SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.14.2.10
Vol.14 (27), No.2, (2018), 239–250

BANACH ALGEBRAS OF ULTRAMETRIC LIPSCHITZIAN FUNCTIONS

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Dedicated to Professor Mirjana Vuković on the occasion of her 70th birtday

ABSTRACT. We examine Banach algebras of bounded uniformly continuous functions and particularly Lipschitzian functions from an ultrametric space $\mathbb{E}$ to a complete ultrametric field $\mathbb{K}$: prime and maximal ideals, multiplicative spectrum, Shilov boundary and topological divisors of zero. We get a new compactification of $\mathbb{E}$ similar to the Banaschewski's one and which is homeomorphic to the multiplicative spectrum. On these algebras, we consider several norms or semi-norms: a norm letting them to be complete, the spectral semi-norm and the norm of uniform convergence (which are weaker), for which prime closed ideals are maximal ideals. When $\mathbb{E}$ is a subset of $\mathbb{K}$, we also examine algebras of Lipschitzian functions that are derivable or strictly differentiable. Finally, we examine certain abstract Banach $\mathbb{K}$-algebras in order to show that they are algebras of Lipschitzian functions on an ultrametric space through a kind of Gelfand transform.

1. INTRODUCTION

Let $\mathbb{K}$ be an ultrametric complete field and $\mathbb{E}$ be an ultrametric space. It is well known that the set of maximal ideals of a Banach $\mathbb{K}$-algebra is not sufficient to describe its spectral properties: we have to consider the set of continuous multiplicative semi-norms often called the multiplicative spectrum. Many studies were made on continuous multiplicative semi-norms on algebras of analytic functions, analytic elements and their applications to holomorphic functional calculus.

In Sections 2, 3, 4 we generalize some of the results previously obtained in [8], [9] and [10] to some Banach algebras of bounded uniformly continuous functions which we call *semi-compatible* and which are related to the *contiguity relation* yet considered in these papers. These results concern maximal ideals, multiplicative spectrum and Shilov boundary. In Section 5, we study the Stone space of some

2010 *Mathematics Subject Classification.* 30D35; 30G06, 46S10.
Key words and phrases. ultrametric Banach algebras, ultrafilters.
This paper was presented at the International Scientific Conference *Modern Algebra and Analysis and their Applications*, ANUBiH - Sarajevo, September 20 - 22, 2018.

Boolean subring of the clopen sets of $\mathbb{E}$ which turn to be a compactification of $\mathbb{E}$ homeomorphic to the multiplicative spectrum.

The algebra $\mathcal{B}$ of uniformly continuous functions is semi-compatible. In Section 6, we consider the algebra $\mathcal{L}$ of Lipschitzian functions and when $\mathbb{E}$ is a subset of $\mathbb{K}$, the algebras $\mathcal{D}$ of derivable functions and $\mathcal{E}$ of strictly differentiable functions. These are semi-compatible algebras provided with a suitable complete norm. In Section 7 we get some properties for $\mathcal{B}$, $\mathcal{L}$, $\mathcal{D}$ or $\mathcal{E}$ in particular about topological divisors of zero.

In Section 8, we can show that a large class of unital ultrametric commutative Banach algebra may be viewed as a class of Lipschitzian functions.

The proofs will be published in a further paper.

2. Basic results in topology

Notations and definitions: *Let $\mathbb{K}$ be a field complete with respect to an ultrametric absolute value $|\ .\ |$, let $\mathbb{E}$ denote a metric space whose distance δ is ultrametric.*

Finally we denote by $|\ .\ |_\infty$ the Archimedean absolute value of $\mathbb{R}$.

If $F \subset \mathbb{E}$, the function u defined on $\mathbb{E}$ by $u(x) = 1$ if $x \in F$ and $u(x) = 0$ if $x \notin F$, will be called the characteristic function of F.

Given a subset F of $\mathbb{E}$ such that $F \neq \emptyset$ and $F \neq \mathbb{E}$, we call codiameter of F the number $\delta(F, \mathbb{E} \setminus F)$. If $F = \emptyset$ or $F = \mathbb{E}$, we say that its codiameter is infinite. The set F will be said to be uniformly open if its codiameter is strictly positive.

We will denote by $\mathbb{G}(\mathbb{E})$ the family of uniformly open subsets of $\mathbb{E}$. In [8], dealing with the Banaschewski compactification of $\mathbb{E}$ the authors considered the Boolean ring of clopen sets of $\mathbb{E}$ (with the usual addition Δ and multiplication $\cap$). In Section 5 we will consider the Boolean ring of uniformly open sets.

Given a normed $\mathbb{K}$-algebra whose norm is $\|\ .\ \|$, we denote by $\|\ .\ \|_{sp}$ the spectral semi-norm that is associated and defined as $\|f\|_{sp} = \lim\limits_{n\to+\infty} \left(\|f^n\|\right)^{\frac{1}{n}}$ [4], [5]. We denote by $\|\ .\ \|_0$ the norm of uniform convergence on $\mathbb{E}$ and we denote by $\mathcal{B}$ the Banach $\mathbb{K}$-algebra of bounded uniformly continuous functions on $\mathbb{E}$ provided with the norm $\|\ .\ \|_0$.

