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ASPECTS OF WEAK, s-CS AND ALMOST INJECTIVE RINGS

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Abstract. It is not known whether right CF-rings (FGF-rings) are right artinian (quasi-Frobenius). This paper gives a positive answer of this question in the case of weak CS (s-CS) and $GC2$ rings. Also we get some new results on almost injective rings.

1. Introduction

A module M is said to satisfy $C1$-condition or called a CS-module if every submodule of M is essential in a direct summand of M. Patrick F. Smith [20] introduced weak CS modules. A right R-module M is called weak CS if every semisimple submodule of M is essential in a summand of M. I. Amin, M. Yousif and N. Zeyada [1] introduced *soc*-injective and strongly *soc*-injective modules. Given two R-modules M and N, M is *soc*-N-injective if any R-homomorphism $f : soc(N) \longrightarrow M$ extends to N. R is called right (self-) *soc*-injective, if the right R-module R_R is *soc*-injective. M is strongly *soc*-injective if M is *soc*-N-injective for any module N. They proved that every strongly *soc*-injective module is weak CS.

N. Zeyada [24] introduced the notion of s-CS, for any right R-module M, M is called s-CS if every semisimple submodule of M is essential in a summand of M.

A ring R is called a right CF ring if every cyclic right R-module can be embedded in a free module. A ring R is called a right FGF ring if every finitely generated right R-module can be embedded in a free right R-module. In section 2, we show that the right CF, weak CS (s-CS) and $GC2$ rings are artinian.

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Zeyada, Hussein and Amin introduced the notions of almost and rad-injectivity [23]. In the third section we make a correction to the result [23, Theorem 2.12] and we get a new result using these notions.

Throughout this paper R is an associative ring with identity and all modules are unitary R-modules. For a right R-module M, we denote the socle of M by $soc(M)$. S_r and S_l are used to indicate the right socle and the left socle of R, respectively. For a submodule N of M, the notations $N \subseteq^{ess} M$ and $N \subseteq^{\oplus} M$ mean that N is essential and a direct summand, respectively. We refer to [2], [5], [7], [12] and [15] for all undefined notions in this paper.

2. Generalizations of CS-modules and rings

Lemma 1. *For a right R-module M, the following conditions are equivalent:*

1. *M is weak CS.*
2. *$M = E \oplus T$ where E is CS with $soc(M) \subseteq^{ess} E$.*
3. *For every semisimple submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $A \subseteq M_1$ and M_2 is a complement of A in M.*

Proof. (1) $\Longrightarrow$ (2). Let M be a weak CS. Then $soc(M)$ is essential in a summand, so $M = E \oplus T$ with $soc(M) \subseteq^{ess} E$. Now if K is a submodule of E, then $soc(K) \subseteq^{ess} L$ where L is a summand of M and $L \subseteq^{ess} (K + L)$. But L is closed, so $K \subseteq L$. Since $E \subseteq^{ess} (L + E)$ and E is closed in M, so $L \subseteq E$ and E is CS.

(2) $\Longrightarrow$ (1). If E is CS and a summand of M with $soc(M) \subseteq^{ess} E$, then every submodule of $soc(M)$ is a summand of E and a summand of M.

($1 \Longrightarrow 3$). Let A be a submodule of $soc(M)$. By (1), there exists $M_1 \subseteq^{\oplus} M$ such that $A \subseteq^{ess} M_1$. Write $M = M_1 \oplus M_2$ for some $M_2 \subseteq M$. Since M_2 is a complement of M_1 in M and A is essential in M_1, then M_2 is the complement of A in M.

($3 \Longrightarrow 1$). Let A be a submodule of $soc(M)$. By (2), there exists a decomposition $M = M_1 \oplus M_2$ such that $A \subseteq M_1$ and M_2 is a complement of A in M. Then $(A \oplus M_2) \subseteq^{ess} M = M_1 \oplus M_2$ and $A \subseteq M_1$ then $A \subseteq^{ess} M_1$. Hence M is weak CS module. □

Recall that, a right R-module M is *s-CS* if every singular submodule of M is essential in a summand [24].

