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FROM KRASNER'S CORPOID AND BOURBAKI'S GRADUATIONS TO KRASNER'S GRADUATIONS AND KRASNER-VUKOVIĆ'S PARAGRADUATIONS

MIRJANA VUKOVIĆ

ABSTRACT. The paper, supplemented with a short historical development of graduation which begins with Krasner's famous notion of a corpoid, introduced in 1940s and general graded groups in Krasner's sense, more general than Bourbaki's, will present some results in the theory of Krasner-Vuković's para- and extra-graded groups including examples of paragraduations which are and which are not graduations, and some proofs of statements that were not given earlier.

1. INTRODUCTION

The paper contains an Introduction with a short history of the notion of homogeneity (Section 1), a review of the Krasner's fundamental notions on a corpoid which originates from [7] (see also [8 – 12]) (Section 2), and general graded groups [3, 5, 13] (see also [15]) (Section 3), as well as a finish with para- and extra- graded groups, which first appear in [15 – 18] (see also [21 – 24]) as a solution of an important problem in the category of general graded structures, known as the problem of closure of graded structures (groups, rings, modules) with respect to the direct sum and the direct product of homogeneous parts of factors (Section 4).

The notion of graduation is, at least in the light of certain examples, quite an old one. It was introduced explicitly into mathematics during XVIIIth century when *Euler* defined the notion of homogeneous polynomials and real functions [4] (see also [14]). However, when applied to the case of single variable, it was already familiar to mathematicians well before Euler (*Diophant*, some Arab mathematicians, *Viète*, *Descartes*). The germ of the notion of homogeneity can be seen in early Greek mathematics ("multiplication of segments" which preceded the *Grassmann's* Aussdehnungslehres theory) and in some documents of Babylonian times [14].

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Other notions of homogeneity and its corresponding grades were introduced afterwards in various algebraic structures: weights of polynomials and functions, dimension of geometrical and topological objects, order of differential operators, and later, the more general notion of $\mathbb{Z}$-graded ring mainly in algebraic geometry and topology (see for example *Samuel* and *Zarisky* [20]).

Graded rings, the grades of which are embedded in an abelian group, were studied by *C. Chevalley* [2]. The first relatively general definition of graded groups and rings was given by *Bourbaki* [1]. But this definition was unnecessary based on the notion of the abelian graded group. It was a very good definition, although there is no need to limit it to the abelian case.

However, a little is known that all started earlier – before Bourbaki, with the abstract notion of a corpoid, introduced during 1940s by Marc Krasner while he investigated valued fields and observed a connection to their valuation rings through the equivalence of valuations. More precisely, a corpoid first appeared in a series of M. Krasner's *Comptes Rendus* notes in 1944 and 1945 ([7-9]).

2. Krasner's corpoid

Let's say now something about a corpoid and Krasner's motivation for studying the homogeneous parts only, which are of special interest to those who are dealing with homogeneous elements and substructures only. The origin of M. Krasner's studying of homogeneous parts only goes back to [7], continued through studying of partial structures obtained by inducing operations of a graded structure to its set of homogeneous elements (named the homogeneous part), i.e. homogroupoids, anneids, and moduloids in the case of: graded groups, graded rings, and graded modules respectively (see [8, 9], [3], [5], [13, 15, 21]), and was motivated by the following observation:

Let K be a field, $|\cdot|$ its valuation onto a totally ordered group T, and observe the family $\{A_\gamma\}_{\gamma\in T}$, where $A_\gamma = \{x \in K \mid |x| \le \gamma\}$. Then $\{A_\gamma\}_{\gamma\in T}$ is total filtration of $(K,+)$. If

$$\overline{A}_\gamma = \cup_{\gamma'<\gamma} A_{\gamma'},\ K_\gamma = A_\gamma/\overline{A}_\gamma \ \textit{ and } \ gr(K) = \bigoplus\nolimits_{\gamma\in T} K_\gamma$$

then, as it is known, $gr(K)$ can be made into a graded ring.

Let $s(K) = \cup_{\gamma\in T} K_\gamma$ be its homogeneous part. Notice that $(s(K),+)$ is a partial structure, since the sum of two homogeneous elements does not have to be homogeneous, however, when it is, we say that those elements are *addible* and we denote that relation by #. $(s(K),+)$ has the following properties:

i) $(s(K),\cdot)$ *is a group with the bi-absorbing element* 0, *i.e.* $\overline{x}0 = 0\overline{x} = 0,\ \overline{x} \in s(K)$.
ii) $(\forall \overline{x},\overline{y},\overline{z} \in s(K))\ \overline{x}\#0;\ \overline{x}\#\overline{x};\ \overline{x}\#\overline{y} \wedge \overline{y}\#\overline{z} \wedge \overline{y} \ne 0 = \overline{x}\#\overline{z}$;
iii) *for all* $0 \ne \overline{a} \in s(K)\ \{\overline{x} \in s(K) \mid \overline{a}\#\overline{x}\}$ *is an additive abelian group*;
iv) $(\forall \overline{x},\overline{y},\overline{z} \in s(K)),\ \overline{x}\#\overline{y} \Rightarrow \overline{z}\,\overline{x}\#\overline{z}\,\overline{y} \wedge \overline{x}\,\overline{z}\#\overline{y}\,\overline{z} \wedge \overline{z}(\overline{x}+\overline{y}) = \overline{z}\,\overline{x}+\overline{z}\,\overline{y} \wedge (\overline{x}+\overline{y})\overline{z} = \overline{x}\,\overline{z}+\overline{y}\,\overline{z}$.

