_{n+1} := K_n \cap K_n t_{m(n)} \subseteq K_n$ completes the inductive step.

By compactness, select s with

$$s \in \bigcap\nolimits_{m\in\mathbb{N}} K_m \subseteq K_{n+1} = K_n \cap K_n t_{m(n)} \qquad (n \in \mathbb{N});$$

choosing $k_n \in K_n \subseteq K$ with $s = k_n t_{m(n)}$ gives $s \in K_0 \subseteq E$, and

$$sz_{m(n)} = st_{m(n)}^{-1} = k_n \in K_n \subseteq K_0 \subseteq E.$$

Finally take $\mathbb{M} := \{m(n) : n \in \mathbb{N}\}$.

As for the final assertion, we follow the idea of the Generic Completeness Principle [BinO1, Th. 3.4] (but with $\mathcal{U}(G)$ for $\mathcal{B}a$ there): define

$$F(H) := \bigcap\nolimits_{n\in\mathbb{N}} \bigcup\nolimits_{m>n} H \cap Ht_m \qquad (H \in \mathcal{U}(G));$$

then $F : \mathcal{U}(G) \to \mathcal{U}(G)$ and F is monotone ($F(S) \subseteq F(T)$ for $S \subseteq T$); moreover, $s \in F(H)$ iff $s \in H$ and $sz_m \in H$ for infinitely many m. We are to show that $E_0 := E \backslash F(E)$ is left Haar null. Suppose otherwise. Then renaming g and K_0 as necessary, w.l.o.g. both $\mu(E_0) > 0$ and $K_0 \subseteq E_0$ (and $\mu(K_0) > 0$). But then, as above, $\emptyset \neq F(K_0) \cap K_0 \subseteq F(E) \cap E_0$, a contradiction, since $F(E) \cap E_0 = \emptyset$. □

Remarks. 1. In the setting of Th. 3 any non-empty open set U is not left Haar null (as $\{dU : D \in D\}$ with D countable dense covers G), hence neither is $U \backslash H$ for $H \in \mathcal{HN}$. So the (Hashimoto ideal) topology generated by such sets includes $\mathcal{HN}$ among its negligible sets.
2. Recently the special *abelian* case of Th. 3 has been independently established by Banakh and Jabłońska in [BanJ]. A similar result extends to the Haar-meagre sets of Darji [Dar]; cf. [Jab]. See also [BinO10, §8.9].

Corollary 3. *For G Polish and amenable at 1_G and z_n null, there is $\mu \in \mathcal{P}(G)$ such that for $K \in \mathcal{K}_+(\mu)$*

$$K \cap Kz_m^{-1} \in \mathcal{M}_+(\mu) \qquad \text{for infinitely many } m \in \mathbb{N},$$

iff for μ-quasi all $s \in K$ there is an infinite $\mathbb{M} \subseteq \mathbb{N}$ with

$$\{sz_m : m \in \mathbb{M}\} \subseteq K.$$

Proof. We will refer to the function F of the preceding proof. First proceed as in the proof of Th. 3 above, taking $t_n := z_n^{-1}$ and $g = 1_G$ (so that $\mu = \sigma$). Fix K with $\mu(K) > 0$. For the forward direction, continue as in the proof of Th. 3 with $K_0 = K$ and observe that the proof above needs only that $s_{k(n)} \in K_n^{-1}K_n$ occurs infinitely often whenever $\mu(K_n) > 0$. This yields the desired conclusion that $\mu(K \backslash F(K)) = 0$. For the converse direction, suppose that $\mu(F(K)) > 0$. Since for each $n \in \mathbb{N}$

$$F(K) \subseteq \bigcup\nolimits_{m>n} K \cap Kt_m,$$

we have $\mu(K \cap Kt_m) > 0$ for some $m > n$; so

$$K \cap Kt_m \in \mathcal{M}_+(\mu) \qquad \text{for infinitely many } m. \qquad \square$$

Remark. With E as in the Shift-compactness Theorem, if $z_n \in B_{1/n} \backslash E^{-1}E$, then z_n is null; so, for some $s \in E$, $sz_m \in E$ for infinitely many m. Then, for any such m,

$$z_m \in E^{-1}E,$$

contradicting the choice of z_m. So $1_G \in \text{int}(E^{-1}E)$, i.e. E has the Steinhaus-Weil property, as before.