Proposition 1. *If M is a right R-module, then the following conditions are equivalent:*

1. *M is s-CS.*
2. *The second singular submodule $Z_2(M)$ is CS and a summand of M.*
3. *For every singular submodule A of M, there is a decomposition $M = M_1 \oplus M_2$ such that $A \subseteq M_1$ and M_2 is a complement of A in M.*

Proof. (1) $\Longleftrightarrow$ (2). [24, Proposition 14].

(1) $\Longleftrightarrow$ (3). A similar argument as in the proof of the above Lemma. $\square$

Given a right R-module M we will denote by $\Omega(M)$ [respectively $C(M)$] a set of representatives of the isomorphism classes of the simple quotient modules (respectively simple submodules) of M. In particular, when $M = R_R$, then $\Omega(R)$ is a set of representatives of the isomorphism classes of simple right R-modules.

Lemma 2. *Let R be a ring, and let P_R be a finitely generated quasi-projective CS-module, such that $|\Omega(P)| \leq |C(P)|$. Then $|\Omega(P)| = |C(P)|$, and P_R has finitely generated essential socle.*

Proof. See [17, Lemma 7.28]. $\square$

Proposition 2. *Let R be a ring. Then R is a right PF-ring if and only if R_R is a cogenerator and R is weak CS.*

Proof. Every right PF-ring is right self-injective and is a right cogenerator by [17, Theorem 1.56]. Conversely, if R is weak CS and R is cogenerator then $R = E \oplus T$ where E is CS with $S_r \subseteq^{ess} E$. By the above Lemma, E has a finitely generated, essential right socle. Since E is right finite dimensional and R_R is a cogenerator, let $S_r = S_1 \oplus S_2 \oplus ..... \oplus S_m$ and $I_i = I(S_i)$ be the injective hull of S_i, then there exists an embedding $\sigma : I_i \longrightarrow R^I$ for some set I. Then $\pi \circ \sigma \neq 0$ for some projection $\pi : R^I \longrightarrow R$, so $(\pi \circ \sigma)|S_i \neq 0$ and hence is monic. Thus $\pi \circ \sigma : I_i \longrightarrow R$ is monic, and so $R = E_1 \oplus .... \oplus E_m \oplus T$ where $soc\,(T) = 0$. So R is a right PF-ring. $\square$

Proposition 3. [24, Proposition 16] *Let R be a ring. Then R is a right PF-ring if and only if R_R is a cogenerator and $\left(Z_r^2\right)_R$ is CS.*

Proposition 4. *The following conditions are equivalent:*

(1) *Every right R-module is weak CS.*
(2) *Every right R-module with essential socle is CS.*
(3) *For every right R-module M, $M = E \oplus K$ where E is CS with $soc\,(M) \subseteq^{ess} E$.*

Proposition 5. *The following conditions are equivalent:*

(1) *Every right R-module is s-CS.*
(2) *Every Goldie torsion right R-module is CS.*
(3) *For every right R-module M, $M = Z_2\,(M) \oplus K$ where $Z_2\,(M)$ is CS.*

Dinh Van Huynh, S. K. Jain and S. R. López-Permouth [11] proved that if R is simple such that every cyclic singular right R-module is CS, then R is right noetherian.

Corollary 1. *If R is simple such that every cyclic right R-module is s-CS, then R is right noetherian.*

Proposition 6. *If R is a weak CS and $GC2$, and right Kasch, then R is semiperfect.*

