SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.16.01.01
Vol.16 (29), No.1, (2020), 5 – 11

ON NONCOMMUTATIVE PARAGRADED RINGS

MIRJANA VUKOVIĆ

Dedicated to the Memory of my dear Professor Academician Fikret Vajzović with deep esteem

ABSTRACT. Successful study of Bourbaki – Krasner's graded structures problems opened the way to paragraded structures for me. In this paper we study paragraded modules over noncommutative paragraded rings and as a main result prove the paragraded version of Schur's Lemma.

1. INTRODUCTION

It is well known that the first relatively general definition of graded groups and rings was given by Bourbaki [1], although his definition was unnecessarily based on the notion of the abelian graded group. There are surprisingly not much records that all started earlier – before Bourbaki with the Krasner's abstract notion of a corpoid, introduced during 1940s first appearing in a series of his Comptes Rendus notes [8-11], see also [14]. Namely, origins of the notion of homogeneity can be seen in Krasner's earliest works along this line dealt with graded fields when, studying valued fields and observing connection between their valuation rings, by using the equivalence of valuations. This investigations led him to the abstract notion of a corpoid – homogeneous part of a graded field, graded by an arbitrary set, with induced operations among which the induced addition is, naturally, a partial operation, since the sum of two homogeneous elements does not have to be homogeneous ([14], [18]).

Krasner, starting from his corpoid and Borbaki's definition of graded group discovered general graded theory as well as the theory of homogroupoids, anneids and moduloids where corpoid, as a special case of an anneid, is viewed as a homogeneous part of a graded field.

Thus Marc Krasner first introduced the theory of general graded structures (groupes, rings, modules), assuming for the set of grades only its nonemptiness, since, in his definitions, additive graduation and structures of rings and modules will imply operations (generally partial) in the set of grades ([14], see also [2] and [3].

2010 *Mathematics Subject Classification.* 16N20, 16W50.
Key words and phrases. Paragraded ring, Ireducible paragraded module.

It is well known that the graded structures (groups, rings, modules) compose the categories that are not closed with respect to direct sum and direct product. The aim of a new concept in my joint works ([15]) and monograph ([18]) (see also [16, 17, 19]) with M. Krasner was to introduce the algebraic structures which generalize the classic graded structures and have in each of the three cases: groups, rings, and modules, the property of closure with respect the direct sum and the direct product in the sense that the support of the homogeneous subset of the composition (direct sum or direct product) is the restricted direct sum or product of the components.

Strengthening and generalizing certain ideas of M. Krasner in 1980s we obtained a wonderful result – paragraded structures, which generalize not only, the theory of graded structures as exposed in Bourbaki ([1]), but also the previous results of M. Krasner ([14]) and M. Chadeyras ([2]. In this way I obtained one of my most important results.

2. Preliminaries

In this section, we first introduce some notions terminology and a few basic facts about paragraded rings and modules:

Definition 2.1. ([18]) *The mapping* $\pi : \Delta \to \mathrm{Sg}(G)$, $\pi(\delta) = G_\delta,\ \delta \in \Delta$, *of partially ordered set* $(\Delta, <)$, *which is from bellow complete semi-lattice and from beyond inductively ordered, to the set* $\mathrm{Sg}(G)$ *of subgroups of the group G, is called paragraduation if it satisfies the following six-axiom system*:

i) $\pi(0) = G_0 = \{0\}$, *where* $0 = \inf \Delta$; $\delta < \delta' \Rightarrow G_\delta \subseteq G_{\delta'}$;

Remark 2.1. $H = \bigcup_{\delta\in\Delta} G_\delta$ *is called the homogeneous part of G with respect to* π.