Proposition 2. 1 is classical:

Proposition 2.1. *Let A be a commutative unital Banach $\mathbb{K}$-algebra of bounded functions defined on $\mathbb{E}$. Then $\|f\|_0 \leq \|f\|_{sp} \leq \|f\|\ \ \forall f \in A$. Moreover, given $f \in A$ satisfying $\|f\|_{sp} < 1$, then $\lim\limits_{n\to+\infty} \|f^n\| = 0$.*

Definition. *We will call semi-compatible algebra a unital commutative Banach $\mathbb{K}$-algebra S of uniformly continuous bounded functions f from $\mathbb{E}$ to $\mathbb{K}$ satisfying the two following properties:*

1) *every function $f \in S$ such that $\inf_{x\in E} |f(x)| > 0$ is invertible in S,*

2) *for every subset $F \subset \mathbb{E}$, the characteristic function of F belongs to S if and only if F is uniformly open.*

   Moreover, a semi-compatible algebra S will be said to be C-compatible if it satisfies

3) *the spectral semi-norm of S is equal to the norm $\| \, . \, \|_0$.*

Given a subset X of S, we call spectral closure of X denoted by $\widetilde{X}$ the closure of X with respect to the norm $\| \, . \, \|_{sp}$ and X will be said to be spectrally closed if $X = \widetilde{X}$.

Throughout the lecture, we will denote by S a semi-compatible $\mathbb{K}$-algebra.

Let $f \in \mathcal{B}$ be such that $\inf\{ \, |f(x)| \mid x \in \mathbb{E}\} > 0$, it is clear that $\dfrac{1}{f}$ belongs to $\mathcal{B}$ and we can check that the spectral norm $\| \, . \, \|_{sp}$ is just $\| \, . \, \|_0$ and that a subset F of $\mathbb{E}$ is uniformly open if and only if its characteristic function is uniformly continuous. Therefore, the following statement is almost immediate:

Theorem 2.2. *The Banach $\mathbb{K}$-algebra $\mathcal{B}$ is C-compatible.*

More notations and definitions: *Let $\mathcal{F}$ be a filter on $\mathbb{E}$. Given a function f from $\mathbb{E}$ to $\mathbb{K}$ admitting a limit along $\mathcal{F}$, we will denote by $\lim\limits_{\mathcal{F}} f(x)$ this limit.*

Given a filter $\mathcal{F}$ on $\mathbb{E}$, we will denote by $\mathcal{J}(\mathcal{F},S)$ the ideal of the $f \in S$ such that $\lim_{\mathcal{F}} f(x) = 0$. Notice that the unity does not belong to $\mathcal{J}(\mathcal{F},S)$, so $\mathcal{J}(\mathcal{F},S) \neq S$.

Given $a \in \mathbb{E}$, we will denote by $\mathcal{J}(a,S)$ the ideal of the $f \in S$ such that $f(a) = 0$.

We will denote by $Max(S)$ the set of maximal ideals of S and by $Max_{\mathbb{E}}(S)$ the set of maximal ideals of S of the form $\mathcal{J}(a,S)$, $a \in \mathbb{E}$.

Given a set F, we will denote by $U(F)$ the set of ultrafilters on F.

Two ultrafilters $\mathcal{F}$, $\mathcal{G}$ on $\mathbb{E}$ will be said to be contiguous if for every $H \in \mathcal{F}$, $L \in \mathcal{G}$, we have $\delta(H,L) = 0$. We will denote by $(\mathcal{R})$ the relation defined on $U(\mathbb{E})$ as $\mathcal{U}(\mathcal{R})V$ if $\mathcal{U}$ and $\mathcal{V}$ are contiguous.

Remark 1: T he contiguity relation on ultrafilters on $\mathbb{E}$ is a particular case of the relation on ultrafilters defined by Labib Haddad and in other terms by Pierre Samuel in a uniform space. This relation on a uniform space actually is an equivalence relation [12], [14].

Proposition 2.3. *Every maximal ideal $\mathcal{M}$ of S is closed with respect to the norm $\| \, . \, \|_0$ and hence is spectrally closed.*

Proposition 2.4 now is easy:

Proposition 2.4. *Given an ultrafilter $\mathcal{U}$ on $\mathbb{E}$, $\mathcal{J}(\mathcal{U},S)$ is a prime ideal closed with respect to the norm $\| \, . \, \|_0$.*

Corollary 2.4.a: *Given an ultrafilter $\mathcal{U}$ on $\mathbb{E}$, $\mathcal{J}(\mathcal{U},S)$ is a spectrally closed prime ideal.*

We have the following Proposition 2.5 which is important:

Proposition 2.5. *Let $\mathcal{U}$, V be ultrafilters on* $\mathbb{E}$. *Then $\mathcal{U}$ and $\mathcal{V}$ are contiguous if and only if they contain the same uniformly open sets.*

Corollary 2.5.a: *Relation $(\mathcal{R})$ is an equivalence relation on $U(\mathbb{E})$.*

As a consequence we can derive the following theorem:

Theorem 2.6. *Let $\mathcal{U}$, V be ultrafilters on* $\mathbb{E}$. *Then $\mathcal{J}(\mathcal{U},S) = \mathcal{J}(\mathcal{V},S)$ if and only if $\mathcal{U}$ and $\mathcal{V}$ are contiguous.*

We must now state a theorem that is essential for all further results and is often called the Corona Theorem [15]:

Theorem 2.7. *Let $f_1, ..., f_q \in S$ satisfy*

$$\inf_{x\in\mathbb{E}} (\max_{1\leq j\leq q} |f_j(x)|) > 0.$$

Then there exists $g_1, ..., g_q \in S$ such that

$$\sum_{j=1}^{q} f_j(x) g_j(x) = 1 \ \forall x \in \mathbb{E}.$$

Notation: Given $f \in S$ and $\varepsilon > 0$, we put $D(f,\varepsilon) = \{x \in \mathbb{E} \mid |f(x)| \leq \varepsilon\}$.