Marc Krasner named such structure a corpoid [7] (see also notes [8,9]).

Thus, a corpoid, in accordance with the previous represents the homogeneous part of a graded field, graded by an arbitrary nonempty set, with induced operations among which the induced addition is, naturally, a partial operation, since the sum of two homogeneous elements does not have to be homogeneous.

The notion of a corpoid results from the notion of a graded field of which the homogeneous subsets are logically graded through the filtration of their valuation (which can be *Krull's* valuation). The "corpoids", called skeletons (of the corresponding graded fields) have the property that the group of their grades is without torsion.

Furthermore, the commutativity of the valued field implies the commutativity of its skeleton. An overview of corpoids, skeletons and their extensions as well as the theory of vector spaces on these structures, as published in [14], Krasner summarized in 1954 (see [11] and [12]).

Commutative corpoids without torsion have a theory of extensions based on the tailored construction of polynomials on these corpoids similar to (but richer than) *Steinitz's* theory of field – extension, and a good *Galois's* theory.

It is interesting that in a commutative, even in a noncommutative case (when graded group is commutative), Krasner's graded fields (mostly corpoids) are included (as a special case), in the theory of graded rings in the Bourbaki sense. However, they neither pointed that out, nor cited his papers, although, Krasner introduced the notion of a corpoid and announced the greater part of the theory related to them in 1944 ([7] and [8]), and together with proofs published in papers [10] and [11] in 1949, i. e. 1953-54, therefore, much earlier than Bourbaki.

3. Graded groups in Krasner's sense

The abstract notion of a corpoid led Krasner to the development of a general graded theory.

Krasner, starting from his corpoid and Bourbaki's definition of graduation, giving up his hypothesis of commutativity – original unnecessary condition, discovered general theory of graded structures (*groups, rings, modules*) and together with them the theory of *homogroupoids – homogeneous parts of groups, anneids – homogeneous parts of rings, and moduloids – homogeneous parts of modules*, where a corpoid, as a special case of an anneid, is viewed as a homogeneous part of a graded field [13].

However, graded fields are not to be understood as fields with graduation, but fields are to be understood as graded fields with trivial graduation.

Note that Krasner's approach to graduations differs from the Bourbaki approach: Krasner did not restricts himself to the case when the set of grades Δ is a group, but he assumed it to be only a nonempty set in which the associativity, the commutativity and the existence of the neutral element is also not assumed.

Now we will give definitions of a graded group by Krasner called *non-homogeneous aspect* and of a homogroupoid which are essential notions in this part of the paper.

Definition 3.1. *Let G be a multiplicative group with the neutral element e and let Δ be a nonempty set. A mapping* $\gamma : \Delta \to Sg(G)$, $\gamma(\delta) = G_\delta$ $(\delta \in \Delta)$, *such that*

$$G = \bigoplus_{\delta\in\Delta} G_\delta, \qquad (*)$$

where $Sg(G)$ is the set of all subgroups of G, is called a graduation by Krasner, or grading of G[1].

A group G, with graduation γ is called a graded group.

A graded group is called abelian if it is abelian as an abstract group.

The essential features of the graded groups from which some will be expressed through the following Remarks 3.1.

1. A graduation is called strict if $G_\delta \neq \{e\}$, for all $\delta \in \Delta$ and empty if $G_\delta = \{e\}$.
2. If $\Delta^* = \{\delta \in \Delta \mid G_\delta \neq \{e\}\}$, then $\gamma^* = \gamma|\Delta^*$ is a strict graduation of G and is called the strict kernel of γ.
3. The set $H = \cup_{\delta\in\Delta} G_\delta$ $(= \cup_{\delta\in\Delta^*} G_\delta$ if $G \neq \{e\})$ is called a homogeneous part of the graded group G, and elements $x \in H$ are called homogeneous elements of G. Elements $\delta \in \Delta$ are called grades or degrees and the corresponding G_δ are called homogeneus components.
4. For a homogeneous elements $x \neq e$, there is the unique $\alpha \in \Delta^*$, for which $x \in G_\alpha$, is called the grade of x and denoted by $\delta(x)$.
5. Element e in general does not have a degree, but if the empty grades exist, i.e. $\Delta \neq \Delta^*$ it is useful to associate one empty grade to e, which we denote by 0, and call it a zero grade [13]. Sometimes, when there are no empty grades, it is useful to adjoin to Δ a special ”ideal” grade denoted 0, and to put $G_0 = \{e\}$.
6. If $\Delta = \Delta^* \cup \{0\}$, and if we put $\delta(e) = 0$, the graduation is called proper [13].
7. When two subgroups G_δ and $G_{\delta'}$ are different they have only e as common element, i.e. $\delta \neq \delta' \Rightarrow G_\delta \cap G_{\delta'} = \{e\}$. So, every $x \in H$, $x \neq e$, belongs to one and only one G_δ, $\delta \in \Delta$.

Note that from $(*)$ follows that subgroups $G_\delta, \delta \in \Delta$ are normal subgroups of the group G, and so elements $x \in G_\delta$ and $x' \in G_{\delta'}$ commute for $\delta \neq \delta'$.

Another important property of Krasner's graded structures is that these structures are characterized by both the underlying abstract group and the homogeneous part (or even only by the homogeneous part) equipped with the operation(s) induced by operation(s) of the structure as well as that characterization axioms give a way to three study methods of graded structures in principle equivalent: *i*) *the non-homogeneous*; *ii*) *the semi-homogeneous*; and *iii*) *the homogeneous method*, which is not defined for Bourbaki's graded structures.