Remark 2.2. If $x \in H$, *we say that* $\delta(x) = \inf\{\delta \in \Delta \,|x \in G_\delta\}$ *is the grade of x. We have* $\delta(x) = 0$ *iff* $x = e$. *Elements* $\delta(x), x \in H$, *are called principal grades and they form a set which we will denote by* Δ_p.

ii) $\theta \subseteq \Delta \Rightarrow \bigcap_{\delta\in\theta} G_\delta = G_{\inf\theta}$;

iii) *If* $x, y \in H$ *and* $yx = zxy$, *then* $z \in H$ *and* $\delta(z) \le \inf\{\delta(x), \delta(y)\}$;

iv) *Homogeneous part H is a generating set of G*;

v) *Let* $A \subseteq H$ *be a subset such that for all* $x, y \in A$ *there exists an upper bound for* $\delta(x), \delta(y)$. *Then there exists an upper bound for all* $\delta(x), x \in A$;

vi) *G is generated by H with the set of relations*:

- $xy = z$ (*H-inner*);
- $yx = z(x, y)xy$ (*left commutation*).

A group with paragraduation is called a *paragraded group*.

Definition 2.2. ([18]) *The ring* $(R, +, \cdot)$ *is called paragraded if its additive group* $(R, +)$ *is a paragraded group, with paragraduation* π *and set of grades* Δ *and if*

$$(\forall \xi, \eta \in \Delta)(\exists \zeta \in \Delta)\ R_\xi R_\eta \subseteq R_\zeta\,. \tag{2.1}$$

This definition gives a binary operation on Δ:

$$\xi\eta = \sup\{\delta(z) | z \in R_\xi R_\eta\}$$

called the minimal multiplication [15-18], see also [19] and so $R_\xi R_\eta \subseteq R_{\xi\eta}$.

Proposition 2.1. *If R is a paragraded ring with homogeneous part H and $\delta(x)$ is the degree of an element $x \in H$, then condition* (2.1) *is equivalent with the following two conditions:*

(1) $x, y \in H \Rightarrow xy \in H$, *i.e.* $H^2 \subseteq H$;
(2) *if* $\xi, \eta \in \Delta$, *then for* $x, x', y, y' \in H$ *such that* $\delta(x), \delta(x') \leq \xi$ *and* $\delta(y), \delta(y') \leq \eta$, *we have that* $xy + x'y' \in H$.

Remark 2.3. In the case of graded rings, analogues of conditions (1) and (2) are not equivalent we can only prove that (2) is a consequence of (1) ([1]).

Definition 2.3. ([18]) *Let R_1 and R_2 be two paragraded rings and $f : R_1 \to R_2$ a homomorphism. We say that this homomorphism is a quasihomogeneous if it is a quasihomogeneous homomorphism from a paragraded group $(R_1, +)$ to a paragraded group $(R_2, +)$, i.e. if*

$$(\forall x \in H_1) \quad f(x) \in H_2\,.$$

A homomorphism is called quasihomogeneous if an image of a homogeneous element is a homogeneous element too.

Definition 2.4. ([18]) *A subring (ideal) I of a paragraded ring R is called homogeneous if $(I, +)$ is generated by $I \cap H$.*

Proposition 2.2. ([18]) *Let I be a honogeneous ideal of paragraded ring R. Then R/I is also a paragraded ring.*

Proposition 2.3. ([18]) *If $I \subseteq R$ and $h(I) = I \cap H$, then I is a homogeneous subring of paragraded ring R iff $h = h(I)$ satisfies the following three conditions:*

$a)$ $x \in h \Rightarrow -x \in h$;
$b)$ $x, y \in h$ *and* $x + y \in H \Rightarrow x + y \in h$;
$c)$ $x, y \in h \Rightarrow xy \in h$.

Lemma 2.1. ([18]) *The subset $I \subseteq R$ is a left rep. right, homogeneous ideal of R iff I satisfies conditions $a)$ and $b)$ of Proposition 2.2. and the condition*
$c')$ $x \in H$ *and* $y \in h \Rightarrow xy \in h$ *resp.* $c'')$ $x \in h$ *and* $y \in H \Rightarrow xy \in h$.

Now, we will give the definition of a paragraded module.