Corollary 2.7.a: *Let I be an ideal of S different from S. The family of sets*

$$\{D(f,\varepsilon),\ f \in I,\ \varepsilon > 0\}$$

generates a filter $\mathcal{F}_{I,S}$ on $\mathbb{E}$ *such that $I \subset \mathcal{J}(\mathcal{F}_{I,S},S)$.*

3. MAXIMAL AND PRIME IDEALS OF S

Theorem 3.1. *Let $\mathcal{M}$ be a maximal ideal of S. There exists an ultrafilter $\mathcal{U}$ on* $\mathbb{E}$ *such that $\mathcal{M} = \mathcal{J}(\mathcal{U},S)$. Moreover, $\mathcal{M}$ is of codimension* 1 *if and only if every element of S converges along $\mathcal{U}$. In particular if $\mathcal{U}$ is convergent, then $\mathcal{M}$ is of codimension* 1.

Remark 2: I f $\mathbb{K}$ is not locally compact, a maximal ideal of codimension 1 of S is not necessarily of the form $\mathcal{J}(\mathcal{U},S)$ where $\mathcal{U}$ is a converging ultrafilter. Suppose that $\mathbb{E}$ admits a sequence $(a_n)_{n\in\mathbb{N}}$ such that either it satisfies $|a_n - a_m| = r\ \forall n \neq m$, or the sequence $|a_{n+1} - a_n|$ is strictly increasing. Let $\mathcal{U}$ be an ultrafilter thinner than the sequence $(a_n)_{n\in\mathbb{N}}$.

Consider now a function $f \in S$ and let $\mathcal{W}$ be the filter admitting for basis $f(\mathcal{U})$. Then $\mathcal{W}$ is an ultrafilter again and hence it converges in $\mathbb{K}$ to a point $b \in \mathbb{K}$. In that way, we can define a homomorphism χ from S onto $\mathbb{K}$ as $\chi(g) = \lim_{\mathcal{U}} g(x)$ and therefore $\mathbb{K}$ is the quotient $\dfrac{S}{Ker(\chi)}$.

Corollary 3.1.a: *Let* $\mathbb{K}$ *be a locally compact field. Then every maximal ideal of S is of codimension 1.*

Notation: We will denote by $Y_{(\mathcal{R})}(\mathbb{E})$ the set of equivalence classes on $U(\mathbb{E})$ with respect to the relation $(\mathcal{R})$.

Corollary 3.1.b: *Let $\mathcal{M}$ be a maximal ideal of S. There exists a unique $\mathcal{H} \in Y_{(\mathcal{R})}(\mathbb{E})$ such that $\mathcal{M} = \mathcal{J}(\mathcal{U},S)$ for every $\mathcal{U} \in \mathcal{H}$.*

Conversely, Theorem 3.2 now characterizes all maximal ideals of S.

Theorem 3.2. *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{E}$. Then $\mathcal{J}(\mathcal{U},S)$ is a maximal ideal of S.*

By Corollary 3.1.b and Theorem 3.2., we derive the following Corollary 3.2.a:

Corollary 3.2.a: *The mapping that associates to each maximal ideal $\mathcal{M}$ of S the class with respect to $(\mathcal{R})$ of ultrafilters $\mathcal{U}$ such that $\mathcal{M} = \mathcal{J}(\mathcal{U},S)$, is a bijection from $Max(S)$ onto $Y_{(\mathcal{R})}(\mathbb{E})$.*

Theorem 3.3. *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{E}$ and let $\mathcal{P}$ be a prime ideal included in $\mathcal{J}(\mathcal{U},S)$. Let $L \in \mathcal{U}$ be uniformly open and let $H = \mathbb{E} \setminus L$. Then the characteristic function u of H belongs to $\mathcal{P}$.*

Theorem 3.4. *Let $\mathcal{M}$ be a maximal ideal of S and let $\mathcal{P}$ be a prime ideal contained in $\mathcal{M}$. Then $\mathcal{M}$ is the closure of $\mathcal{P}$ with respect to the norm $\| \cdot \|_0$.*

Corollary 3.4.a: *A prime ideal of S is a maximal ideal if and only if it is closed with respect to the norm $\| \cdot \|_0$.*

Corollary 3.4.b: *Let S be C-compatible. A prime ideal of S is a maximal ideal if and only if it is spectrally closed.*

Corollary 3.4.c: *A prime ideal of S is included in a unique maximal ideal of S.*

4. MULTIPLICATIVE SPECTRUM

The multiplicative spectrum of a Banach $\mathbb{K}$-algebra was first introduced by B. Guennebaud and was at the basis of Berkovich's theory [1], [11].

