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POSINORMALITY, COPOSINORMALITY, AND SUPRAPOSINORMALITY FOR SOME TRIANGULAR OPERATORS

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Abstract. The range inclusion criterion for posinormality is studied in order to classify more examples of lower triangular factorable matrices as posinormal operators or not. Also, coposinormality is shown to be a hereditary property for lower triangular operators on ℓ^2, and this leads to some results involving the posispectrum. Finally, sufficient conditions are given for lower triangular factorable matrices to be supraposinormal, and an example is given of a lower triangular factorable matrix that is supraposinormal but neither posinormal nor coposinormal. The last two sections also contain more general results that apply to operators on abstract Hilbert spaces.

1. Introduction

If $B(H)$ denotes the set of all bounded linear operators on a Hilbert space H, then $A \in B(H)$ is said to be *supraposinormal* if $AQA^* = A^*PA$ for some pair of positive operators Q, $P \in B(H)$, where at least one of P, Q has dense range (see [11]). The operator A is *posinormal* (see [3], [4], [5]) if $AA^* = A^*PA$ for some positive operator $P \in B(H)$, called the *interrupter*. The operator A is *coposinormal* if A^* is posinormal. Some key facts about posinormal operators appear in the following results found in [7, Theorem 2.1 and Corollary 2.3].

Proposition 1.1. *For $A \in B(H)$, the following statements are equivalent:*

(a) *A is posinormal;*
(b) *$RanA \subseteq RanA^*$;*
(c) *$AA^* \leq \gamma^2 A^*A$ for some $\gamma \geq 0$; and*
(d) *There exists a bounded operator B on H such that $A = A^*B$.*

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If $Ran(A-\lambda I) \subseteq Ran(A-\lambda I)^*$ for all λ in the spectrum of A, then A is *dominant* [12]. If statement (c) is satisfied for $\gamma = 1$, then A is *hyponormal*, so hyponormal operators are necessarily posinormal and dominant.

Proposition 1.2. *If $A \in B(H)$ is posinormal, then $KerA \subseteq KerA^*$.*

In earlier papers, attention was focused primarily on statement (d) of Proposition 1.1 because that formulation could sometimes be used to prove that an operator is hyponormal. In this paper, attention will be focused on statement (b), concerning range inclusion. This approach will allow us to classify more examples than before, and it also turns out to be particularly useful in studying coposinormality of triangular operators.

We note that although most of this paper deals with triangular operators on ℓ^2, the results at the beginning of Sections 4 and 5 are concerned with more general operators on abstract Hilbert spaces H as well.

2. Using the range inclusion criterion

A square matrix is called *lower triangular* if all the entries above the main diagonal are zero. The lower triangular infinite matrix $M = [m_{ij}]$, acting through multiplication to give a bounded linear operator on ℓ^2, is *factorable* if its entries are

$$m_{ij} = \begin{cases} a_i c_j & if \quad j \leq i \\ 0 & if \quad j > i \end{cases}$$

where a_i depends only on i and c_j depends only on j. The factorable matrix M is *terraced* if $c_j = 1$ for all j. The elements of ℓ^2 will be presented as row vectors, although in computations we frequently need to use the transpose.

2.1. Necessary and sufficient conditions for a lower triangular factorable matrix to be a posinormal operator.

Theorem 2.1. *Assume that $a_n, c_n > 0$ for all n. The lower triangular factorable matrix $M = [a_i c_j] \in B(\ell^2)$ is posinormal if and only if the following condition holds: For each $x :\equiv < x_0, x_1, x_2, ... > \in \ell^2$, it is true that $y :\equiv < y_0, y_1, y_2, ... > \in \ell^2$ also, where*

$$y_n :\equiv (\frac{1}{c_n} - \frac{a_{n+1}}{c_{n+1} a_n}) \sum_{i=0}^{n} c_i x_i - \frac{a_{n+1}}{a_n} x_{n+1}$$

for each n and

$$x_0 = \frac{1}{a_0} \sum_{i=0}^{\infty} a_i y_i.$$

Proof. By Proposition 1.1, M is posinormal if and only if $RanM \subseteq RanM^*$. This means that for each $x \in \ell^2$, there is a $y \in \ell^2$ such that $Mx = M^*y$. Therefore, for each n, it is true that

$$a_n \sum_{i=0}^{n} c_i x_i = c_n \sum_{i=n}^{\infty} a_i y_i.$$

Consequently, we have

$$c_n \sum_{i=n}^{\infty} a_i y_i - \frac{c_n}{c_{n+1}} c_{n+1} \sum_{i=n+1}^{\infty} a_i y_i = a_n \sum_{i=0}^{n} c_i x_i - \frac{c_n}{c_{n+1}} a_{n+1} \sum_{i=0}^{n+1} c_i x_i,$$

or

$$c_n a_n y_n = (a_n - \frac{c_n}{c_{n+1}} a_{n+1}) \sum_{i=0}^{n} c_i x_i - c_n a_{n+1} x_{n+1},$$

and this gives the result. □

Corollary 2.2. *If $M \in B(\ell^2)$ is a lower triangular factorable matrix associated with sequences such that $c_n = a_n^p$ for all n where $p \geq 1$ and $\{a_n\}$ is positive and decreasing, then M is not posinormal.*