[1] Although with multiplicative operation, the direct product (restricted) will be denoted by $\bigoplus$ (Kurosh, [19]).

Definition 3.2. [13] *Let G be a graded group with two graduations:*

$$\gamma : \Delta \to Sg(G), \gamma(\delta) = G_\delta \ (\delta \in \Delta) \ \text{ and } \ \gamma' : \Delta' \to Sg(G), \gamma'(\delta') = G_{\delta'}(\delta' \in \Delta').$$

We say that these graduations are:

i) *equivalent if there exists a bijective mapping*

$$\varphi : \Delta^* \to \Delta'^* \ \text{ such that } \ \gamma'(\varphi(\delta)) = \gamma(\delta) \ (\forall \delta \in \Delta)$$

holds, i.e. if its proper parts are equal to the names of grades.

ii) *weakly equivalent if their strict kernels are equivalent.*

Obviously, strict graduations are determined up to an equivalence by the set $\{G_\delta | \delta \in \Delta\}$, while graduations are determined up to the weak equivalence by the set $\{G_\delta | \delta \in \Delta^*\}$.

The homogeneous part H of a group G, together with the group structure of G, determines a corresponding graduation up to a weak equivalence. Indeed, if

$$a \in H^* = H \backslash \{e\}, b \in H, \ \text{ and if } \ a \in G_\delta, \ \text{ then } \ b \in G_\delta \ \text{ iff } \ ab \in H.$$

Hence, if for $a \in H^*$, we define $G(a) = \{x \in H \mid ax \in H\}$, then sets

$$\{G_\delta | \delta \in \Delta^*\} \text{ and } \{G(a) | a \in H^*\} \text{ are coincides.}$$

This observation was for M. Krasner a motivation for defining a graded group from *semi-homogeneous aspect* as an ordered pair (G,H), where $H \subseteq G$ is the homogeneous part with respect to some graduation of G, in the previous sense.

This, however, required the characterisation of the homogeneous part with the following system of six axioms:

Theorem 3.1. [13] *A nonempty subset H of a group G is the homogeneous part of G with respect to some graduation of G if and only if the following conditions are satisfied:*

i) $e \in H$;

ii) $x \in H \Rightarrow x^{-1} \in H$;

iii) $x,y,z,xy,yz \in H$ *and* $y \neq e \Rightarrow xz \in H$;

iv) $x,y \in H$ *and* $xy \notin H \Rightarrow xy = yx$;

v) *H generates G;*

vi) *If $n \geq 2$ and if $x_1, x_2, \ldots, x_n \in H$ are such that for all i, j, $1 \leq i, j \leq n$, $i \neq j$, we have $x_i x_j \notin H$, then $x_1 x_2 \cdots x_n \neq e$.*

So, for a given pair $\{G, H \subseteq G\}$, if H *satisfies the axioms i*) – *vi*), we can reconstruct a corresponding graduation (up to an equivalence). For this reason, every such pair $\{G,H\}$ will be called a graduation (up to an equivalence).

Let G be a graded group with the homogeneous part H. Multiplicative operation on G induces a partial operation on H. Namely, since

$$\text{if } \ x,y \in H, \ \text{ then } \ x \cdot y \ \text{ is defined in } \ H \ \text{ iff } \ x \cdot y \in G \ \text{ is the element from } \ H$$

and in that case the result is the same.

In this case, we say that elements x, y are *composable* (*addible* in the case of an additive operation) and we denote it by $x\#y$ ([13], see also [15]).

It is clear that

$$x\#y \text{ iff } x,y \text{ are from the same subgroup } G(a)=\{x\in H \mid ax\in H\},\ a\in H^*.$$

Such partial structures $(H,\cdot)$ are called *homogrupoids*, and the corresponding graded groups $G\supseteq H$ are called (graded) group closures.

In the case, when H, with the induced operation from G is given, we may reconstruct G up to H-isomorphism. Indeed, if $a\in H^*$, then $G(a)$ may be defined as $G(a)=\{x\in H \mid a\#x\}$.

In this case G is the direct sum of different subgroups $G(a)$ and H is obviously homogeneous part of G. The group G, obtained in this way, is called the *linearization* of $(H,\cdot)$ and we denote it by $\overline{H}=\bigoplus_{a\in H^*} G(a)$ ([13], see also [15]).

It is natural now to define a graded group using the corresponding homogeneous part, at least up to H-isomorphism. In order to do that, we need to characterize the structure $(H,\cdot)$ which is the homogeneous part of some graded group, with the partial operation induced from the operation of that group.

This characterisation, by M. Krasner called *homogeneous aspect*, can be pronounced in the following theorem:

Theorem 3.2. [13] *The structure $(H,\cdot)$ is the homogeneous part of some graded group G with partial operation induced from that group, if and only if the following conditions hold:*

i) $(\exists e\in H)\,(\forall x\in H)\ x\#e$ *and* $x\cdot e=x$ (*axiom about neutral element*)*;*

ii) $(\forall x\in H)\ x\#x$ (*axiom about autocomposability*)*;*

iii) $(\forall x,y,z\in H)\ x\#y,\ y\#z$ *and* $y\neq e\Rightarrow x\#z$ (*axiom of almost transitivity of composability*)*,*

iv) *For all* $a\in H^*$, $H(a)=\{x\in H \mid a\#x\}$ *is a group with respect to operation* ”$\cdot$”*, where* $x\#y$ *means that* $x\cdot y$ *exists.*

Definition 3.3. *The structure $(H,\cdot)$ which satisfies conditions i) – iv) is called a homogroupoid.*

Therefore, the structures occurring are coincident (up to a H-isomorphism) with the *linearization* of the homogroupoids.