Notations and definitions: *Let G be a normed $\mathbb{K}$-algebra. We denote by $Mult(G, \| \cdot \|)$ the set of continuous multiplicative algebra semi-norms of G provided with the topology of pointwise convergence, which means that a basic neighborhood of some $\psi \in Mult(G, \| \cdot \|)$ is a set of the form $W(\psi, f_1, ..., f_q, \varepsilon)$, with $f_j \in G$ and $\varepsilon > 0$, which is the set of $\phi \in Mult(G, \| \cdot \|)$ such that $|\psi(f_j) - \phi(f_j)|_\infty \leq \varepsilon \ \forall j = 1, ..., q$. The topological space $Mult(G, \| \cdot \|)$ is then compact [11].*

Given $\phi \in Mult(G, \| \cdot \|)$, we call kernel of ϕ the set of the $x \in S$ such that $\phi(x) = 0$ and we denote it by $Ker(\phi)$. It is a prime closed ideal of G with respect to the norm $\| \cdot \|$ [4], [11].

We denote by $Mult_m(G, \| \cdot \|)$ the set of continuous multiplicative semi-norms of G whose kernel is a maximal ideal and by $Mult_1(G, \| \cdot \|)$ the set of continuous multiplicative semi-norms of G whose kernel is a maximal ideal of codimension 1.

Theorem 4.1 is classical:

Theorem 4.1. *Let A be a unital commutative ultrametric Banach $\mathbb{K}$-algebra. For each $f \in A$, $\|f\|_{sp} = \sup\{\phi(f) \mid \phi \in Mult(A, \|\ .\ \|)\}$.*

More notations: For any ultrafilter $\mathcal{U} \in U(\mathbb{E})$ and any $f \in S$, $|f(x)|$ has a limit along $\mathcal{U}$ since f is bounded. Given $a \in \mathbb{E}$ we denote by φ_a the mapping from S to $\mathbb{R}$ defined by $\varphi_a(f) = |f(a)|$ and for any ultrafilter $\mathcal{U} \in U(\mathbb{E})$, we denote by $\varphi_{\mathcal{U}}$ the mapping from S to $\mathbb{R}$ defined by $\varphi_{\mathcal{U}}(f) = \lim_{\mathcal{U}} |f(x)|$. These maps belong to $Mult(S, \|\ .\ \|)$ since $\|\ .\ \|_0 \le \|\ .\ \|_{sp} \le \|\ .\ \|$.

We denote by $Mult_{\mathbb{E}}(S, \|\ .\ \|)$ the set of multiplicative semi-norms of S of the form φ_a, $a \in \mathbb{E}$.

Proposition 4.2. *Let $a \in \mathbb{E}$. Then $\mathcal{J}(a, S)$ is a maximal ideal of S of codimension 1 and φ_a belongs to $Mult_1(S, \|\ .\ \|)$.*

Theorem 4.3. *Let $\mathcal{U}$ be an ultrafilter on $\mathbb{E}$. Then $\varphi_{\mathcal{U}}$ belongs to the closure of $Mult_{\mathbb{E}}(S, \|\ .\ \|)$.*

Theorem 4.4. *For each $\phi \in Mult(S, \|.\ \|)$, $Ker(\phi)$ is a prime spectrally closed ideal.*

Corollary 4.4.a: *If S is C-compatible, then $Mult(S, \|\ .\ \|) = Mult_m(S, \|\ .\ \|)$.*

Theorem 4.5 is classical [11].

Theorem 4.5. *Let G be a commutative unital ultrametric Banach $\mathbb{K}$-algebra. For every maximal ideal $\mathcal{M}$ of G, there exists $\phi \in Mult_m(S, \|\ .\ \|)$ such that $\mathcal{M} = Ker(\phi)$.*

Definition. Recall that a $\mathbb{K}$-Banach algebra is said to be *multbijective* if every maximal ideal is the kernel of only one continuous multiplicative semi-norm.

Remark 3: T here exist some rare cases of ultrametric Banach algebras that are not multbijective [4].

Theorem 4.6. *Suppose S is C-compatible. Then S is multbijective. Precisely if $\psi \in Mult(S, \|\ .\ \|)$ and $Ker(\psi) = \mathcal{M}$ then $\psi = \varphi_{\mathcal{U}}$ for every ultrafilter $\mathcal{U}$ such that $\mathcal{M} = \mathcal{J}(\mathcal{U}, S)$.*

Corollary 4.6.a: *Suppose S is C-compatible. For every $\phi \in Mult(S, \|\ .\ \|)$ there exists a unique $\mathcal{H} \in Y_{(\mathcal{R})}(\mathbb{E})$ such that $\phi(f) = \lim_{\mathcal{U}} |f(x)|\ \forall f \in S,\ \forall \mathcal{U} \in \mathcal{H}$.*

Moreover, the mapping Ψ that associates to each $\phi \in Mult(S, \|\ .\ \|)$ the unique $\mathcal{H} \in Y_{\mathcal{R}}(\mathbb{E})$ such that $\phi(f) = \lim_{\mathcal{U}} |f(x)|\ \forall f \in S,\ \forall \mathcal{U} \in \mathcal{H}$, is a bijection from $Mult(S, \|\ .\ \|)$ onto $Y_{(\mathcal{R})}(\mathbb{E})$.