The following sharpens a result due (for Lebesgue measure on $\mathbb{R}$) to Mospan [Mos] by providing the converse below; it is antithetical to Lemma 1 (and so to Theorem 3).

Proposition 6 (Mospan property)**.** *For G a metric group, $\mu \in \mathcal{P}_{fin}(G)$ and compact $K \in \mathcal{K}_+(\mu)$:*

i) if $1_G \notin \text{int}(K^{-1}K)$, then $\mu_-(K) = 0$, i.e. μ is maximally discontinuous; equivalently, there is a null sequence $t_n \to 1_G$ with $\lim_n \mu(Kt_n) = 0$;

ii) conversely, if $\mu(K) > \mu_-(K) = 0$, then there is a null sequence $t_n \to 1_G$ with $\lim_n \mu(Kt_n) = 0$, and there is a compact $C \subseteq K$ with $\mu(K \backslash C) = 0$ with $1_G \notin \text{int}(C^{-1}C)$.

Proof. The first assertion follows from Lemma 1. For the converse, as in [Mos]: suppose that $\mu(Kt_n) = 0$, for some sequence $t_n \to 1_G$. By passing to a subsequence, we may assume that $\mu(Kt_n) < 2^{-n-1}$. Put $D_m := K \backslash \bigcap_{n \geq m} Kt_n \subseteq K$; then $\mu(K \backslash D_m) \leq \sum_{n \geq m} \mu(Kt_n) < 2^{-m}$, so $\mu(D_m) > 0$ provided $2^{-m} < \mu(K)$. Now choose compact $C_m \subseteq D_m$, with $\mu(D_m \backslash C_m) < 2^{-m}$. So $\mu(K \backslash C_m) < 2^{1-m}$. Also $C_m \cap C_m t_n = \emptyset$, for each $n \geq m$, as $C_m \subseteq K$; but $t_n \to 1_G$, so the compact set $C_m^{-1}C_m$ contains no interior points. Hence, by Baire's theorem, neither does $C^{-1}C$, since $C = \bigcup_m C_m$, which differs from K by a null set. $\square$

Proposition 7. *A (regular) Borel measure μ on a locally compact metric topological group G has the Steinhaus-Weil property iff either*

i) for each $K \in \mathcal{K}_+(\mu)$, the map $m_K : t \to \mu(Kt)$ is subcontinuous at 1_G;
or

ii) for each $K \in \mathcal{K}_+(\mu)$, there is no sequence $t_n \to 1_G$ with $\mu(Kt_n) \to 0$.

Remark. This is immediate from Prop. 6 (cf. [Mos]).

We now prove a strengthening of the Subcontinuity Theorem obtained by assuming a 'concentration property'. That this property holds in a abelian Polish group emerges from an inspection of Solecki's proof of his theorem that an abelian Polish group is amenable at 1.

Definitions. *Say that a null sequence* **t** *is regular if* **t** *is non-trivial, $||t_k||$ is non-increasing, and*

$$||t_k|| \le r(k) := 1/[2^k(k+1)] \qquad (k \in \mathbb{N}).$$

For regular $\mathbf{t}$*, put*

$$\mu_k = \mu_k(\mathbf{t}) := 2^{k-2}\sum_{m\ge k} 2^{-m}(\delta_{t_m^{-1}} + \delta_{t_m}) = \tfrac{1}{4}\delta_{t_k} + \tfrac{1}{4}\delta_{t_k^{-1}} + \tfrac{1}{8}\delta_{t_{k+1}^{-1}} + \dots \ .$$