We saw that the characterisation axioms in Krasner’s graded structures give way to three study methods of graded groups in principle equivalent:

1. *non-homogeneous*, where G is considered as a group with determined graduation;
2. *semi-homogeneous*, where G is considered as a pair $\{G,H\}$ of a group and its homogeneous part $H\subseteq G$, called *homogroupoid*, where a graduation of the group G can be reconstructed (determined up to the equivalence) using the homogeneous part H;

3. *homogeneous*, where the *homogroupoid* $(H,\cdot)$ is considered and where its linearization $(\overline{H,\cdot})$ can be reconstructed, i.e. the graded group for which $(H,\cdot)$ is the homogeneous part.

Analogous aspects exist for the structures richer than groups (rings and modules) too, as well as for those structures that generalize graded structures: groups, rings, modules, called paragraded: groups, rings, modules.

It is well known that the category of graded groups, rings and modules and their homogeneous parts: homogroupoids, anneids, moduloids, respectively are not closed with respect to the direct products and the direct sums, where the homogeneous part is the direct product of the homogeneous parts of the factors.

4. Krasner-Vuković's paragraded groups

Since the category of graded structures (groups, rings, modules) has no property of closure with respect to the direct sum and the direct product, it was a motivation for M. Krasner and myself to focus on this important problem in the theory of graded structures and we were the first who overcame it. In this way we discovered a theory of paragraded structures [15 - 18] which are at the same time a generalization of the classical graduation as defined by Bourbaki [1] and an extension of the earlier works done by M. Krasner [8 - 14] and his pupils M. Chadeyras [3] and E. Halberstadt [5].

This section highlights a non-homogeneous aspect of the paragraded groups.

Definition 4.1. (Krasner-Vuković, see [15]) *Let G be a group (multiplicative), with e as neutral element. The mapping $\pi : \rightarrow Sg(G)$, $\pi(\delta) = G_\delta$ $(\delta \in \Delta)$, of a partially ordered set $(\Delta, <)$, which is from bellow a complete semi-lattice and from above inductively ordered, to the set $Sg(G)$ of subgroups of the group G is called a paragraduation if it satisfies the following six-axiom system:*

- *i)* $\pi(0) = G_0 = \{e\}$, *where* $0 = \inf\{\Delta\}$; $\delta < \delta' \Rightarrow G_\delta \subseteq G_{\delta'}$;

  Remarks 4.1.

  1. *$H = \cup_{\delta\in\Delta} G_\delta$ is called the homogeneous part of the group G with respect to π, and elements $x \in H$ are called homogeneous elements of G.*
  2. *If $x \in H$ and $\Delta(x) = \{\delta \in \Delta \,|\, x \in G_\delta\}$, we say that $\delta(x) = \inf\{\Delta(x)\}$ is a grade of x. We have $\delta(x) = 0$ iffs $x = e$. Elements $\delta(x), x \in H$, are called the principal grades and they form a set denoted by Δ_p which is called the principal part of Δ. Restriction $\pi|\Delta_p$ is called the proper kernel of π.*
- *ii)* *If* $\theta \subseteq \Delta$, *then* $\cap_{\delta\in\Delta} G_\delta = G_{\inf\theta}$;
- *iii)* *If* $x, y \in H$ *and* $yx = zxy$, *then* $z \in H$ *and* $\delta(z) \leq \inf(\delta(x), \delta(y))$;
- *iv)* *Homogeneous part H is a generating set of G;*
- *v)* *Let $A \subseteq H$ be a subset such that, for all $x, y \in A$ there exists an upper bound for $\delta(x), \delta(y)$. Then there exists an upper bound for all $\delta(x), x \in A$;*
- *vi)* *G is generated by H with the set R of H-inner and left commutation relation:*

– $xy = z$ (H-*inner relations*),
– $yx = z(x,y)xy$ (*left commutation relations*).

The group with paragraduation is called paragraded group.

Definition 4.2. (Krasner-Vuković, see [15]) *The mapping* π *from Definition* 4.1 *which satisfies axioms* $i)$ - $v)$ *and instead* $vi)$

$vi')$ *If* $\delta_1,\ldots,\delta_s \in \Delta_p$ *are pairwise incomparable and if* $x_i, x_i' \in H$, $i = 1,\ldots,s$ *are such that*
$x_1x_2\ldots x_s = x_1'x_2'\ldots x_s'$ *and* $x_i, x_i' \in G_\delta$, $\forall i = 1,\ldots,s$, *then* $\delta(x_i^{-1}x_i') < \delta_i$

is called extragraduation.

The group with extragraduation is called an extragraded group.

Remark 4.2. The axiom v) is equivalent to the axiom:

$v')$ *Let* $A \subseteq H$ *be a subset such that for all* $x, y \in A$, *we have* $xy \in H$. *Then there exists* $\delta \in \Delta$ *such that* $A \in G_\delta$.

Later we will see that every extragraduation is a paragraduation.

Definition 4.3. [15] *Let* $g \subseteq H$ *be a subgroup of* G. *Such a subgroup* g *is called the subgroup of the homogeneous part* H *and the set of all subgroups of the homogeneous part* H *is denoted by* $Sg(H)$.