Definition 2.5. ([18]) *Let R be a paragraded ring with paragraduation π and set of grades Δ, M a commutative paragraded group with paragraduation π' and set of grades D and in addition let M be an R-module. Denote*

$$\pi(\delta) \text{ by } R_\delta \text{ and } \pi'(d) \text{ by } M_d, \text{ where } \delta \in \Delta,\ d \in D.$$

R-module M is called left resp. right paragraded if

$$(\forall\delta\in\Delta)(\forall d\in D)(\exists t\in D)\ R_\delta M_d\subseteq M_t,\ \ resp.\ \ M_dR_\delta\subseteq M_t.$$

This definition also gives a minimal multiplication

$$d\delta\in\Delta=\sup\{\delta(z)\,|\,z\in M_dR_\delta\}\ \text{ where }\ d(z)=\inf\{d\in D\,|\,z\in M_d\}.$$

In this paper we continue to do research in the theory of paragraded rings by giving the paragraded version of Schur's Lemme using the approach given by I.N. Herstein in [4].

3. Paragraded Modules and Schur's Lemma

Let M be the right paragraded R-module with set of grades Δ, and where R is the paragraded ring with the same set of grades Δ. From now on we will call this kind of modules the *paragraded modules of type* Δ and if not stated all modules will be assumed to be of that kind.

Definition 3.1. ([18]) *Let M and M' be paragraded R-modules of type Δ. For a homomorphism $f: M\to M'$ we say that it is a morphism of grade δ if*

$$(\forall\delta'\in\Delta)\ f(M_{\delta'})\subseteq M'_{\delta'\delta},$$

where $\delta'\delta$ is a minimal multiplication. The set of all morphisms of grade δ we denote by $\hom(M,M')_\delta$ and define $\mathrm{HOM}(M,M')$ by

$$\mathrm{HOM}(M,M')=\left\langle\bigcup\nolimits_{\delta\in\Delta}\hom(M,M')_\delta\right\rangle.$$

Remark 3.1. If $M=M'$, then, instead of $\mathrm{HOM}(M,M')$, we write $\mathrm{END}(M)$.

We will prove that $\mathrm{END}(M)$ is a paragraded ring of type Δ with respect to composition of morphisms, under the assumption of associativity of minimal multiplication.

Here f_a is the "natural" example of a morphism of grade $\delta\in\Delta$. Namely, if M is a paragraded R-module of type Δ and $a\in R_\delta$, for some $\delta\in\Delta$, then the mappimg

$$f_a: M\to M,\text{ defined by } f_a(m)=ma\ (m\in M),$$

represents a morphism of grade δ. Indeed, if $\xi\in\Delta$, then

$$f_a(M_\xi)=M_\xi a\subseteq M_\xi R_\delta\subseteq M_{\xi\delta}.$$

Proposition 3.1. *Let minimal multiplication be associative and let M be a paragraded R-module of type Δ. Then $\mathrm{HOM}(M,M)$ is the paragraded ring which we denote shortly by $\mathrm{END}(M)$.*

Proof. It is an easy excersise to check that under assumptions that minimal multiplication is associative and that M is a paragraded R-module of type Δ, it follows that $(\mathrm{Hom}(M.M),+)=(\mathrm{END}(M),+)$ is the commutative paragraded group.

In the rest of proof, define the multiplication of morphisms as follows.

If for morphisms $f_1, f_2 \in \mathrm{END}(M)$ of grades δ_1, δ_2 respectively, we define $f_1 \cdot f_2$ by $f_1 \cdot f_2 := f_2 \circ f_1$, we obtain

$$(f_2 \circ f_1)(M_\delta) = f_2(f_1(M_\delta)) = f_2(M_{\delta\delta_1}) \subseteq M_{(\delta\delta_1)\delta_2},\ \delta \in \Delta.$$

In accordance with associativity of minimal multiplication it is

$$M_{(\delta\delta_1)\delta_2} = M_{\delta(\delta_1\delta_2)}, \quad for\ \ \delta \in \Delta.$$

Thus, we have

$$(f_2 \circ f_1)(M_\delta) \subseteq M_{\delta(\delta_1\delta_2)}, \quad for\ all\ \ \delta \in \Delta,$$

i.e. $f_2 \circ f_1$ is morphism of grade $\delta_1\delta_2$.