Then $\mu_k(B_{r(k)}) = 1$ *for* $k \in \mathbb{N}$*. Merging* $\mathbf{t}^{-1}$ *with* $\mathbf{t}$ *by alternation of terms if necessary, it is now convenient to assume that* $\mathbf{t}$ *contains as successive pairs inverses of its terms. So, if* $\nu_k \ll \mu_k$*, then*

$$\nu_k := \sum\nolimits_{m\ge k} a_{km}\delta_{t_m^{-1}},$$

for some non-negative sequence $\mathbf{a}_k := \{a_{kk,}a_{k,k+1,}a_{k,k+2},...\}$ *of unit* ℓ_1*-norm. Say that* $\{\mathbf{a}_k\}$ *has the coefficient concentration property if for some index* j *and some* $\alpha > 0$

$$a_{k,k+j} \ge \alpha > 0 \qquad \text{for all large } k;$$

then say the measures $\{\nu_k\}$ *have the concentration property. (This will fail if* $\mathbf{a}_k$ *has* $a_{k,k+k} = 1$*, which concentrates measure in an unbounded fashion.)*

Definition. *Say that a group* G *is strongly amenable at 1 if* G *is amenable at 1, and for each regular* $\mathbf{t}$ *a selective measure* $\sigma(\mathbf{t})$ *exists with associated measures* $\sigma_k(\mathbf{t}) \ll \mu_k(\mathbf{t})$ *having the concentration property.*

Theorem 4 (Strong amenability at 1, after [Sol2, Prop. 3.3(i)]). *Any abelian Polish group* G *is strongly amenable at 1.*

Proof. This follows the construction in [Sol2] of the reference measure in the case of $\mu_k(\mathbf{t})$ above. First define the normalized restriction

$$\sigma_k := \mu_k|B_{r(k)}/\mu_k(B_{r(k)})$$

and then set

$$\sigma := *_{k=1}^{\infty}\rho_k \text{ for } \rho_k := \frac{1}{k+1}\sum\nolimits_{i=0}^{k}\sigma_k^i.$$

(Convolution powers intended here.) Then the argument in [Sol2] shows that $\sigma_k * \sigma(K) \to \sigma(K)$ (for K compact). However, as $\mathbf{t}$ is regular, $\mu_k \equiv \sigma_k$. But $a_{kk} = 1/2$ (all k), so the measures σ_k here have the concentration property. □

Definitions (Sequence and measure symmetrization):

1. *Merging* $\mathbf{t}^{-1}$ *with* $\mathbf{t}$ *by alternation of terms yields the regular sequence* $\mathbf{s} = (s_1, s_2, ...) := (t_1, t_1^{-1}, t_2, t_2^{-1}, ...)$*; we term this the* ***symmetrized sequence*** *of* $\mathbf{t}$*. (It is 'symmetric' in the sense only that* $||s_{2k-1}|| = ||s_{2k}||$*.)*

2. *For odd* k*, as* $(\mu_k(\mathbf{t}) + \mu_k(\mathbf{t}^{-1}))/2$ *is symmetric as a measure, taking* $\sigma_{2k}(\mathbf{s}) = \sigma_{2k-1}(\mathbf{s}) := (\mu_k(\mathbf{t}) + \mu_k(\mathbf{t}^{-1}))/2$ *in lieu of* $\mu_k(\mathbf{t})$ *above yields each* ρ_k *symmetric. So, in the abelian context of Theorem 4 above, the limiting convolution* σ *is a symmetric selective measure* $\sigma(\mathbf{t})$*.*

Remark. Performing the symmetrization of the Definition above gives in the proof of Theorem 4 above that $a_{2k-1,2k-1} = a_{2k-1,2k} = a_{2k,2k} = a_{2k,2k+1} = 1/4$, which presents *simultaneous concentration* along $\mathbf{t}$ and $\mathbf{t}^{-1}$.