Definition 4.4. *The subgroup* $g \in Sg(H)$ *is called saturated with respect to* π *if*

$$x \in g \Rightarrow G_{\delta(x)} \subseteq g.$$

Remark 4.3. [15] Every subgroup G_δ, $\delta \in \Delta$, is saturated.

Now we will introduce some notions terminology that we will use in the proof of the next theorem:

Definition 4.5. *Every finite sequence* $x_1,\ldots,x_s$ *of the elements from* H *or elements whose inverses are from* H *is called a word of* H.

Remark 4.4. Let m be the word of H. We can associate to word m the product of its elements (in given order) in G. That product is denoted by (m).

Definition 4.6. *If* $m = x_1\ldots x_s$ *is a word, then the array*

$$d(m) = (\delta(x_1),\ldots,\delta(x_s))$$

is called the array of grades of that word.

Definition 4.7. *The word* $x_1\ldots x_s$ *is called reduced if for every pair* x_i, x_j *there exists no common upper bound of elemnts* $\delta(x_i), \delta(x_j)$, $i \neq j$ *(in other words, if for all* $i, j = 1,\ldots,s$, $i \neq j$, *we have* $x_ix_j \notin H$*).*

The following theorem is very important.

Theorem 4.1. [15] *An extragraded group is a paragraded group, i.e. an extragraded group* G *is generated by* H *with set relations* R.

Proof. Let P be the set of elements of the form $x_1 \dots x_s$, $x_i \in H$, observed as words of H. We will assume that two words from P are equal if they are equal as elements of G. Let $(\operatorname{mod} R)$ be the congruence in P, and C some class of that congruence. It is obvious that all words from C are equal and that two classes are equal if their words are equal.

Suppose that $m = x_1 \dots x_s$ is a word and that $d(m) = (\delta(x_1), \dots, \delta(x_s))$ is the array of grades of that word.

Let $m = x_1 \dots x_s \in C$ be the shortest word. We claim that it is reduced. Assume otherwise, i.e. assume that there exist elements $x_i, x_j, i < j$, such that $x_i x_j \in H$.

Let

$$x_i x_q = x'_q x_i, \text{ for } i < q < j.$$

Hence, $x'_q \in H$ and

$$x_1 \dots x_{i-1} x_i x_{i+1} \dots x_j \dots x_s \equiv x_1 \dots x_{i-1} x'_{i+1} x_i x_{i+2} \dots x_j \dots x_s \ (\operatorname{mod} R),$$
$$x_1 \dots x_{i-1} x_i x_{i+1} \dots x_j \dots x_s \equiv x_1 \dots x_{i-1} x'_{i+1} \dots x'_q x_i x_{q+1} \dots x_j \dots x_s \ (\operatorname{mod} R),$$

and, for $q = j-1$

$$x_1 \dots x_{i-1} x_i x_{i+1} \dots x_j \dots x_s \equiv x_1 \dots x_{i-1} x'_{i+1} \dots x'_{j-1} x_i x_j \dots x_s \ (\operatorname{mod} R).$$

Since $x_i x_j \in H$, the last word is congruent modulo R to the word $y_1 \dots y_{s-1}$, whose length is less than s, where

$$y_q = x_q, \ \ if \ \ 1 \leq q < i, \ \ y_q = x'_{q+1}, \ \ if \ \ i \leq q < j-1, \ \ \textit{and}$$
$$y_{j-1} = x_i x_j, \ y_q = x_{q-1}, \ \ if \ \ j \leq q < s.$$

Thus, in case when m is not a reduced word, it doesn't belong to C and is not the shortest one.

On the other hand, let us notice that for word $m \in C$ there exists a word $m' \in C$ whose array of grades $d(m')$ is obtained from the array $d(m) = (\delta_1, \dots, \delta_s)$ of the word $m = x_1 \dots x_s$, where $\delta_i = \delta(x_i)$, by applying an arbitrary permutation on it.

Indeed, if

$$1 \leq i < s \ \ \textit{and} \ \ x_i x_{i+1} = x'_{i+1} x_i, \ \ \textit{then the word} \ \ m = x_1 \dots x_{i-1} x'_{i+1} x_i x_{i+2} \dots x_s,$$

has an array of grades $(\delta_1, \dots, \delta_{i-1}, \delta'_{i+1}, \delta_i, \delta_{i+2}, \dots, \delta_s)$ which is obtained from $d(m)$ by transposing arbitrary neighboring elements δ_i, δ_{i+1}. The set

$$\Delta(m) = \{\delta(x_i) | \, i = 1, \dots, s\} \ \textit{is called the set of grades of m.}$$

Let us prove that equally reduced words $m = x_1 \dots x_s$ and $n = y_1 \dots y_t$ have the same length, i.e. that $s = t$, and that the corresponding sets of grades $\Delta_1 = \Delta(m)$ and $\Delta_2 = \Delta(n)$ are totaly ordered in sense that the corresponding elements are comparable.

Let Δ^* be the set of maximal elements of $\Delta_1 \cup \Delta_2$. Observe some total ordering on Δ^*, with $(\delta^*_1, \dots, \delta^*_u)$ as the array of elements of Δ^* written with respect to the ordering.

If $\delta \in \Delta_1 \cup \Delta_2$ is not from Δ^*, then there exist elements $\delta_i^* \in \Delta^*$ which are an upper bound of δ. Let $\delta^*(\delta)$ be the first of those upper bounds.

If $\delta, \delta' \in \Delta_1 \cup \Delta_2$, but $\delta, \delta' \notin \Delta^*$, then we cannot have $\delta^*(\delta) = \delta^*(\delta')$. Namely, for otherwise, δ and δ' would be from one of the sets Δ_1, Δ_2 and would have a common upper bound, which is impossible, since m and n are reduced words.