Hence, if $\delta_1, \delta_2 \in \Delta$, then

$$\mathrm{END}(M)_{\delta_1}\mathrm{END}(M)_{\delta_2} = \hom(M,M)_{\delta_1}\hom(M,M)_{\delta_2} \subseteq \hom(M,M)_{\delta_1\delta_2}$$

which concludes the proof. □

Definition 3.2. *The set* $A(M)_\delta = \{r \in R_\delta | Mr = \{0\}\}$ *is called* δ*-annihilator for some* $\delta \in \Delta$.

Now, we will observe the set

$$A(M) = \langle \cup_{\delta\in\Delta} A(M)_\delta \rangle.$$

Lemma 3.1. *Let M be a paragraded R-module of type Δ and let minimal multiplication be associative. Then $A(M)$ is isomorphic to a subring of* $\mathrm{END}(M)$.

Proof. Let $\varphi : R \to \mathrm{END}(M)$ be the mapping defined by $\varphi(a) = f_a$ $(a \in R)$. The mapping $f_a \in \mathrm{END}(M)$, because a is generated by homogeneous elements, sey a_δ, and as we saw earlier $f_a \in \hom(M,M)_\delta$.

Since, for $\delta \in \Delta$ and $a \in R_\delta$, we have

$$\varphi(a) = f_a \in \hom(M,M)_\delta,$$

φ is ring homomorphism and it is morphism in the category of paragraded rings of type Δ. Because it is $\ker(\varphi) = A(M), R/A(M)$ is isomorphic to a subring $\mathrm{END}(M)$, which concludes the proof. □

Definition 3.3. *For* $\delta \in \Delta$, *we define*

$$C(M)_\delta = \{f \in \hom(M,M)_\delta \,|\, f f_a = f_a f\ (a \in R)\}\ \ and\ \ C(M) = \langle \bigcup_{\delta\in\Delta} C(M)_\delta \rangle.$$

Remark 3.2. If M is a paragraded R-module of type Δ and if minimal multiplication is associative then it is clear that $C(M)$ is a homogeneous subring of $\mathrm{END}(M)$. Moreover, like in the abstract case [4], $C(M)$ is the ring of all module endomorphisms of M in the category of paragraded modules of type Δ if we make one more assumption – the commutativity of minimal multiplication.

Indeed, let $f \in C(M)$. Then $f \in \sum_\delta f_\delta$, where $f_\delta \in C(M)_\delta$. If $m \in M_\xi$ and $a \in R_\eta$, for some $\xi, \eta \in \Delta$, then

$$f_\delta(m)a = f_a(f_\delta(m)) = f_\delta(f_a(m)) = f_\delta(ma),$$

$$f_\delta(m)a \in M_{\xi\delta}a \subseteq M_{\xi\delta\eta}, \text{ and } f_\delta(ma) \in f_\delta(M_{\xi\eta}) \subseteq M_{\xi\eta\delta} = M_{\xi\delta\eta}.$$

In order to prove the paragraded version of Schur's Lemma we introduce the notion of irreducibility.

Definition 3.4. *We call M an ireducible paragraded R-module of type Δ if there exist $\xi, \eta \in \Delta$ such that $M_\xi R_\eta \neq \{0\}$ and if the only homogeneous submodules of M are $\{0\}$ and M.*

Theorem 3.2. (Schur's Lemma) *Let M be an irreducible paragraded R-module of type Δ and let minimal multiplication be associative. Then $C(M)$ is a paragraded field of type Δ.*

Proof. As in non-paragraded case [4], it is enough to prove that nonzero elemet from $C(M)$ is invertible in $\mathrm{END}(M)$. if $0 \neq f \in C(M)$, then $f(M)$ is honmogeneous paragraded submodule [18]. Since $f \neq 0$, the irreducibilty of M implies $f(M) = M$, i.e. f is surjective. The irreducibility also gives us that f is injective. Thus f is invertibile in $\mathrm{END}(M)$. □

References

[1] N. Bourbaki, *Algèbre*, Chap. II, 3e édit, Paris, Hermann, 1962.

[2] M. Chadeyras, *Essai d'une théorie noetherienne pour les anneaux commutatifs, dont la graduation est aussi générale que possible*, Bull. de la S.M.F., Supplément, Mémoire No. 22, Paris, 1970.