We now re-run the proof of Theorem 1 with improved estimates to yield:

Theorem 1S (Strong Subcontinuity Theorem). *For G a Polish group that is strongly amenable at 1, if $\mathbf{t}$ is regular and $\sigma = \sigma(\mathbf{t})$ is a selective measure – then for $K \in \mathcal{K}_+(\sigma)$*

$$\sigma(K) = \lim_n \sigma(Kt_n) = \sigma_-^{\mathbf{t}}(K).$$

Likewise, passing to the symmetrized sequence of $\mathbf{t}$ as above and to a symmetric selective measure $\sigma(\mathbf{t})$ with the simultaneous concentration property (for $\mathbf{t}$ and $\mathbf{t}^{-1}$) corresponding to an abelian context:

$$\sigma(K) = \lim_n \sigma(t_n K).$$

Proof. Fix $\mathbf{t}$ and a corresponding selective measure $\sigma(\mathbf{t})$ and its associated sequence ν_k, which as in Th. 4 has the concentration property. Write $\sigma_k := \sum_{m\geq k} a_{km}\delta_{t_m^{-1}}$; as σ_k has the concentration property, there are $n_0,\ j$ and $\alpha > 0$ with

$$a_{kk+j} \geq \alpha > 0 \qquad (k \geq n_0).$$

Now fix K compact with $\sigma(K) > 0$ and $\varepsilon > 0$. Put

$$\delta := \varepsilon \big/ \left(\tfrac{2}{\alpha} - 1\right) > 0,$$

as $\alpha \leq 1$. Then, by upper semicontinuity and by 'amenability at 1' (i.e. $\sigma_k * \sigma(K) \to \sigma(K)$), there is $n_1 = n_1(\varepsilon, K) > n_0$ with

$$\sigma(Kt_k) \leq \sigma(K) + \delta \text{ and } \sigma_k * \sigma(K) \geq \sigma(K) - \delta \qquad (k \geq n_1).$$

So (by upper semicontinuity) for $k \geq n_1$

$$\sum\nolimits_{m\geq k, m\neq k+j} a_{km}\sigma(Kt_m) \leq \sum\nolimits_{m\geq k, m\neq k+j} a_{km}(\sigma(K)+\delta) = (\sigma(K)+\delta)(1-a_{kk+j}).$$

Also (by 'amenability at 1') for $k \geq n_1$

$$\begin{aligned} a_{kk+j}\sigma(Kt_{k+j}) &\geq \sigma(K) - \delta - \sum\nolimits_{m\geq k, m\neq k+j} a_{km}\sigma(Kt_m) \\ &\geq \sigma(K) + \delta - 2\delta - (\sigma(K)+\delta)(1-a_{kk+j}) \\ &= a_{kk+j}\sigma(K) - \delta(2 - a_{kk+j}). \end{aligned}$$

So for $m = k + j > n_1 + j$

$$\sigma(Kt_m) = \sigma(Kt_{k+j}) \geq \sigma(K) - \delta\left(\frac{2}{a_{kk+j}} - 1\right) \geq \sigma(K) - \delta\left(\frac{2}{\alpha} - 1\right) = \sigma(K) - \varepsilon.$$

As for the final assertion concerning symmetrization, note that $\sigma(t_n K) = \sigma(K^{-1}t_n^{-1}) \to \sigma(K^{-1}) = \sigma(K)$, by symmetry of σ. □

We note an immediate corollary, needed in Part II §2.

Corollary 4. *For $G, \mathbf{t}$ and σ as in Th.*1S *above, and $K, H \in \mathcal{K}_+(\sigma), \delta > 0$: if $0 < \Delta < \sigma(K)$ and $0 < D < \sigma(H)$, then there is n with*

$$B_\delta^{K,\Delta} \cap B_\delta^{H,D} \supseteq \{t_m : m \geq n\}.$$

Proof. Take $\varepsilon := \min\{\sigma(K)-\Delta, \sigma(H)-D\} > 0$. As $K, H \in \mathcal{K}_+(\sigma)$, there is n such that $||t_m|| < \delta$ for $m \geq n$ and

$$\sigma(Kt_m) \geq \sigma(K) - \varepsilon \geq \Delta, \qquad \sigma(Ht_m) \geq \sigma(H) - \varepsilon \geq D \qquad (m \geq n). \qquad \square$$

4. Acknowledgement

We are delighted to have the opportunity to contribute our paper to this memorial issue dedicated to Prof. Harry Miller a fascinating man, a friend and collaborator, a scholar and a gentleman, and thank the Editors for their kind invitation to do so.