Let $\delta^*(\delta) = \delta$ if $\delta \in \Delta^*$. Aline the elements $\delta \in \Delta_1$ in the same way as the corresponding $\delta^*(\delta)$ are and do the same for the elements of the set Δ_2. We notice that there exist words

$$m' = x_1' \ldots x_s' \equiv m \pmod R \text{ and } n' = y_1' \ldots y_s' \equiv n \pmod R$$

whose arrays of grades $d(m')$ and $d(n')$ are the arrays of grades of elements of the set Δ_1, respectively Δ_2 and $m' = n'$.

If $1 \leq j \leq u$,

$$\textit{put } \bar{x}_j = x_i', \textit{ if } \delta^*(\delta(x_i')) = \delta_j^* \textit{ and put } \bar{x}_j = 1, \textit{ if such } i \textit{ doesn't exist},$$

and then, in the same way, define·elements $\bar{y}_j$.

Since the grades of Δ^* are elements of Δ_p and pairwise incomparable, grades $\delta_1^*, \ldots, \delta_u^*$ and elements $\bar{x}_j, \bar{y}_j$, $j = 1, \ldots, u$ satisfy the assumptions $vi')$ of Definition 4.2. On the other hand, since $\delta_j^* \in \Delta_1 \cup \Delta_2$, one of the common upper bounds of elements $\delta(\bar{x}_j)$, $\delta(\bar{y}_j)$ is δ_j^*, and hence, $\delta(\bar{x}_j^{-1}\bar{y}_j) < \delta_j^*$. However, if one of the elements $\bar{x}_j, \bar{y}_j$ equals 1, then

$$\delta(\bar{x}_j^{-1}\bar{y}_j) = \max\{\delta(\bar{x}_j),\ \delta(\bar{y}_j)\} = \delta_j^*,$$

contrary to the previous inequality.

Hence, none of the elements $\bar{x}_j, \bar{y}_j$ equals 1, and every δ_j^* has the form $\delta^*(\delta)$, for some $\delta \in \Delta_1$. Thus, the mapping $\delta \to \delta^*(\delta)$ is surjection, hence, bijection $\Delta_1 \to \Delta^*$ and $s = u$. From the reasons, $t = u$. We conclude that $s = t$. It is clear that the corresponding elements of sets Δ_1, Δ_2, alined in this way, are comparable.

Assume there exist equal words which are not congruent modulo R. According to already proven, there exist reduced words of the same length for which the corresponding elements of their arrays of grades are comparable.

Let $m = x_1 \ldots x_s$ and $m' = x_1' \ldots x_s'$ be two such words with shortest length. Then s ≥ 2, since two words of lenght 1 are equal only if they coincide. Put $\delta_i = \max\{\delta(x_i), \delta(x_i')\}$. Then the words m and m' satisfy $vi')$ of Definition 4.2 for array $\delta_1, \ldots, \delta_s$. Hence, if $y_1 = x_1^{-1}x_1'$ and if we put $n = x_1 x_2 \ldots x_s$ and $n' = y_1 x_2' \ldots x_s'$ then, since

$$m = m', \textit{ we have } n = n', \textit{ but } n \not\equiv n' \pmod R,$$

for otherwise, we would have $m \equiv m' \pmod R$, because

$$m \equiv x_1 n \pmod R \textit{ and } m' \equiv x_1 n' \pmod R, \textit{ hence}, \ n = x_1^{-1} m \textit{ and } n' = x_1^{-1} m'.$$

The word n is reduced with length $s - 1$.

The word $n' = n$, cannot be reduced, since it has length $s > s-1$, but it can be proven that there exists a word $n'' \equiv n' (\mathrm{mod}\, R)$, which is reduced, of length $s-1$, for which one can assume that the corresponding elements of arrays $d(n)$ and $d(n'')$ are comparable. Contradiction, since the least length of a word is s. □

With this we have proved that every extragraduation is at the same time a paragraduation.

It is known that there is a large class of extragraduations (paragraduations) which are graduations [15]. However, there are examples of paragraduations which are not graduations. Here we provide such class of paragraduations.

4.1. **Examples of paragraduations**

Example 4.1. Assume that $\Delta = \Delta^* \cup \{0\}$ is the set in which all elements $\delta \in \Delta^*$ are greater than 0 and pairwise incomparable. We thus have

$$\inf(\delta,\delta') = 0, \ \text{ for all } \ \delta \neq \delta' \ (\delta,\delta' \in \Delta^*).$$

Hence,

$$G_\delta \cap G_{\delta'} = \{e\} \ \textit{ and, for } \ x \in G_\delta, \ \ y \in G_{\delta'}, \ \delta(z(x,y)) = 0 \Rightarrow$$
$$z(x,y) = e, \ \ i.e. \ \ yx = xy.$$

Also, if $\delta_1, \delta_2, \ldots \delta_n$ are distinct elements of Δ^*, they are pairwise incomparable and if

$$x_1 x_2 \ldots x_n = x_1' x_2' \ldots x_n' \ \ (x_i, x_i' \in H) \ \ \textit{and} \ \ \delta(x_i), \delta(x_i') < \delta_i \ \ (i = 1, \ldots, n),$$

then, according to Definition 4.2 vi'), we have

$$\delta(x_i^{-1} x_i') < \delta_i \Rightarrow \delta(x_i^{-1} x_i') = 0, \ \ \textit{and} \ \ x_i^{-1} x_i' = e \ \Rightarrow x_i = x_i'.$$

This means that G is a direct sum

$$G = \bigoplus\nolimits_{\delta \in \Delta^*} G_\delta = \bigoplus\nolimits_{\delta \in \Delta} G_\delta$$

thus meaning that such extragraduation is a graduation whose empty grades are assumed to be the grades of e.