Proof. Since R is a weak CS, so E is CS by Lemma 1 and $R = E \oplus K$ for some right ideal K of R and so E is a finitely generated projective module. By Lemma 2, E has a finitely generated essential socle. Then, by hypothesis, there exist simple submodules $S_1, \cdots, S_n$ of E such that $\{S_1, \cdots, S_n\}$ is a complete set of representatives of the isomorphism classes of simple right R-modules. Since E is CS, there exist submodules $Q_1, \cdots, Q_n$ of E such that $Q_1, \cdots Q_n$ is an direct summands of E and $(S_i)_R \subseteq^{ess} (Q_i)_R$ for $i = 1, \cdots, n$. Since Q_i is an indecomposable projective and $GC2$ R-module, it has a local endomorphism ring; and since Q_i is projective, $J(Q_i)$ is maximal and small in Q_i. Then Q_i is a projective cover of the simple module $Q_i/J(Q_i)$. Note that $Q_i \cong Q_j$ clearly implies $Q_i/J(Q_i) \cong Q_j/J(Q_j)$; and the converse also holds because every module has at most one projective cover up to isomorphism. It is clear that $Q_i \cong Q_j$ if and only if $S_i \cong S_j$ if and only if $i = j$. Thus, $\{Q_1/J(Q_1), \cdots, Q_n/J(Q_n)\}$ is a complete set of representatives of the isomorphism classes of simple right R-modules. Hence every simple right R-module has a projective cover. Therefore R is semiperfect. $\square$

The following example shows that the proof given in [24, Proposition 13] is not true, since the endomorphism ring of an indecomposable projective module which is an essential extension of a simple module may be not a local ring. So we add an extra condition that R is right $GC2$ to prove the Proposition.

Example 1. Let R be the ring of triangular matrices, $R = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a \in Z, b, c \in Q \right\}$. Take $P_1 = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a \in Z, b \in Q \right\}$ and $P_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} : c \in Q \right\}$, we see that P_1 is indecomposable projective module with simple essential socle and P_2 is a projective simple module. The socle of P_1 is isomorphic to P_2. Thus, the endomorphism ring of P_1 is isomorphic to Z which is not local.

Proposition 7. *If R is right s-CS and $GC2$, and right Kasch, then R is semiperfect.*

Proof. Since R is a weak CS, so E is CS by Lemma 1 and $R = E \oplus K$ for some right ideal K of R and so E is a finitely generated projective module. By Lemma 2, E has a finitely generated essential socle. Then, by hypothesis, there exist simple submodules $S_1, \ldots, S_n$ of E such that $\{S_1, \ldots, S_n\}$ is a complete set of representatives of the isomorphism classes of simple right R-modules. Since E is CS, there exist submodules $Q_1, \ldots, Q_n$ of E such that $Q_1, \ldots Q_n$ is a direct summands of E and $(S_i)_R \subseteq^{ess} (Q_i)_R$ for $i = 1, \ldots, n$. Since Q_i is an indecomposable projective and $GC2$ R-module, it has a local endomorphism ring; and since Q_i is projective, $J(Q_i)$ is maximal and small in Q_i. Then Q_i is a projective cover of the simple module $Q_i/J(Q_i)$. Note that $Q_i \cong Q_j$ clearly implies $Q_i/J(Q_i) \cong Q_j/J(Q_j)$; and the converse also holds because every module has at most one projective cover up to isomorphism. It is clear that $Q_i \cong Q_j$ if and only if $S_i \cong S_j$ if and only if $i = j$. Thus, $\{Q_1/J(Q_1), \ldots, Q_n/J(Q_n)\}$ is a complete set of representatives of the isomorphism classes of simple right R-modules. Hence every simple right R-module has a projective cover. Therefore R is semiperfect. □

Lemma 3. *Let R be a semiperfect, left Kasch and left min-CS ring. Then the following statements hold:*

(1) *$S_l \subseteq^{ess}_R R$ and $soc(Re)$ is simple and essential in Re for all local idempotents $e \in R$.*
(2) *R is right Kasch if and only if $S_l \subseteq S_r$.*
(3) *If $\{e_1, \ldots, e_n\}$ are basic local idempotents in R then $\{soc(Re_1), \ldots, soc(Re_n)\}$ is a complete set of distinct representatives of the simple left R-modules.*

Proof. See [17, Lemma 4.5]. □

Recall that a ring R is right minfull if it is semiperfect, right mininjective, and $soc(eR) \neq 0$ for each local idempotent $e \in R$.