Proof. Consider $x :\equiv < x_0, x_1, x_2, ... > \in \ell^2$ where $x_n = \frac{1}{n+1}$ for all n. Then

$$y_n \leq -\frac{a_{n+1}}{a_n} \frac{1}{n+2}$$

for all n. Since

$$\frac{1}{a_0} \sum_{i=0}^{\infty} a_i y_i \leq -\frac{1}{a_0} \sum_{i=0}^{\infty} \frac{a_{i+1}}{i+2} < 0$$

and $x_0 = 1$, M cannot be posinormal. □

Corollary 2.3. *Assume that $M \in B(\ell^2)$ is a terraced matrix with $a_n > 0$ for all n. If $L :\equiv lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and $0 \leq L < 1$, then M is not posinormal.*

Proof. Assume $x_n = \frac{1}{n+1}$ for all n. Then $x :\equiv < x_0, x_1, x_2, ... > \in \ell^2$, but

$$y_n = (1 - \frac{a_{n+1}}{a_n}) \sum_{i=0}^{n} \frac{1}{i+1} - \frac{a_{n+1}}{a_n} \frac{1}{n+1} \to (1-L) \sum_{i=0}^{\infty} \frac{1}{i+1} \neq 0,$$

so $y \notin \ell^2$. □

The *p-Cesàro matrices* and and the *generalized Cesàro matrices of order one* are the terraced matrices associated with the sequences

$$a_n = \frac{1}{(n+1)^p} \quad , \quad p > 0$$

and

$$a_n = \frac{1}{k+n} \ , \ \ k > 0$$

respectively. Previously, these operators on ℓ^2 have been shown to be both posinormal and coposinormal [7, 8]. The only terraced matrices that have heretofore been identified as non-posinormal involved a leading entry of zero (that is, $a_0 = 0$). The next example identifies some non-posinormal terraced matrices with a nonzero leading entry.

Example 2.4. By Corollary 2.3, the following terraced matrices are not posinormal operators on ℓ^2.

(a) M associated with the sequence $a_n = \alpha^n$ for all n, where $0 < \alpha < 1$.
(b) M associated with the sequence $a_n = \frac{1}{(n+1)!}$ for all n.
(c) M associated with the sequence $a_n = \frac{1}{(n+1)^{n+1}}$ for all n.

We note that [9, Theorem 4] can be used to show that the operator in part (a) is coposinormal.

2.2. **Necessary and sufficient conditions for a lower triangular factorable matrix to be a coposinormal operator.**

Theorem 2.5. *Assume that $a_n, c_n > 0$ for all n. The lower triangular factorable matrix $M = [a_i c_j] \in B(\ell^2)$ is coposinormal if and only if the following condition holds: For each $x :\equiv < x_0, x_1, x_2, ... > \in \ell^2$, it is true that $y :\equiv < y_0, y_1, y_2, ... > \in \ell^2$ also, where*

$$y_0 = \frac{1}{a_0} \sum_{i=0}^{\infty} a_i x_i$$

and

$$y_{n+1} :\equiv \left(\frac{1}{a_{n+1}} - \frac{c_n}{c_{n+1} a_n}\right) \sum_{i=n+1}^{\infty} a_i x_i - \frac{c_n}{c_{n+1}} x_n$$

for each n.

Proof. By Proposition 1.1, M is coposinormal if and only if $RanM^* \subseteq RanM$. This means that for each $x \in \ell^2$, there is a $y \in \ell^2$ such that $M^*x = My$. Therefore, for each n, it is true that

$$c_n \sum_{i=n}^{\infty} a_i x_i = a_n \sum_{i=0}^{n} c_i y_i.$$

It follows that

$$y_0 = \frac{1}{a_0} \sum_{i=0}^{\infty} a_i x_i$$

and

$$a_{n+1}\sum_{i=0}^{n+1} c_i y_i - \frac{a_{n+1}}{a_n} a_n \sum_{i=0}^{n} c_i y_i = c_{n+1}\sum_{i=n+1}^{\infty} a_i x_i - \frac{a_{n+1}}{a_n} c_n \sum_{i=n}^{\infty} a_i x_i,$$

or

$$a_{n+1}c_{n+1}y_{n+1} = (c_{n+1} - \frac{a_{n+1}c_n}{a_n})\sum_{i=n+1}^{\infty} a_i x_i - a_{n+1}c_n x_n,$$

and this gives the result. □

Example 2.6. Let M denote the lower triangular factorable matrix associated with the sequences $\{a_i\}$, $\{c_j\}$ where $a_i = \frac{1}{i+1}$ and $c_j = \frac{1}{[(j+1)!]^{1/4}}$ for all i, j.

(a) If $x_n = \frac{1}{(n+1)^{3/4}}$ for all n, then
$y_n = (n+1)^{1/4}[(n+1)^{3/4} - n]\sum_{k=n}^{\infty}\frac{1}{(k+1)^{7/4}} - (n+1)^{1/4}n^{-3/4} < -\frac{1}{\sqrt{n}}$ for $n \geq 3$, so $\{y_n\} \notin \ell^2$. By Theorem 2.5, M is not coposinormal.