References

[ArhT] A. Arhangelskii, M. Tkachenko, *Topological groups and related structures*. World Scientific, 2008.

[BanJ] T. Banakh, E. Jabłońska, *Null–finite sets in metric groups and their applications*. Israel J. Math. 230 (2019), 361–386.

[BeeV] G. Beer, L. Villar, *Borel measures and Hausdorff distance*. Trans. Amer. Math. Soc. **307** (1988), 763–772.

[Bin] N. H. Bingham, *Finite additivity versus countable additivity*. Electronic J. History of Probability and Statistics, **6** (2010), 35p.

[BinGT] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular variation*. 2nd ed., Cambridge University Press, 1989 (1st ed. 1987).

[BinO1] N. H. Bingham and A. J. Ostaszewski, *Kingman, category and combinatorics*. Probability and mathematical genetics (Sir John Kingman Festschrift, ed. N. H. Bingham and C. M. Goldie), 135-168, London Math. Soc. Lecture Notes in Mathematics **378**, CUP, 2010.

[BinO2] N. H. Bingham and A. J. Ostaszewski, *Normed groups: Dichotomy and duality*. Dissert. Math. **472** (2010), 138p.

[BinO3] N. H. Bingham and A. J. Ostaszewski, *Dichotomy and infinite combinatorics: the theorems of Steinhaus and Ostrowski*. Math. Proc. Camb. Phil. Soc. **150** (2011), 1-22.

[BinO4] N. H. Bingham and A. J. Ostaszewski, *Homotopy and the Kestelman-Borwein-Ditor theorem*. Canad. Math. Bull. **54** (2011), 12–20.

[BinO5] N. H. Bingham and A. J. Ostaszewski, *Category-measure duality: convexity, midpoint convexity and Berz sublinearity*. Aequationes Mathematicae, **91** (2017), 801-836 (arXiv:1607.05750).

[BinO6] N. H. Bingham and A. J. Ostaszewski, *Beyond Lebesgue and Baire IV: Density topologies and a converse Steinhaus-Weil theorem*. Topology and its Applications **239** (2018), 274-292 (arXiv:1607.00031).

[BinO7] N. H. Bingham and A. J. Ostaszewski, *Additivity, subadditivity and linearity: Automatic continuity and quantifier weakening*. Indag. Math. (N.S.) **29** (2018), 687–713 (arXiv 1405.3948v3).

[BinO8] N. H. Bingham and A. J. Ostaszewski, *Set theory and the analyst*. Eur. J. Math., **5** (2019), 2–48 (arXiv:1801.09149v2).

[BinO9] N. H. Bingham and A. J. Ostaszewski, *Beyond Haar and Cameron-Martin: the Steinhaus support*. Topology Appl. **260** (2019), 23–56 arXiv: 1805.02325v2.

[BinO10] N. H. Bingham and A. J. Ostaszewski, The Steinhaus-Weil property and its converse: subcontinuity and amenability, arXiv:1607.00049.

[BinO11] N. H. Bingham and A. J. Ostaszewski, The Steinhaus-Weil property: II. The Simmons-Mospan converse. Preprint.

[BinO12] N. H. Bingham and A. J. Ostaszewski, The Steinhaus-Weil property: III. Weil topologies. Preprint.

[BinO13] N. H. Bingham and A. J. Ostaszewski, The Steinhaus-Weil property: IV. Other interior-point properties. Preprint.

[Bog1] V. I. Bogachev, *Gaussian Measures*, Math. Surveys & Monographs **62**, Amer Math Soc., 1998.

[Bog2] V. I. Bogachev, *Measure theory, I, II*. Springer-Verlag, Berlin, 2007.