Corollary 2. *If R is commutative s-CS (weak CS) and Kasch, then R is minfull.*

Proof. Since every Kasch ring is $C2$, so R is semiperfect by Proposition 6 (Proposition 7). Thus using the above Lemma and [17, Proposition 4.3] R is minfull. □

Theorem 1. *If R is right weak CS (s-CS), $GC2$ and every cyclic right R-module can be embedded in a free module (right CF ring) then R is right artinian*

Proof. If R is right weak CS (s-CS) right CF, then by Lemma 1 $R = E \oplus K$ where E is CS and $soc(K) = 0$ ($Z(K) = 0$). Thus by Proposition 6 (Proposition 7), R is semiperfect. The above Lemma gives $S_r \subseteq^{ess} R_R$, so $K = 0$. Hence R is CS and R is right artinain by [9, Corollary 2.9]. □

Proposition 8. *Let R be a right FGF, right weak CS (s-CS) and right $GC2$ ring. Then R is QF.*

Proof. It is clear by Proposition 6 (Proposition 7) , and [8, Theorem 3.7]. □

3. Almost injective modules

Definition 1. A right R-module M is called almost injective, if $M = E \oplus K$ where E is injective and K has zero radical. A ring R is called right almost injective, if R_R is almost injective.

In [23], the statement of Theorem 2.12 is not true. The following Proposition is the true version of [23, Theorem 2.12]. Moreover, the rest of this section will be devoted to do the necessary changes of the related results in [23]. Also we get new results in view of the following Proposition.

Proposition 9. *For a ring R the following statements are true:*

(1) *R is semisimple if and only if every almost-injective right R-module is injective.*
(2) *If R is semilocal, then every rad-injective right R-module is injective.*

Proof. (1). Assume that every almost-injective right R-module is injective, then every right R-module with zero radical is injective. Thus every semisimple right R-module is injective and R is right V-ring. Hence, every right R-module has a zero radical. Therefore, every right R-module is injective and R is semisimple. The converse is clear.

(2). Let R be a semilocal ring and M be a rad-injective right R-module. Consider a homomorphism $f : K \longrightarrow M$ where K is a right ideal of R. Since R is semilocal, there exists a right ideal L of R such that $K + L = R$ and $K \cap L \subseteq J$ [13]. Then there exists a R-homomorphism $g : R \longrightarrow M$ such that $g(x) = f(x)$ for every $x \in K \cap L$. Define $F : R \longrightarrow M$ by $F(x) = f(k) + g(l)$ for any $x = k + l$ where $k \in K$ and $l \in L$. It is clear that F is a well-defined R-homomorphism such that $F|_K = f$. i.e. F extends f. Therefore M is injective. □

A ring R is called quasi-Frobenius (QF) if R is right (or left) artinian and right (or left) self-injective. Also, R is QF if and only if every injective right R-module is projective.

Theorem 2. *R is a quasi-Frobenius ring if and only if every rad-injective right R-module is projective.*

Proof. If R is quasi-Frobenius, then R is right artinian, and by Proposition 9 (2), every rad-injective right R-module is injective. Hence, every rad-injective right R-module is projective. Conversely, if every rad-injective right R-module is projective, then every injective right R-module is projective. Thus, R is quasi-Frobenius. □

Recall that a ring R is called a right pseudo-Frobenius ring (right PF-ring) if the right R-module R_R is an injective cogenerator.

Proposition 10. [17, Theorem 1.56] *The following conditions are equivalent:*

(1) *R is a right PF-ring.*
(2) *R is a semiperfect right self-injective ring with essential right socle.*
(3) *R is a right finitely cogenerated right self-injective ring.*
(4) *R is a right Kasch right self-injective ring.*