(b) Next we apply Theorem 2.1 to show that M is also not posinormal. If $x_n = \frac{1}{(n+1)^{3/4}}$ for all n, then
$y_n = ([(n+1)!]^{1/4}(1 - \frac{n+1}{(n+2)^{3/4}})\sum_{i=0}^{n}\frac{1}{[(i+1)!]^{1/4}}\frac{1}{(i+1)^{3/4}} - \frac{n+1}{n+2}\frac{1}{(n+1)^{3/4}}$
$< -0.06066([(n+1)!]^{1/4}$ for all $n \geq 2$, so $\{y_n\} \notin \ell^2$.

We will see in Section 5 that the matrix M from Example 2.6 is a supraposinormal operator on ℓ^2, although it is neither posinormal nor coposinormal.

Proposition 2.7. *Suppose the lower triangular factorable matrix* $M = [a_i c_j] \in B(\ell^2)$ *satisfies* $a_n = c_n > 0$ *for all* n *and* $\{\frac{c_n}{c_{n+1}} : n \geq 0\}$ *is bounded. Then* M *is coposinormal.*

Proof. Define $Z = [z_{ij}]$ by

$$z_{ij} = \begin{cases} \frac{c_i}{c_j} & if \quad j = 0 \\ -\frac{c_{j-1}}{c_j} & if \quad j = i+1 \\ 0 & if \quad j > 0, \ \ j \neq i+1 \end{cases}.$$

Clearly $Z \in B(\ell^2)$. It is straightforward to verify that $M = ZM^*$, so $M^* = MZ^*$. Therefore M^* is posinormal by Proposition 1.1. □

Note that the previous proposition settles a case where, even with the additional assumption that $\{a_n\}$ is decreasing, [9, Theorems 3 and 4] would not help since $\frac{a_n}{c_n} = 1$ for all n, so $\frac{a_n}{c_n} \nrightarrow 0$.

Example 2.8. If M is the lower triangular factorable matrix associated with the sequences $\{a_i\}$, $\{c_j\}$ specified below, then M is coposinormal by Proposition 2.7.

(a) The sequences defined by $a_n = c_n = \alpha^n$ for all n, where $0 < \alpha < 1$.
(b) The sequences defined by $a_n = c_n = \frac{1}{(n+1)^p}$ for all n, where $p \geq 1$.

Note that the operator in part (a) fails to be posinormal by Corollary 2.2, as does the operator in part (b) in the special case when $p = 1$.

While no example has been presented here of a lower triangular factorable matrix M which satisfies the requirement that M is posinormal but not coposinormal, it should be noted that the unilateral shift $U \in B(\ell^2)$ is a lower triangular operator satisfying that requirement.

3. Heredity

Throughout this section we assume that $M = [a_{ij}] \in B(\ell^2)$ is a lower triangular infinite matrix with complex entries. The entries a_{ij} should not be confused with the entries $a_i c_j$ from earlier sections, since M is not necessarily factorable here. At the end of the previous section, U was used to denote the unilateral shift, and we continue that usage throughout this section. We observe that U^*MU is the lower triangular matrix that is obtained when the first row and first column are deleted from M. In [10] it was shown that posinormality and hyponormality, among other properties, are inherited from M by U^*MU; however, the question regarding coposinormality was not settled there. Our different approach here will allow that question to be settled now.

3.1. **Coposinormality is inherited by triangular operators.**

Proposition 3.1. *Suppose that the lower triangular matrix $M = [a_{ij}]$ is a coposinormal operator on ℓ^2. If $a_{00} = 0$, then it must hold true that $a_{i0} = 0$ for all i.*

Proof. Let $z :\equiv < 0, a_{10}, a_{20}, a_{30}, ... >$ and note that $z \in \ell^2$. Since M is coposinormal, the property $RanM^* \subseteq RanM$ holds, so there exists an $x \in \ell^2$ such that $Mx = M^*z$. Therefore, $\sum_{i=1}^{\infty} |a_{i0}|^2 = (M^*z)(0) = (Mx)(0) = 0$, so $a_{i0} = 0$ for all i. $\square$

Theorem 3.2. *If M is coposinormal, then U^*MU is also coposinormal.*

Proof. It suffices to show that $Ran(U^*MU)^* \subseteq Ran(U^*MU)$, by Proposition 1.1.

(a) First we consider the case $a_{00} \neq 0$. Suppose that $x :\equiv < x_1, x_2, x_3, \cdots > \in \ell^2$. Take $x_0 :\equiv -\frac{1}{\overline{a_{00}}} \sum_{i=1}^{\infty} \overline{a_{i0}} x_i$. If we take $\hat{x} :\equiv < x_0, x_1, x_2, x_3, \cdots >$, then since M is coposinormal, there exists a $\hat{y} :\equiv < y_0, y_1, y_2, y_3, \cdots > \in \ell^2$ such that $M^*\hat{x} = M\hat{y}$. Observe that $a_{00}y_0 = (M\hat{y})(0) = (M^*\hat{x})(0) = \sum_{i=0}^{\infty} \overline{a_{i0}} x_i = 0$, so $y_0 = 0$; therefore, $\sum_{j=1}^{n} a_{nj} y_j = \sum_{j=0}^{n} a_{nj} y_j = (M\hat{y})(n) = (M^*\hat{x})(n) = \sum_{i=n}^{\infty} \overline{a_{in}} x_i$ for each $n \geq 1$. This means that if $y :\equiv < y_1, y_2, y_3, \cdots >$, then we have $(U^*MU)y = (U^*MU)^*x$, as needed.