[Bog3] V. I. Bogachev, *Differentiable measures and the Malliavin calculus*. Math. Surv. Mon. **164**, American Math. Soc, 2010.

[Bord] K. C. Border, *Fixed point theorems with applications to economics and game theory*. Cambridge University Press, 2nd ed. 1989 (1st ed.1985).

[Bou] A. Bouziad, Continuity of separately continuous group actions in p-spaces. *Topology Appl.* **71** (1996), 119-124.

[Chr1] J. P. R. Christensen, On sets of Haar measure zero in abelian Polish groups. Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972). *Israel J. Math*. **13** (1972), 255–260 (1973).

[Chr2] J. P. R. Christensen, *Topology and Borel structure. Descriptive topology and set theory with applications to functional analysis and measure theory*. North-Holland Mathematics Studies **10**, 1974.

[CieJ] K. Ciesielski, J. Jasiński, Topologies making a given ideal nowhere dense or meager. *Topology Appl.* **63** (1995), 277–298.

[Dar] U. B. Darji, On Haar meager sets. *Topology Appl*.**160** (2013), 2396–2400.

[DieS] J. Diestel, A. Spalsbury, *The joys of Haar measure*. Grad. Studies in Math. **150**. Amer. Math. Soc., 2014.

[Eng] R. Engelking, *General topology*. 2nd ed., Heldermann, 1989 (1st ed. PWN, 1977).

[FenN] J. E. Fenstad, D. Normann, On absolutely measurable sets. *Fund. Math*. **81** (1973/74), 91–98.

[For] M. K. Fort, Jr., A unified theory of semi-continuity. *Duke Math. J.* **16** (1949), 237–246.

[Fre] D. Fremlin, *Measure theory* Vol. 3: Measure algebras. Corrected 2nd printing of the 2002 original. Torres Fremlin, Colchester, 2004.

[FreNR] D. Fremlin, T. Natkaniec, I. Recław, *Universally Kuratowski-Ulam spaces*. Fund. Math. **165** (2000), 239–247.

[Ful] R. V. Fuller, *Relations among continuous and various non-continuous functions*. Pacific J. Math. **25** (1968), 495–509.

[Gow1] C. Gowrisankaran, *Radon measures on groups*. Proc. Amer. Math. Soc. **25** (1970), 381–384.

[Gow2] C. Gowrisankaran, *Quasi-invariant Radon measures on groups*. Proc. Amer. Math. Soc. **35** (1972), 503–506.

[GroE] K.-G. Grosse-Erdmann, *An extension of the Steinhaus-Weil theorem*. Colloq. Math. **57** (1989), 307–317.

[Hal] P. R. Halmos, *Measure theory*, Grad. Texts in Math. **18**, Springer 1974 (1st ed. Van Nostrand, 1950).

[Hey] H. Heyer, *Probability measures on locally compact groups*. Ergebnisse Math. **94**, Springer, 1977.

[HolN] L. Holá, B. Novotný, *Subcontinuity*. Math. Slov. **62** (2012), 345–362.

[Jab] E. Jabłońska, *Some analogies between Haar meager sets and Haar null sets in abelian Polish groups*. J. Math. Anal. Appl. **421** (2015), 1479–1486.

[Kal] O. Kallenberg, *Foundations of modern probability*. 2nd ed., Springer, 2002 (1st ed. 1997).

[Kec] A. S. Kechris, *Classical descriptive set theory*. Graduate Texts in Mathematics **156**, Springer, 1995.

[Kem] J. H. B. Kemperman, *A general functional equation*. Trans. Amer. Math. Soc. **86** (1957), 28–56.

[Kuc] M. Kuczma, *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality*. 2nd ed., Birkhäuser, 2009 (1st ed. PWN, Warszawa, 1985).

[LiuR] T. S. Liu, A. van Rooij, *Transformation groups and absolutely continuous measures*, lndag. Math. **71** (1968), 225-231.

[LiuRW] T. S. Liu, A. van Rooij, J-K Wang, *Transformation groups and absolutely continuous measures II*. Indag. Math. **73** (1970), 57–61.