Corollary 4.6.b: *Suppose S is C-compatible. $Mult_{\mathbb{E}}(S, \|\ .\ \|)$ is dense in $Mult(S, \|\ .\ \|)$.*

Remark 4: I t is not clear whether a semi-compatible $\mathbb{K}$-algebra is multbijective, in general.

Theorem 4.7. *The topological space $\mathbb{E}$, provided with its distance δ, is homeomorphic to $Mult_{\mathbb{E}}(S, \| \,.\, \|)$ provided with the restricted topology from that of $Mult(S, \| \,.\, \|)$.*

Corollary 4.7.a: *$Mult(S, \| \,.\, \|)$ is a compactification of the topological space $\mathbb{E}$.*

Theorem 4.8. *Let $\varphi_{\mathcal{U}} \in Mult_m(S, \| \,.\, \|)$, with $\mathcal{U}$ an ultrafilter on $\mathbb{E}$, let Γ be the field $\dfrac{S}{Ker(\phi)}$ and let θ be the canonical surjection from S onto Γ. Then, the mapping defined on Γ by $|\theta(f)| = \phi(f), \forall f \in S$ is the quotient norm $\| \,.\, \|'$ of $\| \,.\, \|_0$ defined on Γ and is an absolute value on Γ. Moreover, if $Ker(\varphi_{\mathcal{U}})$ is of codimension 1, then this absolute value is the one defined on $\mathbb{K}$ and coincides with the quotient norm of the norm $\| \,.\, \|$ of S.*

Corollary 4.8.a: *Suppose that S is C-compatible. Let $\phi \in Mult(S, \| \,.\, \|)$, let Γ be the field $\dfrac{S}{Ker(\phi)}$ and let θ be the canonical surjection from S onto Γ. Then, the mapping defined on Γ by $|\theta(f)| = \phi(f), \forall f \in S$ is the quotient norm $\| \,.\, \|'$ of $\| \,.\, \|_0$ on Γ and is an absolute value on Γ. Moreover, if $Ker(\phi)$ is of codimension 1, then this absolute value is the one defined on $\mathbb{K}$ and coincides with the quotient norm of the norm $\| \,.\, \|$ of S.*

Definition and notation: *Given a $\mathbb{K}$-normed algebra G, we call Shilov boundary of G a closed subset F of $Mult(G, \| \,.\, \|)$ that is minimum with respect to inclusion, such that, for every $x \in G$, there exists $\phi \in F$ such that $\phi(x) = \|x\|_{sp}$ [6], [7].*

Let us recall the following Theorem [6], [4]:

Theorem 4.9. *Every normed $\mathbb{K}$-algebra admits a Shilov boundary.*

Notation: Given a $\mathbb{K}$-normed algebra G, we denote by $Shil(G)$ the Shilov boundary of G.

Theorem 4.10. *Suppose S is C-compatible. The Shilov boundary of S is equal to $Mult(S, \| \,.\, \|)$.*

5. THE STONE SPACE OF $\mathbb{G}(\mathbb{E})$

It was proved in [8] that for the algebra $\mathcal{A}$ of continuous bounded functions from $\mathbb{E}$ to $\mathbb{K}$, the Banaschewski compactification of $\mathbb{E}$ is homeomorphic to $Mult(\mathcal{A}, \| \,.\, \|_0)$. Here we get some similar version for C-compatible algebras.

We have defined the Boolean ring $\mathbb{G}(\mathbb{E})$ of uniformly open subsets of $\mathbb{E}$ provided with the laws Δ for the addition and $\cap$ for the multiplication. Let $\Sigma(\mathbb{E})$ be the set of non-zero ring homomorphisms from $\mathbb{G}(\mathbb{E})$ onto $\mathbb{F}_2$ provided with the topology of pointwise convergence. This is the Stone space of the Boolean ring $\mathbb{G}(\mathbb{E})$, it is a compact space (see for example [16] for further details).

For every $\mathcal{U} \in U(\mathbb{E})$, we denote by $\zeta_{\mathcal{U}}$ the ring homomorphism from $\mathbb{G}(\mathbb{E})$ onto $\mathbb{F}_2$ defined by $\zeta_{\mathcal{U}}(O) = 1$ for every $O \in \mathbb{G}(\mathbb{E})$ that belongs to $\mathcal{U}$ and $\zeta_{\mathcal{U}}(O) = 0$ for every $O \in \mathbb{G}(\mathbb{E})$ that does not belong to $\mathcal{U}$.

Particularly, given $a \in \mathbb{E}$, we denote by ζ_a the ring homomorphism from $\mathbb{G}(\mathbb{E})$ onto $\mathbb{F}_2$ defined by $\zeta_a(O) = 1$ for every $O \in \mathbb{G}(\mathbb{E})$ that contains a and $\zeta_a(O) = 0$ for every $O \in \mathbb{G}(\mathbb{E})$ that does not contain a.

Throughout this section we suppose that S **is a C-compatible algebra**.

Remark 5: L et $\Sigma'(\mathbb{E})$ be the set of $\zeta_a,\ a \in \mathbb{E}$. The mapping that associates ζ_a to $a \in \mathbb{E}$ defines a surjective mapping from $\mathbb{E}$ onto $\Sigma'(\mathbb{E})$. That mapping is also injective because given $a,\ b \in \mathbb{E}$, there exists a uniformly open subset F such that $a \in F$ and $b \notin F$.