(b) Now we consider the case $a_{00} = 0$. By Proposition 3.1, it must then be true that $a_{i0} = 0$ for all i. Once again, suppose that $x :\equiv < x_1, x_2, x_3, \cdots > \in \ell^2$. If x_0 is a (any) complex number and $\hat{x} :\equiv < x_0, x_1, x_2, x_3, \cdots >$, then there exists a $\hat{y} :\equiv < y_0, y_1, y_2, y_3, \cdots > \in \ell^2$ such that $M^*\hat{x} = M\hat{y}$. Note that for each $n \geq 1$, $\sum_{i=n}^{\infty} \overline{a_{in}} x_i = \sum_{j=1}^{n} a_{nj} y_j$. As before, if we take $y :\equiv < y_1, y_2, y_3, \cdots >$, then we obtain $(U^*MU)y = (U^*MU)^*x$.

This completes the proof. □

If k is a positive integer, then $(U^*)^k MU^k$ is obtained by deleting the first k rows and the first k columns from M.

Corollary 3.3. *If M is coposinormal, then $(U^*)^k MU^k$ is coposinormal for all positive integers k.*

This corollary provides a useful tool for extending Proposition 3.1.

Proposition 3.4. *Suppose that the lower triangular matrix $M = [a_{ij}]$ is a coposinormal operator on ℓ^2. If $a_{nn} = 0$ for some n, then it must hold true that $a_{in} = 0$ for all i.*

Proof. Since M is coposinormal, Corollary 3.3 tells us that $(U^*)^k MU^k$ is coposinormal for all k. Now apply Proposition 3.1 to $(U^*)^n MU^n$ to justify the assertion that $a_{in} = 0$ for all i. □

We note that the conclusion of Proposition 3.4 does not hold for posinormal operators; to see this, consider the unilateral shift U, which is known to be posinormal but not coposinormal.

Proposition 3.4 will help us settle questions about the posispectrum in the next section of this paper.

3.2. **Concerning heredity and non-posinormality.** The following example involving the generalized Cesàro operators of order one shows that non-posinormality is not inherited in general by lower triangular operators.

Example 3.5. For fixed $k > 0$, let C_k denote the terraced matrix associated with the sequence $\{1/(k+n) : n \geq 0\}$. Then $C_k - (1/k) \cdot I$ is not posinormal for $0 < k < 1/2$ (see [8,Theorem 2.4]), but $U^*(C_k - (1/k) \cdot I)U = C_{k+1} -(1/k) \cdot I$ is posinormal since C_{k+1} is hyponormal when $k > 0$.

In other words, the previous example illustrates that the posinormality of U^*MU does not imply that the triangular operator M is posinormal. However, the following result does hold.

Theorem 3.6. *Suppose $M :\equiv [a_i c_j] \in B(\ell^2)$ is a lower triangular factorable matrix satisfying*

$$\sum_{n=0}^{\infty}(\frac{1}{c_n} - \frac{a_{n+1}}{c_{n+1}a_n})^2 < \infty$$

with $a_n, c_n > 0$ for all n and $\frac{a_n}{c_n} \to 0$. If U^*MU is posinormal, then M is posinormal also.

Proof. Since U^*MU is posinormal, Proposition 1.1 guarantees a $B = [b_{ij}]_{i,j \geq 1} \in B(\ell^2)$ such that $U^*MU = (U^*M^*U)B$. Define $Z = [z_{ij}]$ by

$$z_{ij} = \begin{cases} c_0(\frac{1}{c_i} - \frac{a_{i+1}}{c_{i+1}a_i}) & for \quad j = 0, i \geq 0 \\ -\frac{a_1}{a_0} & for \quad j = 1, i = 0 \\ 0 & for \quad j > 1, i = 0 \\ b_{i-1,j-1} & for \quad i, j \geq 1 \end{cases} .$$

It is straightforward to verify that $Z \in B(\ell^2)$ and $M = M^*Z$, so M is posinormal by Proposition 1.1. □

Corollary 3.7. *If $M :\equiv [a_i c_j] \in B(\ell^2)$ is a lower triangular factorable matrix satisfying*

$$\sum_{n=0}^{\infty}(\frac{1}{c_n} - \frac{a_{n+1}}{c_{n+1}a_n})^2 < \infty$$

with $a_n, c_n > 0$ for all n and $\frac{a_n}{c_n} \to 0$, then M is posinormal if and only if U^*MU is posinormal.

Recall that the *p-Cesàro matrices* are the terraced matrices associated with the sequences

$$a_n = \frac{1}{(n+1)^p} \ , \ p > 0.$$

Example 3.8. Suppose M is a terraced matrix with a leading entry $a_0 > 0$ and U^*MU is a p-Cesàro matrix C_p for $p \geq 1$. Since C_p is known to be posinormal and

$$1 - \frac{(n+1)^p}{(n+2)^p} \leq \frac{p}{n+2}$$

for all n and for $p > 1$ by [2, Theorem 42, 2.15.3, page 40], M must also be posinormal.