Conversely, a graduation which has all homogeneus elements with grades is an extragraduation with appropriate set of grades.

Example 4.2. Assume that $\Delta = \Delta^* \cup \{\omega, 0\}$, where all elements $\delta \in \Delta^*$ are greater than ω, pairwise incomparable where $\omega > 0$. Hence, if elements $\delta, \delta' \in \Delta^*$, are distinct, then

$$\inf(\delta,\delta') = \omega, \ \text{ so, } \ G_\delta \cap G_{\delta'} = G_\omega,$$

and hence

$$\text{if } \ x \in G_\delta, y \in G_{\delta'}, \ \delta(z(x,y)) \leq \omega \Rightarrow z(x,y) \in G_\omega.$$

So, we have that G_ω commutes with all elements of G_δ, for every $\delta \in \Delta^*$, and so does with all elements of $H = \cup_{\delta \in \Delta^*} G_\delta$, and since H is a genereting set of G, it commutes with all elements of G. This implies that G_ω is normal subgroup of G.

On the other hand, all elements $x, y \in G$ commute modulo G_ω. Hence,

$$G/G_\omega = \bigoplus_{\delta \in \Delta^*} (G_\delta / G_\omega).$$

This means that G is a quasigraded group whose center is $C = G_\omega$ and

$$E : \Delta_p \to Sg(G) \ \text{ its quasigraduation}.$$

On the other hand, it is obvious that an extragraduation of quasigraded group has considered form.

Example 4.3. Let A be a ring and observe the ring of upper triangular matrices

$$R = \begin{pmatrix} A & A \\ A & 0 \end{pmatrix}, \text{ and let } R_{\delta_1} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, R_{\delta_2} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, R_{\delta_3} = \begin{pmatrix} A & A \\ 0 & 0 \end{pmatrix}.$$

If $R_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, denote by Δ the set $\{0, \delta_1, \delta_2, \delta_3\}$. For convenience, let $\delta_0 = 0$. Set Δ is a partially ordered set, which is from below complete semi-lattice and from above inductively ordered, with respect to

$$\delta_i < \delta_j \Leftrightarrow R_{\delta_i} \subseteq R_{\delta_j} \ i, j = 0, 1, 2, 3.$$

It is easy to see that the mapping $\pi : \delta_i \to R_{\delta_i}, \delta_i \in \Delta$, is paragraduation, but it is not graduation, since, for instance

$$R_{\delta_1} \cap R_{\delta_3} = R_{\delta_1} \neq R_0.$$

From these examples we saw that there are paragraduations which are and which are not graduations.

However, each graduation is a special paragraduation. Namely, it is sufficient that the set of empty grades is well ordered, taking the empty grade 0 as its first element, and after all of them to put mutually incomparable essential grades of the given graduation.

4.2. **Direct product and direct sum of paragraded groups**

For the family of the paragraded groups, the direct product and the direct sum, in the natural way, are defined. Those are, again paragraded groups, and the direct sum is homogeneous subgroup of the direct product.

The direct product and the direct sum of the paragraded groups are extragraded groups, if and only if all of the factors are extragraded groups. However, the direct product of the graded groups is only extragraded group, except if it is at most one of the factors with nontrivial graduation, that is $\Delta_\alpha \neq \{0\}$.

Now we will introduce the direct product and the direct sum of pargraded groups.

Let $\{(G_\alpha, \cdot, \pi_\alpha) | \alpha \in A\}$ be a family of paragraded groups with paragraduations $\pi_\alpha : \Delta_\alpha \to S_g(G_\alpha)$, $\alpha \in A$, where $(\Delta_\alpha, <)$ is the set of grades of $(G_\alpha, \cdot, \pi_\alpha)$ and 1_α its neutral element.

Let us observe the direct product $G = \prod_{\alpha\in A} G_\alpha$ and the direct sum $\bigoplus_{\alpha\in A} G_\alpha$ of paragraded groups $(G_\alpha|\alpha \in A)$ respectively, with neutral element $1 = (1_\alpha|\alpha \in A)$.

Obviously, both, the direct product and the direct sum are groups with respect to the operation "$\cdot$", defined for

$$x = (x_\alpha|\,\alpha \in A),\; y = (y_\alpha|\,\alpha \in A) \in G \;\; with \;\; x\cdot y = (x_\alpha \cdot y_\alpha|\,\alpha \in A).$$

Next, we focus on question whether these groups are paragraded and according to the following theorem, (formulated in [15] without proof) they are.

Theorem 4.2. *Groups $\prod_{\alpha\in A} G_\alpha$ and $\bigoplus_{\alpha\in A} G_\alpha$ are paragraded with*

$$\pi : \Delta = \prod_{\alpha\in A} \Delta_\alpha \to Sg\big(\prod_{\alpha\in A}\big), \;\; where \;\; \pi(\delta) = (\pi_\alpha(\delta_\alpha)|\,\alpha \in A) = \prod_{\alpha\in A} \pi_\alpha(\delta_\alpha),$$

as their paragraduation mappings.