We have a bijection Ψ from $Mult(S, \|\ .\ \|)$ onto $Y_{(\mathcal{R})}(\mathbb{E})$ associating to each $\phi \in Mult(S, \|\ .\ \|)$ the unique $\mathcal{H} \in Y_{(\mathcal{R})}(\mathbb{E})$ such that $\phi(f) = \lim_{\mathcal{U}} |f(x)|$, $\mathcal{U} \in \mathcal{H},\ f \in S$, i.e. $\phi = \varphi_{\mathcal{U}}$ for every $\mathcal{U} \in \mathcal{H}$.

On the other hand, let us take some $\mathcal{H} \in Y_{(\mathcal{R})}(\mathbb{E})$ and ultrafilters $\mathcal{U},\ V$ in $\mathcal{H}$. Since $\mathcal{U}, \mathcal{V}$ own the same uniformly open subsets of $\mathbb{E}$, we have $\zeta_{\mathcal{U}} = \zeta_{\mathcal{V}}$ and hence we can define a mapping Ξ from $Y_{\mathcal{R}}(\mathbb{E})$ into $\Sigma(E)$ which associates to each $\mathcal{H} \in Y_{\mathcal{R}}(\mathbb{E})$ the $\zeta_{\mathcal{U}}$ such that $\mathcal{U} \in \mathcal{H}$.

Lemma 5.1. Ξ *is a bijection from* $Y_{\mathcal{R}}(\mathbb{E})$ *onto* $\Sigma(\mathbb{E})$.

We put $\Phi = \Xi \circ \Psi$ and hence Φ is a bijection from $Mult(S, \|\ .\ \|)$ onto $\Sigma(\mathbb{E})$.

Notice that for every ultrafilter $\mathcal{U}$, $\Psi(\varphi_{\mathcal{U}})$ is the class $\mathcal{H}$ of $\mathcal{U}$ with respect to $(\mathcal{R})$ and $\Xi(\mathcal{H}) = \zeta_{\mathcal{U}}$ so $\Phi(\varphi_{\mathcal{U}}) = \zeta_{\mathcal{U}}$.

Theorem 5.2. Φ *is an homeomorphism once* $\Sigma(\mathbb{E})$ *and* $Mult(S, \|\ .\ \|)$ *are provided with topologies of pointwise convergence.*

Corollary 5.2.a: *The space* $\Sigma(\mathbb{E})$ *is a compactification of* $\mathbb{E}$ *which is equivalent to the compactification* $Mult(S, \|\ .\ \|)$.

Remark 6: F or a C-compatible algebra S, the compactification $\Sigma(\mathbb{E})$ coincides with the Guennebaud-Berkovich multiplicative spectrum.

Generally, it can be proved that this compactification is not equivalent to the usual Banaschewski compactification. This last one is the Stone space associated to the Boolean ring of clopen sets of $\mathbb{E}$.

6. ALGEBRAS $\mathcal{B}$, L, D, E

We denote by $\mathcal{L}$ the set of bounded Lipschitzian functions from $\mathbb{E}$ to $\mathbb{K}$. Whenever $\mathbb{E}$ is a subset of $\mathbb{K}$, we denote by $\mathcal{D}$ the subset of $\mathcal{L}$ of derivable functions in $\mathbb{E}$ and by $\mathcal{E}$ the subset of $\mathcal{L}$ of functions such that for every $a \in \mathbb{E}$, $\dfrac{f(x)-f(y)}{x-y}$ has limit when x and y tend to a separately. Following [10] the functions of $\mathcal{E}$ are called *strictly differentiable*.

Given $f \in \mathcal{L}$, we put $\|f\|_1 = \sup\limits_{\substack{x,y\in\mathbb{E}\\ x\neq y}} \dfrac{|f(x)-f(y)|}{\delta(x,y)}$ and $\|f\| = \max(\|f\|_0, \|f\|_1)$. In particular, if $f \in \mathcal{D}$, then $\|f\|_1 = \sup\limits_{\substack{x,y\in\mathbb{E}\\ x\neq y}} \dfrac{|f(x)-f(y)|}{|x-y|}$.

Remark 7: I f $\mathbb{E} \subset \mathbb{K}$, then $\mathcal{E} \subset \mathcal{D} \subset \mathcal{L}$

As noticed in section 2, $(\mathcal{B}, \|\ .\ \|_0)$ is a semi-compatible algebra. In [10], it was proved that the algebra here denoted by $\mathcal{E}$ is a Banach $\mathbb{K}$-algebra with respect to the norm $\|\ .\ \|$.