[LukMZ] J. Lukeš, J. Malý, L. Zajíček, *Fine topology methods in real analysis and potential theory*. Lecture Notes Math. **1189**, Springer, 1986.

[Mic] E. Michael, *Topologies on the spaces of subsets*. Trans. Amer. Math. Soc. **71** (1951), 152-182.

[MilO] H. I. Miller and A. J. Ostaszewski, *Group action and shift-compactness*. J. Math. Anal. App. **392** (2012), 23-39.

[Mos] Y. V. Mospan, *A converse to a theorem of Steinhaus-Weil*. Real An. Exch. **31** (2005), 291-294.

[Mun] J. R. Munkres, *Topology, a first course*, Prentice-Hall, 1975.

[Myc] J. Mycielski, *Finitely additive measures*. Coll. Math. **42** (1979), 309-318.

[Ost] A. J. Ostaszewski, *Families of compact sets and their universals*. Mathematika **21** (1974), 116–127.

[Oxt1] J. C. Oxtoby, *Invariant measures in groups which are not locally compact*. Trans. Amer. Math. Soc. **60** (1946), 215–237.

[Oxt2] J. C. Oxtoby, *Measure and category*, 2nd ed. Graduate Texts in Math. 2, Springer, 1980 (1st ed. 1972).

[Par] K. R. Parthasarathy, *Probability measures on metric spaces*. Academic Press, 1967.

[Pat] A. L. T. Paterson, *Amenability*. Math. Surveys and Mon. **29**, Amer. Math. Soc., 1988

[Pet] B. J. Pettis, *On continuity and openness of homomorphisms in topological groups*. Ann. of Math. (2) **52** (1950), 293–308.

[Pic] S. Piccard, *Sur les ensembles de distances des ensembles de points d'un espace Euclidien*. Mém. Univ. Neuchâtel **13**, 212 pp. 1939.

[Rog] C. A. Rogers et al., *Analytic sets*. Academic Press, 1980.

[Sch] L. Schwartz, *Radon measures on arbitrary topological spaces and cylindrical measures*. Tata Institute of Fundamental Research Studies in Mathematics **6**, Oxford University Press, 1973.

[Sho] R.M. Shortt, *Universally measurable spaces: an invariance theorem and diverse characterizations*. Fundamenta Math. **121** (1984), 169-176.

[Sol1] S. Solecki, *Size of subsets of groups and Haar null sets*. *Geom. Funct. Anal*. **15** (2005), 246–273.

[Sol2] S. Solecki, *Amenability, free subgroups, and Haar null sets in non-locally compact groups*. Proc. London Math. Soc. (3) **93** (2006), 693–722.

[Sol3] S. Solecki, A Fubini theorem. *Topology Appl*. **154** (2007), 2462–2464.

[Ste] H. Steinhaus, *Sur les distances des points de mesure positive*. Fund. Math. **1** (1920), 83-104.

[vDo] E. van Dowen, *Fubini's theorem for null sets*. Amer. Math. Monthly **96** (1989), 718-721.

[Wei] A. Weil, *L'intégration dans les groupes topologiques*, Actualités Scientifiques et Industrielles **1145**, Hermann, 1965 (1st ed. 1940).

[Xia] D. X. Xia, *Measure and integration theory on infinite-dimensional spaces. Abstract harmonic analysis*. Pure and App. Math. **48**. Academic Press, 1972.

[Zak] E. Zakon, *A remark on the theorems of Lusin and Egoroff*. Canad. Math. Bull. **7** (1964), 291–295.

(Received: November 17, 2019)
(Revised: March 09, 2020)

Nicholas H. Bingham
Mathematics Department
Imperial College
London SW7 2AZ
e-mail: *n.bingham@ic.ac.uk*
and
Adam J. Ostaszewski
Mathematics Department
London School of Economics
Houghton Street, London WC2A 2AE
e-mail: *A.J.Ostaszewski@lse.ac.uk*