4. Posispectral operators

In this section we do not restrict ourselves to triangular operators, and we consider an operator property that, as it has turned out, may not be all that important but is nevertheless interesting. Before the posispectrum is defined, we present some notation that will be employed here.

For $A \in B(H)$, let $\rho(A)$ denote the resolvent set of A,

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is invertible}\},$$

let $\sigma(A)$ denote the spectrum of A,

$$\sigma(A) = \mathbb{C} \setminus \rho(A) = \{\lambda : \lambda I - A \text{ is not invertible}\},$$

let $\pi_0(A)$ denote the point spectrum (i.e., the set of all eigenvalues) of A,

$$\pi_0(A) = \{\lambda : Ker(\lambda I - A) \neq \{0\}\},$$

and consider the following part $\pi_1(A)$ of the point spectrum,

$$\pi_1(A) = \{\lambda \in \pi_0(A) : Ran(\lambda I - A) = H\},$$

which is an open set in $\mathbb{C}$.

Definition 4.1. For $A \in B(H)$, the *posispectrum* $\sigma_{po}(A)$ is the set

$$\sigma_{po}(A) = \{\lambda : \lambda I - A \text{ is not posinormal}\}.$$

The following proposition records a few very basic facts concerning $\sigma_{po}(A)$.

Proposition 4.2. *The following properties of the posispectrum $\sigma_{po}(A)$ hold true.*

(a) *A is dominant if and only if $\sigma_{po}(A) = \emptyset$. In particular, if A is normal or hyponormal, then $\sigma_{po}(A) = \emptyset$.*
(b) *A is posinormal if and only if $0 \notin \sigma_{po}(A)$.*
(c) $\pi_1(A) \subseteq \sigma_{po}(A) \subseteq \sigma(A)$.
(d) *There exist operators for which the posispectrum is topologically large in the sense that it may contain a nonempty open set.*
(e) $\sigma_{po}(A) = \sigma_{po}(A^*)^* \iff$
$$\{Ran(\lambda I - A) \subseteq Ran(\overline{\lambda} I - A^*) \iff Ran(\overline{\lambda} I - A^*) \subseteq Ran(\lambda I - A)\}$$
for all $\lambda \in \mathbb{C}$,
which implies that
$$\{Ran(\lambda I - A) \subseteq Ran(\overline{\lambda} I - A^*) \iff Ran(\overline{\lambda} I - A^*) = Ran(\lambda I - A)\}$$
for all $\lambda \in \mathbb{C}$.
(f) $\sigma_{po}(A) = \sigma_{po}(A^*) = \emptyset \iff Ran(\overline{\lambda} I - A^*) = Ran(\lambda I - A)$ *for every $\lambda \in \mathbb{C}$, which means that A is dominant and codominant.*

Proof. Parts (a) and (b) are clear: Indeed, by the range inclusion criterion we get

$$\begin{aligned}\sigma_{po}(A) &= \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not posinormal}\} \\ &= \{\lambda \in \mathbb{C} : Ran(\lambda I - A) \nsubseteq Ran(\overline{\lambda} I - A^*)\}.\end{aligned}$$

Since A is dominant if and only if $\lambda I - A$ is posinormal for all $\lambda \in \mathbb{C}$, which means by the range inclusion criterion that $Ran(\lambda I - A) \subseteq Ran(\overline{\lambda} I - A^*)$ for all $\lambda \in \mathbb{C}$, it follows that

$$A \text{ is dominant} \iff \sigma_{po}(A) = \emptyset, \quad A \text{ is posinormal} \iff 0 \notin \sigma_{po}(A).$$

In particular, if A is normal or hyponormal, then $\sigma_{po}(A) = \emptyset$.

(c) Since every invertible operator is posinormal, and since every $\lambda \in \mathbb{C}$ for which $\lambda I - A$ is invertible lies in the complement of the spectrum, it follows that

$$\sigma_{po}(A) \subseteq \sigma(A).$$

Also note that

$$\pi_0(A) = \big\{\lambda \in \mathbb{C} : Ker(\lambda I - A) \neq \{0\}\big\} = \big\{\lambda \in \mathbb{C} : Ran(\overline{\lambda} I - A^*)^- \neq H\big\},$$

and that

$$\begin{aligned}\pi_1(A) &= \big\{\lambda \in \pi_0(A) : Ran(\lambda I - A) = H\big\} \\ &= \big\{\lambda \in \mathbb{C} : Ran(\overline{\lambda} I - A^*)^- \subset Ran(\lambda I - A) = H\big\}.\end{aligned}$$

Since $Ran(A)$ is closed if and only if $Ran(A^*)$ is closed for every $A \in B(H)$, the above proper inclusion can be rewritten as

$$\pi_1(A) = \big\{\lambda \in \mathbb{C} : Ran(\overline{\lambda} I - A^*) \subset Ran(\lambda I - A) = H\big\}.$$

Therefore

$$\pi_1(A) \subseteq \sigma_{po}(A),$$

so the proof of (c) is complete.