Theorem 3. *If R is right Kasch right almost-injective, then R is semiperfect.*

Proof. Let R be right Kasch and $R_R = E \oplus T$, where E is injective and T has zero radical. If $J = 0$, then every simple right ideal of R is projective and R is semiperfect (for R is right Kasch). Now suppose that $J \neq 0$. Clearly, every simple singular right R-module embeds in E. In particular, every simple quotient of E is isomorphic to a simple submodule of E, and so E is a finitely generated injective and projective module containing a copy of every simple quotient of E. By [8, Lemma 18], E has a finitely generated essential socle. Then by hypothesis, there exist simple submodules $S_1, \ldots, S_n$ of E such that $\{S_1, \ldots, S_n\}$ is a complete set of representatives of the isomorphism classes of simple singular right R-modules. Since E is injective, there exist submodules $Q_1, \ldots, Q_n$ of E such that $Q_1 \oplus \cdots \oplus Q_n$ is a direct summand of E and $(S_i)_R \subseteq^{ess} (Q_i)_R$ for $i = 1, 2, \ldots, n$. Since Q_i is an indecomposable injective R-module, it has a local endomorphism ring. The projectivity of Q_i implies that $J(Q_i)$ is maximal and small in Q_i. Then Q_i is the projective cover of the simple module $Q_i/J(Q_i)$. Note that $Q_i \cong Q_j$ clearly implies $Q_i/J(Q_i) \cong Q_j/J(Q_j)$ and the converse also holds because every module has at most one projective cover up to isomorphism. But it is clear that $Q_i \cong Q_j$ if and only if $S_i \cong S_j$, if and only if $i = j$. Moreover, every $Q_i/J(Q_i)$ is singular. Thus, $\{Q_1/J(Q_1), \ldots, Q_n/J(Q_n)\}$ is a complete set of representatives of the isomorphism classes of the simple singular right R-modules. Hence, every simple singular right R-module has a projective cover. Since every non-singular simple right R-module is projective, we conclude that R is semiperfect. □

Proposition 11. *The following conditions are equivalent:*

(1) *R is a right PF-ring.*
(2) *R is a semiperfect right rad-injective ring with soc$(eR) \neq 0$ for each local idempotent e of R.*
(3) *R is a right finitely cogenerated right rad-injective ring.*
(4) *R is a right Kasch right rad-injective ring.*

(5) *R is a right rad-injective ring and the dual of every simple left R-module is simple.*

Proof. (1) $\Leftrightarrow$ (2) By Proposition 9 (2).

(1) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Since R is a right rad-injective ring, it follows from [23, Proposition 2.5] that $R = E \oplus K$, where E is injective and K has zero radical. Since R is a right finitely cogenerated ring, K is a finitely cogenerated right R-module with zero radical. Hence, K is semisimple. Therefore, by [22, Corollary 8], R is a right PF-ring.

(1) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (1) If R is right Kasch right rad-injective, then R is right almost-injective ([23, Proposition 2.5]). Thus R is semiperfect (3). Hence R is injective by Proposition 9 (2). Therefore, R is right PF.

(1) $\Rightarrow$ (5) Since every right PF-ring is left Kasch and left mininjective, the dual of every simple left R-module is simple by [16, Proposition 2.2].

(5) $\Rightarrow$ (1) By [23, Proposition 2.10], R is a right $min-CS$ ring (i.e. every minimal right ideal of R is essential in a summand). Thus, by [10, Theorem 2.1], R is semiperfect with essential right socle. Proposition 9 (2) entails that R is right self-injective, and hence right PF by Proposition 10.

(1) $\Longleftrightarrow$ (4) and (1) $\Longleftrightarrow$ (5) are direct consequences of [23, Proposition 2.5]. □

A result of Osofsky [18, Proposition 2.2] asserts that a ring R is QF if and only if R is a left perfect, left and right self-injective ring. This result remains true for rad-injective rings.

Proposition 12. *The following conditions are equivalent:*

(1) *R is a quasi-Frobenius ring.*
(2) *R is a left perfect, left and right rad-injective ring.*

Proof. (1) $\Rightarrow$ (2) It is well known.

(2) $\Rightarrow$ (1) By hypothesis, R is a semiperfect right and left rad-injective ring. By Proposition 9 (2), R is right and left injective, hence R is quasi-Frobenius. □

Note that the ring of integers Z is an example of a commutative noetherian almost-injective ring which is not quasi-Frobenius.

Definition 2. A ring R is called right CF-ring (FGF-ring) if every cyclic (finitely generated) right R-module embeds in a free module. It is not known whether right CF-rings (FGF-rings) are right artinian (quasi-Frobenius rings). In the next result, a positive answer is given if we assume in addition that the ring R is right rad-injective.