Theorem 6.1. *$\mathcal{L}$, $\mathcal{D}$ and $\mathcal{E}$ are normed $\mathbb{K}$-algebras.*

Theorem 6.2. *$\mathcal{L}$ is a $\mathbb{K}$-Banach algebra with respect to the norm $\|\ .\ \|$.*

Theorem 6.3. *For every $g \in \mathcal{D}$, we have $\|g'\|_0 \leq \|g\|_1$. And if $(f_n)_{n\in\mathbb{N}}$ is a sequence of $\mathcal{D}$ converging to a limit f with respect to the norm $\|\ .\ \|$, then f belongs to $\mathcal{D}$ and the sequence $(f'_n)_{n\in\mathbb{N}}$ converges to f' with respect to the norm $\|\ .\ \|_0$.*

Corollary 6.3.a: *$\mathcal{D}$ is a Banach $\mathbb{K}$-algebra with respect to the norm $\|\ .\ \|$. Moreover, the functions in $\mathcal{D}$ have derivatives which are bounded.*

Theorem 6.4. *$\mathcal{E}$ is closed in $\mathcal{D}$ and hence is a Banach $\mathbb{K}$-algebra with respect to the norm $\|\ .\ \|$. Moreover, each function in $\mathcal{E}$ has a derivative which is bounded and continuous.*

Theorem 6.5. *In each algebra $\mathcal{L}$, $\mathcal{D}$, $\mathcal{E}$, the spectral norm $\|\ .\ \|_{sp}$ is $\|\ .\ \|_0$.*

Theorem 6.6. *Let T be one of the algebras $\mathcal{B}$, L, D, E. An element of T is invertible in T if and only if* $\inf\{|f(x)| \mid x \in \mathbb{E}\} > 0$.

We can now conclude with algebras $\mathcal{B}$, L, D, E:

Theorem 6.7. *The $\mathbb{K}$-algebras $\mathcal{B}$, L, D, E are C-compatible algebras.*

7. PARTICULAR PROPERTIES OF ALGEBRAS $\mathcal{B}$, L, D, E

Notation: We denote by Ł a finite extension of $\mathbb{K}$ provided with the absolute value which extends that of $\mathbb{K}$. Let $T = \mathcal{B}$ (resp. $T = \mathcal{L}$, resp. $T = \mathcal{D}$, resp. $T = \mathcal{E}$). We will denote here by T^* the Ł-algebra of uniformly continuous functions from

$\mathbb{E}$ to Ł (resp. the Ł-algebra of bounded Lipschitzian functions from $\mathbb{E}$ to Ł, resp. the Ł-algebra of bounded derivable functions from $\mathbb{E}$ to Ł, resp. the Ł-algebra of bounded strictly differentiable functions from $\mathbb{E}$ to Ł).

A first specific property of algebras $\mathcal{B}$, L, D, E concerns maximal ideals of finite codimension.

The following Theorem 7.1 is similar to Theorem 25 in [10] and Theorem 4. 5 in [9] but the proof requires some corrections that will be in a further paper.

Theorem 7.1. *Suppose there exists a morphism of $\mathbb{K}$-algebra, χ, from T onto Ł. Then χ has continuation to a surjective morphism of Ł-algebra χ^* from T^* onto Ł.*

We can now state the following Theorem 7.2 whose proof is similar to that of Theorem 26 in [10] but here concerns all algebras $\mathcal{B}, L, D, E$.

Theorem 7.2. *Every maximal ideal of finite codimension of $\mathcal{B}, L, D, E$ is of codimension* 1.

Theorem 7.3. *Suppose $\mathbb{E} \subset \mathbb{K}$, let $S = \mathcal{B}$, L, D, E and let $\mathcal{M} = \mathcal{J}(\mathcal{U}, S)$ be a maximal ideal of S where $\mathcal{U}$ is an ultrafilter on $\mathbb{E}$. If $\mathcal{U}$ is a Cauchy filter, then $\mathcal{M}$ is of codimension* 1. *Else, $\mathcal{M}$ is of infinite codimension.*

Notation: We denote by $Max_1(S)$ the set of maximal ideals of S of codimension 1 and by $Max_{\mathbb{E}}(S)$ the set of maximal ideal of S of the form $\mathcal{J}(a, S)$, $a \in \mathbb{E}$.

Corollary 7.3.a: Suppose $\mathbb{E}$ is a closed subset of $\mathbb{K}$ and let $S = \mathcal{B}$, L, D, E. Then $Max_1(S) = Max_{\mathbb{E}}(S)$.

Definition. *Given a $\mathbb{K}$-normed algebra A whose norm is $\| \,.\, \|$, we call topological divisor of zero an element $f \in A$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of elements of A such that $\inf\limits_{n\in\mathbb{N}} \|u_n\| > 0$ and $\lim\limits_{n\to+\infty} f u_n = 0$.*

Theorem 7.4. *Suppose that $\mathbb{E}$ has no isolated points. Then an element of an algebra $T = \mathcal{B}$, L, D, E is a topological divisor of zero if and only if it is not invertible.*

8. A kind of Gelfand transform

A Gelfand transform is not easy on ultrametric Banach algebras, due to maximal ideals of infinite codimension. However, here we can obtain a kind of Gelfand transform under certain hypotheses on the multiplicative spectrum in order to find again an algebra of bounded Lipschitzian functions on some ultrametric space.

Notations: Let $(A, \| \,.\, \|)$ be a commutative unital Banach $\mathbb{K}$-algebra which is not a field. Let $\Upsilon(A)$ be the set of algebra homomorphisms from A onto $\mathbb{K}$ and let λ_A be the mapping from $\Upsilon(A) \times \Upsilon(A)$ to $\mathbb{R}_+$ defined by $\lambda_A(\chi, \zeta) = \sup\{|\chi(f) - \zeta(f)| \mid \|f\| \le 1\}$.