[3] E. Halberstadt, *Théorie artinienne homogène des anneaux gradués à grades non commutatifs réguliers*, PhD Thesis, University Piere and Marie Curie, Paris, 1971.

[4] I.N. Herstein, *Noncommutative Rings*, Carus Mathematical Monographs 15. Fifth Printing 2005.

[5] E. Ilić-Georgijević, M. Vuković, *The Wedderburn–Artin Theorem for paragraded rings*, Fundam. Prikl. Mat., **19** (6) (2014), 125–139, J. Math. Sci., New York, 221, No. 3 (2017), 391–400.

[6] E. Ilić-Georgijević, M. Vuković, *A note on radicals of paragraded rings*, Sarajevo J. Math., Vol. 12(25), No. 2 (2016), 307–316.

[7] E. Ilić-Georgijević, M. Vuković, *A note on general radicals of paragraded rings*, Sarajevo J. Math., Vol. 12(25), No. 2 (2016), 317–324, Suppl.

[8] M. Krasner, *Une généralisation de la notion de corps-corpoíde. Un corpoíde remarquable de la théorie des corps values*, C. R. Acad. Sci., Paris 219 (1944), 345-347.

[9] M. Krasner, *Hypergroupes moduliformes et extramoduliformes*, C. R. Acad. Sci., Paris, 219 (1944), 473-476.

[10] M. Krasner, *Théorie de la ramification dans les extensions finies des corps values: Hypergroupe d'inertie et de ramification; Théorie extrinsèque de la ramification*, C. R. Acad. Sci., Paris, 220 (1945), 28-30.

[11] M. Krasner, *Quelques méthodes nouvelles dans la théorie des corps values complets*, Algèbre et théorie des nombres (Colloque Int. du C. N. R. S., No. 2, Paris 1949), Edit, C. N. R. S., Paris 1950.

[12] M. Krasner, *Congruences multiplicatives, Squelettes et corpoïdes*. Séminaire Krasner, 1953-54, Vol. 1, exposé No. 4, Secrétariat Math. de la Fac. des Sc. Paris, 1955.

[13] M. Krasner, *Théorie élémentaire des corpoïdes commutatifs sans torsion*, Séminaire Krasner, 1953-54, Vol. 2, exposé No. 5, Secrétariat Math. de la Fac. des Sc. Paris, 1956.

[14] M. Krasner, *Anneaux gradués généraux*, Colloque d'Algèbre Rennes, 1980, 209–308.
[15] M. Krasner, M. Vuković, *Structures paragraduées* (*groupes, anneaux, modules*) I, Proc. Japan Acad., Ser, A, 62 (9) (1986), 350–352.
[16] M. Krasner, M. Vuković, *Structures paragraduées (groupes, anneaux, modules)* II, Proc. Japan Acad., Ser, A, 62 (10) (1986), 389–391.
[17] M. Krasner, M. Vuković, *Structures paragraduées (groupes, anneaux, modules)* III, Proc. Japan Acad., Ser, A, 63 (1) (1987), 10–12.
[18] M. Krasner, M. Vuković, *Structures paragraduées (groupes, anneaux, modules)*, Queen's Papers in Pure and Applied Mathematics, No. 77, Queen's University, Kingston, Ontario, Canada 1987.
[19] M. Vuković, *Structures graduées et paragraduées*, Prepublication de l'Institut Fourier, Université de Grenoble I, No. **536** (2001), pp. 1-40.
[20] M. Vuković, E. Ilić-Georgijević, *Paragraded rings and their ideals*, Fundam. Prikl. Mat., 17 (4) (2012), 83–93, J. Math. Sci., New York 191, No. 5 (2013), 654–660.
[21] M. Vuković, *From Krasner's corpoid and Bourbaki's graduations to Krasner's graduations and Krasner-Vuković's paragraduations*, Sarajevo J. Math., Vol. 14 (27), No. 2 (2018), 175–190.

(Received: December 21, 2019)
(Revised: June 20, 2020)

Mirjana Vuković
Academy of Sciences and Arts of
Bosnia and Herzegovina
Bistrik 7, 71000 Sarajvo, BA
e-mail: *mvukovic@anubih.ba*