Before passing on the proof of the aforementioned theorem, we will introduce some more notions and prove the following lemma:

Lemma 4.3. *Let $\Delta = \prod_{\alpha\in A} \Delta_\alpha$ be the direct product of sets $\Delta_\alpha, \alpha \in A$, ordered as follows: if $\delta = (\delta_\alpha \in \Delta_\alpha|\alpha \in A)$, $\delta' = (\delta'_\alpha \in \Delta_\alpha|\alpha \in A) \in \Delta$, then*

$$\delta \le \delta' \Leftrightarrow \delta_\alpha \le \delta'_\alpha \; (\forall \alpha \in A)\,.$$

The set $(\Delta, <)$ is then as ordered as Δ_α, $\alpha \in A$.

Proof. Indeed, if $\overline{\Delta} \subseteq \Delta$, then

$$\{\inf\{\delta_\alpha|\,\delta = (\delta_\beta\,|\,\beta \in A) \in \overline{\Delta}\}|\alpha \in A\} = \inf_{\delta\in\overline{\Delta}} \delta$$

and, if m_α, $\alpha \in A$, is the upper bound of

$$\{\delta_\alpha|\delta = (\delta_\beta\,|\,\beta \in A) \in T\}\,,$$

where T is the chain in Δ, then $m = (m_\alpha|\,\alpha \in A)$ is the upper bound of T. □

Let's go back to proof of Theorem 4.2.

Proof. It remaines to prove that mapping $\pi : \Delta \to Sg(G)$ satisfies the axioms $i)$ - $vi)$ of Definition 4.1.

$$\pi(0_\alpha \in \Delta_\alpha|\,\alpha \in A) = \prod_{\alpha\in A} \pi_\alpha(0_\alpha) = \prod_{\alpha\in A} 1_\alpha = (1_\alpha|\,\alpha \in A) = 1\,.$$

Let $\delta = (\delta_\alpha \in \Delta_\alpha|\,\alpha \in A)$, $\delta' = (\delta'_\alpha \in \Delta_\alpha|\,\alpha \in A) \in \Delta$ and $\delta < \delta'$. Then

$$\delta_\alpha < \delta'_\alpha\,(\alpha \in A), \;\; and\ so \;\; \pi_\alpha(\delta_\alpha) \subseteq \pi_\alpha(\delta'_\alpha),\; \alpha \in A.$$

Hence,

$$\pi(\delta) \subseteq \pi(\delta')$$

and so $i)$ holds.

Δ is a partially ordered set of grades, which is from bellow a complete semi-lattice and from above inductively ordered and

$$\bigcap_{\delta_\alpha \in \theta_\alpha} G_{\delta_\alpha} = G_{\inf \theta_\alpha} \Leftarrow \theta = \prod_{\alpha \in A} \theta_\alpha \subseteq \Delta,$$

and so *ii*) holds as well.

Let $x, y \in H = \prod_{\alpha \in A} H_\alpha$ and

$$yx = zxy,\ x = (x_\alpha | \, \alpha \in A),\ y = (y_\alpha | \, \alpha \in A),\ z = (z_\alpha | \, \alpha \in A).$$

Then we have

$$y_\alpha x_\alpha = z_\alpha x_\alpha y_\alpha \, (\alpha \in A),$$

and so

$$z_\alpha \in H_\alpha \ (\alpha \in A) \ \ \textit{and} \ \ \delta(z_\alpha) \leq \inf (\delta(x_\alpha), \delta(y_\alpha)) \, (\alpha \in A).$$

It follows that

$$\delta(z) \leq \inf (\delta(x), \delta(y)),$$

which means that *iii*) is satisfied.

Since the sets H_α are generating sets of G_α, we have that H is the generating set of G. Thus *iv*) holds.

Let $A \subseteq H$ be a subset such that

$$xy \in H,\ x = (x_\alpha | \, \alpha \in A),\ y = (y_\alpha | \, \alpha \in A) \in H.$$

Hence, $x_\alpha y_\alpha \in H_\alpha$ $(\alpha \in A)$, and so there exists $\delta_\alpha \in \Delta_\alpha$ such that $A_\alpha \subseteq G_{\delta_\alpha}$. Thus we have

$$A \subseteq G_\delta,\ \delta = \prod_{\alpha \in A} \delta_\alpha,$$

and $v') \Leftrightarrow v)$ is satisfied.

Every group G_α is generated by H_α with set of relations R, which means that the same set of relations generates G. □

We will now state and prove one of the main properties of paragraded groups.

Theorem 4.4. *The homogeneous part H of the direct product and the direct sum of paragraded groups is the direct product and the direct sum of corresponding homogeneous parts respectively.*

Proof. Really, let H_α be a homogeneous part of $G_\alpha.\ \alpha \in A$. Then we have

$$\begin{aligned} H = \bigcup_{\delta \in \Delta} \pi(\delta) = \bigcup_{\delta \in \Delta} (\pi_\alpha(\delta_\alpha) | \alpha \in A) &= \bigcup_{\delta \in \prod_{\alpha \in A} \Delta_\alpha} (\pi_\alpha(\delta_\alpha) | \alpha \in A) \\ &= \prod_{\alpha \in A} \bigcup_{\delta_\alpha \in \Delta_\alpha} (\pi_\alpha(\delta_\alpha)) = \prod_{\alpha \in A} H_\alpha, \end{aligned}$$

which is contrary to the category of graded groups [13]. □

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Mirjana Vuković
Academy of Sciences and Arts of
Bosnia and Herzegovina
Bistrik 7, 71000 Sarajvo
Bosnia and Herzegovina
e-mail: *mirvuk48@gmail.com*