(d) It is known that $\pi_1(A)$ is always an open subset of $\mathbb{C}$, and it is also known that there are Hilbert space operators A for which $\pi_1(A)$ is nonempty, so (d) follows from (c).

(e) Consider the set

$$\sigma_{po}(A^*)^* = \{\lambda \in \mathbb{C} : Ran(\overline{\lambda} I - A^*) \nsubseteq Ran(\lambda I - A)\big\},$$

and take

$$\rho_{po}(A) = \mathbb{C} \setminus \sigma_{po}(A) = \{\lambda \in \mathbb{C} : Ran(\lambda I - A) \subseteq Ran(\overline{\lambda} I - A^*)\},$$

$$\rho_{po}(A^*)^* = \mathbb{C} \setminus \sigma_{po}(A^*)^* = \{\lambda \in \mathbb{C} : Ran(\overline{\lambda} I - A^*) \subseteq Ran(\lambda I - A)\}.$$

Then we note that

$$\sigma_{po}(A) = \sigma_{po}(A^*)^* \iff \rho_{po}(A) = \rho_{po}(A^*)^* \iff$$

$$\big\{Ran(\lambda I - A) \subseteq Ran(\overline{\lambda} I - A^*) \iff Ran(\overline{\lambda} I - A^*) \subseteq Ran(\lambda I - A)\big\}$$

for all $\lambda \in \mathbb{C}$, as needed.

(f) This follows from part (a), since $\sigma_{po}(A) = \sigma_{po}(A^*) = \emptyset$ if and only if both A and A^* are dominant. □

Definition 4.3. If $A \in B(H)$, then

(a) A is *posispectral* if $\sigma_{po}(A^*) = \sigma_{po}(A)$, and
(b) A is *-*posispectral* if $\sigma_{po}(A^*) = \sigma_{po}(A)^*$.

Proposition 4.4. *Suppose $A \in B(H)$ and $\sigma_{po}(A) \subset \mathbb{R}$. Then A is *-posispectral if and only if A is posispectral.*

Proof. Clear. □

Corollary 4.5. *Suppose that the spectrum of $A \in B(H)$ is real. Then A is *-posispectral if and only if A is posispectral.*

Proof. Since $\sigma_{po}(A)$ is a subset of the spectrum of A, this result is also clear. □

It will sometimes be convenient below to use $A - \lambda$ to denote $A - \lambda I$, where $\lambda \in \mathbb{C}$.

Theorem 4.6. *Let H denote a Hilbert space.*

(a) *The collection of all posispectral operators on H is closed under*
  - *multiplication by real numbers,*
  - *translation by real numbers, and*
  - *involution.*

(b) *The collection of all *-posispectral operators on H is closed under*
  - *multiplication by real numbers,*
  - *translation by complex numbers, and*
  - *involution.*

Proof. (a) We show closure under translation by real numbers and leave the other parts to the reader. Assume that $\sigma_{po}(A^*) = \sigma_{po}(A)$ and $r \in \mathbb{R}$. Then $z \in \sigma_{po}((A-r)^*) = \sigma_{po}(A^* - r) \iff A^* - r - z$ is not posinormal $\iff r + z \in \sigma_{po}(A^*) = \sigma_{po}(A) \iff A - r - z$ is not posinormal $\iff z \in \sigma_{po}(A - r)$. Thus $\sigma_{po}((A-r)^*) = \sigma_{po}(A-r)$.

(b) We show closure under translation by complex numbers, and the rest is left to the reader. Assume that $\sigma_{po}(A^*) = \sigma_{po}(A)^*$ and $\lambda \in \mathbb{C}$. Then $z \in \sigma_{po}((A-\lambda)^*) = \sigma_{po}(A^* - \overline{\lambda}) \iff A^* - \overline{\lambda} - z$ is not posinormal $\iff \overline{\lambda} + z \in \sigma_{po}(A^*) = \sigma_{po}(A)^* \iff \lambda + \overline{z} \in \sigma_{po}(A) \iff A - \lambda - \overline{z}$ is not posinormal $\iff \overline{z} \in \sigma_{po}(A - \lambda) \iff z \in (\sigma_{po}(A-\lambda))^*$. Thus $\sigma_{po}((A-\lambda)^*) = (\sigma_{po}(A-\lambda))^*$.

□

Since γA is posispectral for any $\gamma \geq 0$ whenever A is posispectral, it follows that the collection of all posispectral operators is a cone in $B(H)$. A similar statement is true for *-posispectral operators.

Proposition 4.7. *Let H denote a Hilbert space.*

(a) *If $A \in B(H)$ is a normal operator, then A is both posispectral and *-posispectral.*

(b) *If $A \in B(\ell^2)$ is a dominant lower triangular operator, then A is neither posispectral nor *-posispectral unless A is diagonal.*

Proof. (a) If A is normal, then clearly $\sigma_{po}(A) = \emptyset = \sigma_{po}(A^*)$.

(b) Suppose that the lower triangular operator A is not diagonal. Then there exists an $a_{i_0 j_0} \neq 0$ for some (i_0, j_0) with $i_0 > j_0$. Then $A^* - \overline{a_{j_0 j_0}}$ cannot be posinormal by Proposition 3.4, so $\overline{a_{j_0 j_0}} \in \sigma_{po}(A^*)$. But $\sigma_{po}(A) = \emptyset$ since A is dominant. Thus $\sigma_{po}(A^*) \neq \sigma_{po}(A)$ and $\sigma_{po}(A^*) \neq \sigma_{po}(A)^*$.