Proposition 13. *The following conditions are equivalent:*

(1) *R is quasi-Frobenius.*
(2) *R is right CF and right rad-injective.*

Proof. (1) $\Rightarrow$ (2) It is well known.

(2) $\Rightarrow$ (1) Since every simple right R-module embeds in R, R is a right Kasch ring. By Proposition 11, R is right self-injective with finitely generated essential right socle. Thus, every cyclic right R-module has a finitely generated essential socle, and by [21, Proposition 2.2], R is right artinian, hence quasi-Frobenius. $\square$

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References

[1] I. Amin, M. Yousif and N. Zeyada, *Soc-injective rings and modules*, Comm. Algebra, 33 (2005), 4229–4250.
[2] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, Berlin-New York, 1974.
[3] J. E. Bjork, *Rings satisfying certain chain conditions*, J. Reine Angew. Math., 245 (1970), 63–73.
[4] V. Camillo, W. K. Nicholson and M. F. Yousif, *Ikeda-Nakayama rings*, J. Algebra, 226 (2000), 1001–1010.
[5] N. V. Dung, V. D. Huynh, P. F. Smith and R. Wisbauer, *Extending Modules,* Pitman Reasearch Notes in Math. Longman, 1994.
[6] C. Faith, *Rings with ascending chain conditions on annihilators*, Nagoya Math. J., 27 (1966), 179–191.
[7] C. Faith, *Algebra II, Ring Theory*, Springer-Verlag, Berlin-New York, 1976.
[8] J. L. G´omez Pardo and P. A. Guil Asensio, *Essential embedding of cyclic modules in projectives*, Trans. Amer. Math. Soc., 349 (1997), 4343–4353.
[9] J. L. Gómez Pardo and P. A. Guil Asensio, *Rings with finite essential socle*, Proc. Amer. Math. Soc., 125 (1997), 971–977.
[10] J. L. G´omez Pardo and M. F. Yousif, *Semiperfect min-CS rings*, Glasg. Math.J., 41 (1999), 231–238.
[11] Dinh Van Huynh, S. K. Jain and S. R. Lopez-Permouth, *When cyclic singular modules over a simple ring are injective*, J. Algebra, 263 (2003), 188–192.
[12] F. Kasch, *Modules and Rings*, Academic Press, New York, 1982.
[13] C. Lomp, *On semilocal modules and rings*, Comm. Algebra, 27 (4) (1999), 1921–1935.
[14] G. O. Michler and O.E. Villamayor, *On rings whose simple modules are injective*, J. Algebra, 25 (1973), 185–201.
[15] S. H. Mohamed and B. J. Muller, *Continuous and Discrete Modules*, Cambridge University Press, Cambridge, 1990.
[16] W. K. Nicholson and M.F. Yousif, *Mininjective rings*, J. Algebra, 187 (1997), 548–578.
[17] W. K. Nicholson and M.F. Yousif, *Quasi-Frobenius Rings*, Cambridge Tracts in Math., 158, Cambridge University Press, Cambridge, 2003.
[18] B. L. Osofsky, *A generalization of quasi-Frobenius rings*, J. Algebra, 4 (1966), 373–387.

[19] Liang Shen and Jianlong Chen, *New characterizations of quasi-Frobenius rings*, Comm. Algebra, 34 (2006), 2157–2165.
[20] P. F. Smith, *CS-modules and weak CS-modules*, Noncommutative Ring Theory, Springer LNM, 1448 (1990), 99–115.
[21] P. Vamos, *The dual of the notion of 'finitely generated'*, J. London Math. Soc., 43 (1968), 186–209.
[22] M. F. Yousif, Y. Zhou and N. Zeyada, *On pseudo-Frobenius rings*, Canad. Math. Bull., 48 (2) (2005), 317–320.
[23] N. Zeyada, S. Hussein and A. Amin, *Rad-injective and almost-injective modules and rings*, Algebra Colloq., 18 (3) (2011), 411–418.
[24] N. Zeyada and N. Jarboui, *s-CS modules and rings*, Int. J. Algebra, 7 (2) 2013, 49–62.

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