Given $\chi \in \Upsilon(A)$, we denote by $|\chi|$ the element of $Mult(A, \| \,.\, \|)$ defined as $|\chi|(f) = |\chi(f)|$, $f \in A$. Given $D \subset \Upsilon(A)$, we put $|D| = \{|\chi|, \chi \in D\}$.

Lemma 8.1. *λ_A is an ultrametric distance on $\Upsilon(A)$.*

Definition and notation: *Let A be a unital commutative ultrametric Banach $\mathbb{K}$-algebra. We will denote by $Max_1(A)$ the set of maximal ideals of codimension 1 of A. The algebra $(A, \|\ .\ \|)$ will be said to be $\mathcal{L}$-based if it satisfies the following:*

a) *$Mult_1(A, \|\ .\ \|)$ is dense in $Mult(A, \|\ .\ \|)$,*
b) *the spectral semi-norm $\|\ .\ \|_{sp}$ is a norm,*
c) *For every uniformly open subset D of $\Upsilon(A)$ with respect to λ_A, the closures of $|D|$ and $|\Upsilon(A) \setminus D|$ are disjoint open subsets of $Mult(A, \|\ .\ \|)$.*

Theorem 8.2. *Let $(A, \|\ .\ \|)$ be a $\mathcal{L}$-based algebra. Then the algebra A is isomorphic to an algebra $\mathcal{A}$ of bounded Lipschitzian functions from the ultrametric space $\mathbb{E} = (\Upsilon(A), \lambda_A)$ to $\mathbb{K}$. Identifying A with $\mathcal{A}$, the Banach $\mathbb{K}$-algebra $(A, \|\ .\ \|)$ is C-compatible. Moreover there exists a constant $c \geq 1$ such that the Lipschitzian semi-norm defined as $\|f\|_1 = \sup\left\{\dfrac{|f(x) - f(y)|}{\lambda_A(x,y)} \mid x, y \in \mathbb{E},\ x \neq y\right\}$ satisfies $\|f\|_1 \leq c\|f\|$ for all $f \in A$.*

By Theorems 8.2, 3. 1, 4.3 and Corollary 4.4.a we can derive the following corollary:

Corollary 8.2.a: *Let A be a $\mathcal{L}$-based algebra. Then*

$$Mult(A, \|\ .\ \|) = Mult_m(A, \|\ .\ \|).$$

Moreover, A is multbijective. Further, $Shil(A) = Mult(A, \|\ .\ \|)$.

Theorem 8.3. *Let A be $\mathcal{L}$-based algebra. Then $\Upsilon(A)$ is complete with respect to the distance λ_A.*

Lemma 8.4. *Suppose $\mathbb{E}$ is a closed subset of $\mathbb{K}$ and let S be a Banach $\mathbb{K}$- algebra of uniformly continuous functions from $\mathbb{E}$ to $\mathbb{K}$ equal to one of the algebras $\mathcal{L}$, $\mathcal{D}$, $\mathcal{E}$. Let T be the mapping from $\mathbb{E}$ into $\Upsilon(S)$ that associates to each point $a \in \mathbb{E}$ the element of $\Upsilon(S)$ whose kernel is $\mathcal{J}(a, S)$. Then T is a bijection from $\mathbb{E}$ onto $\Upsilon(S)$. Moreover, we have $|b - a| \geq \lambda_S(a, b)\ \forall a, b \in \mathbb{E}$.*

Corollary 8.4.a: *Suppose $\mathbb{E}$ is a closed subset of $\mathbb{K}$ and let S be a Banach $\mathbb{K}$-algebra of uniformly continuous functions from $\mathbb{E}$ to $\mathbb{K}$ equal to one of the algebras $\mathcal{L}$, $\mathcal{D}, \mathcal{E}$. Then every uniformly open subset of $\Upsilon(S)$ with respect to λ_S is a uniformly open subset of $\mathbb{E}$ with respect to the absolute value of $\mathbb{K}$.*

Theorem 8.5. *Suppose $\mathbb{E}$ is a closed subset of $\mathbb{K}$ and let S be a Banach $\mathbb{K}$- algebra of uniformly continuous functions from $\mathbb{E}$ to $\mathbb{K}$ equal to one of the algebras $\mathcal{L}, \mathcal{D}, \mathcal{E}$. Then S is a $\mathcal{L}$-based algebra.*

Notation: Let $(A, \|\ .\ \|)$ be a $\mathcal{L}$-based algebra. We will denote by $A^{\sim}$the algebra of all bounded Lipschitzian functions from the space $\mathbb{E} = (\Upsilon(A), \lambda_A)$ to $\mathbb{K}$.

Theorem 8.6. *Let $\mathbb{E}$ be a closed subset of $\mathbb{K}$. Let $(A, \| \, . \, \|)$ be the algebra of all bounded Lipschitzian functions from $\mathbb{E}$ to $\mathbb{K}$. Then A is a $\mathcal{L}$-based algebra such that $A^{\sim} = A$. Moreover, if $\mathbb{E}$ is bounded, there exists a constant $h \geq 1$ such that $\lambda_A(x,y) \leq \delta(x,y) \leq h\lambda_A(x,y)$ $\forall x,y \in \mathbb{E}$.*

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(Received: October 9, 2018)
(Revised: October 24, 2018)

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