□

Corollary 4.8. *If the lower triangular operator $A \in B(\ell^2)$ is posispectral (or *-posispectral) with empty posispectrum, then A must be normal.*

Does the set of all posispectral operators include more than just the normal operators? The next proposition answers that question.

Proposition 4.9. *The p-Cesàro operators C_p on ℓ^2 are posispectral and *-posispectral with $\sigma_{po}(C_p) = \{1/(n+1)^p : n \geq 0\} = \sigma_{po}(C_p^*)$ for $p > 1$.*

Proof. First we note that C_p and C_p^* are posinormal by [8, Theorem 2.3], so $0 \notin \sigma_{po}(C_p)$ and $0 \notin \sigma_{po}(C_p^*)$. We also note that

$$\sigma(C_p) = \{1/(n+1)^p : n \geq 0\} \cup \{0\} = \sigma(C_p^*)$$

by [6]. It follows from Proposition 3.4 that $C_p - (1/(n+1)^p) \cdot I$ is not coposinormal for each n, so $\{1/(n+1)^p : n \geq 0\} \subseteq \sigma_{po}(C_p^*)$. Since

$$\sigma_{po}(C_p^*) \subseteq \sigma(C_p^*) = \{1/(n+1)^p : n \geq 0\} \cup \{0\},$$

it now follows that $\sigma_{po}(C_p^*) = \{1/(n+1)^p : n \geq 0\}$. Next, the eigenvectors computed in [6, Theorem 3] may be used to demonstrate that

$$\ker(C_p - (1/(n+1)^p) \cdot I) \nsubseteq \ker(C_p - (1/(n+1)^p) \cdot I)^*$$

for fixed $n \geq 0$, so it follows from Proposition 1.2 that $C_p - (1/(n+1)^p) \cdot I$ is not posinormal. Since $\sigma_{po}(C_p) \subseteq \sigma(C_p)$, we conclude that

$$\sigma_{po}(C_p) = \{1/(n+1)^p : n \geq 0\} = \sigma_{po}(C_p^*),$$

so the proof is complete. □

It should be emphasized that Proposition 4.9 does not apply to the Cesàro operator C_1, which is known to be a hyponormal operator on ℓ^2.

Remark 4.10. We note that $0 \notin \sigma_{po}(C_p) = \{1/(n+1)^p : n \geq 0\}$ for $p > 1$, and consequently $\rho_{po}(C_p)$ contains 0 but cannot contain any open neighborhood of 0. Thus we see that there exists an operator $A \in B(\ell^2)$ such that $\rho_{po}(A)$ is not open in $\mathbb{C}$.

Proposition 4.9 has supplied us with a collection of nonnormal compact operators that are posispectral. Does the set of all posispectral operators include all the compact operators? The answer can be found in Example 2.4(a), and the next proposition also addresses that question.

Proposition 4.11. *If A is a unilateral weighted shift with positive weights w_n such that $w_n \to 0$, then A is not posispectral.*

Proof. By [7, Proposition 3.3], $\sigma_{po}(A) = \emptyset$ and $\sigma_{po}(A^*) = \{0\}$, so $\sigma_{po}(A) \neq \sigma_{po}(A^*)$. □

We note that Proposition 4.9 can be used to show that iC_p and $C_p + i \cdot I$ are examples of operators that are *-posispectral but not posispectral for $p > 1$. It is also worth noting that, for $p > 1$, $C_p + I$ is an example of a noncompact posispectral operator with nonempty posispectrum.

We close this section with some interesting and natural questions that have not been settled here:

- Does there exist a posispectral operator with posispectrum having positive planar measure? (Note, for example, that $\sigma_{po}(U^*) = \{\lambda : |\lambda| \leq 1\}$, but $\sigma_{po}(U) = \emptyset$ since the unilateral shift U is hyponormal, so U and U^* are not posispectral.)
- Does there exist a posispectral operator that is not *-posispectral?
- Does there exist a posispectral operator that is not of the form $N+C$ where N is normal and C is compact?

5. Supraposinormal factorable matrices

In this section we will find sufficient conditions for a lower triangular factorable matrix $M = [a_i c_j] \in \ell^2$ to be a supraposinormal operator on ℓ^2. However, the initial results apply more generally to operators on abstract Hilbert spaces.

Definition 5.1. If $A \in B(H)$, then A is *supraposinormal* if there exist positive operators P and Q on H such that $AQA^* = A^*PA$, where at least one of P, Q has dense range. The ordered pair (Q, P) is referred to as an *interrupter pair* associated with A.

Proposition 5.2. *If $A \in B(H)$ is supraposinormal, then $KerA \subseteq KerA^*$ or $KerA^* \subseteq KerA$.*

Proof. See [11]. □

Example 5.3. Suppose $a_0 = 0$ and $\{a_n : n \geq 1\}$ is a positive sequence such that $M :\equiv [a_i \cdot 1] \in \ell^2$. Note that $e_0 \in \ker M^*$ but $e_0 \notin \ker M$, (where $\{e_n\}$ is the standard orthonormal basis for ℓ^2). If $x :\equiv e_0 - e_1$, then $x \in \ker M$ but $x \notin \ker M^*$. It follows from Proposition 5.2 that the terraced matrix M is not supraposinormal.

5.1. **Necessary and sufficient conditions for a Hilbert space operator to be supraposinormal.** If $X \in B(H)$ is injective and has dense range, then X is sometimes referred to as a *quasiaffinity.*

Theorem 5.4. *For $A \in B(H)$, the following are equivalent:*

(a) *A is a supraposinormal operator on H.*
(b) *There exist $X, Y \in B(H)$ satisfying $XA = YA^*$ with at least one of X and Y being a quasiaffinity.*

Proof. (a) implies (b): Suppose that $AQA^* = A^*PA$ with Q having dense range. Note that $||\sqrt{Q}A^*f|| = ||\sqrt{P}Af|| \leq ||\sqrt{P}|| \cdot ||Af||$ for all $f \in H$, so it follows from Douglas's Theorem [1, Theorem 1] that there is a $T \in B(H)$ satisfying $A\sqrt{Q} = A^*T$. Take $X = T^*$ and $Y = \sqrt{Q}$. Since Q is one-to-one, so also is $\sqrt{Q} = Y = Y^*$, as needed. A similar argument works if P has dense range.

(b) implies (a): Without loss of generality we assume that X and X^* are both one-to-one. Then the positive operator $P :\equiv X^*X$ is also one-to-one and consequently has dense range. Since $AQA^* = A^*PA$ for $Q :\equiv Y^*Y$, we see that A is supraposinormal with interrupter pair (Q, P). $\square$

5.2. **Sufficient conditions for a factorable matrix to be supraposinormal.**

Theorem 5.5. *If $\{a_n\}$ is a positive, strictly decreasing sequence, $c_n > 0$ for all n, and moreover, the lower triangular factorable matrix $M = [a_i c_j] \in B(\ell^2)$ satisfies any one of the following conditions, then M is supraposinormal.*

(a) *$\{c_n/c_{n+1}\}$ is bounded.*
(b) *$\{c_n\}$ is bounded.*
(c) *$\{c_{n+1}/c_n\}$ is bounded.*
(d) *$\{1/c_n\}$ is bounded.*

Proof. (a) If $X :\equiv I - W_1$ where W_1 is the unilateral weighted shift with weight sequence $\{a_{n+1}/a_n : n \geq 0\}$ and $Y :\equiv I - W_2^*$ where W_2 is the unilateral weighted shift with weight sequence $\{c_n/c_{n+1} : n \geq 0\}$, then $XM = diag\{c_n a_n : n \geq 0\} = YM^*$. Since $\{1/a_n\} \notin \ell^2$, X^* is one-to-one. Clearly X is also one-to-one. By Theorem 5.4, M is supraposinormal.

(b) If $X :\equiv diag\{c_n : n \geq 1\} - W_1$ where W_1 is the unilateral weighted shift with weight sequence $\{c_{n+2}a_{n+1}/a_n : n \geq 0\}$ and $Y :\equiv diag\{c_n : n \geq 1\} - W_2^*$ where W_2 is the unilateral weighted shift with weight sequence $\{c_n : n \geq 0\}$, then $XM = diag\{c_n c_{n+1} a_n : n \geq 0\} = YM^*$. Since $\{1/(c_{n+1}a_n)\} \notin \ell^2$, X^* is one-to-one. Clearly X is also one-to-one.

(c) If $X :\equiv diag\{c_{n+1}/c_n : n \geq 0\} - W$ where W is the unilateral weighted shift with weight sequence $\{(c_{n+2}a_{n+1})/(c_{n+1}a_n) : n \geq 0\}$ and $Y :\equiv diag\{c_{n+1}/c_n : n \geq 0\} - U^*$, then $XM = diag\{c_{n+1}a_n : n \geq 0\} = YM^*$. Since $\{c_n/(c_{n+1}a_n)\} \notin \ell^2$, X^* is one-to-one. Again it is clear that X is also one-to-one.

(d) If $X :\equiv diag\{1/c_n : n \geq 0\} - W_1$ where W_1 is the unilateral weighted shift with the weight sequence $\{a_{n+1}/(c_{n+1}a_n) : n \geq 0\}$ and $Y :\equiv diag\{1/c_n : n \geq 0\} - W_2^*$ where W_2 is the unilateral weighted shift with weight sequence $\{1/c_n : n \geq 1\}$, then $XM = diag\{a_n : n \geq 0\} = YM^*$. Since $\{c_n/a_n\} \notin \ell^2$, X^* is one-to-one. Once again, it is clear that X is also one-to-one.

□

Remark 5.6. Note that if an additional hypothesis – that $\{c_n\} \notin \ell^2$ – is added to Theorem 5.5 (a) through (c), then in all three cases, Y is one-to-one; clearly Y^* is also one-to-one. For part (d), $\{c_n\} \notin \ell^2$ is guaranteed by the original hypothesis, so Y and Y^* are both one-to-one in that case also.

Remark 5.7. For Example 2.6, the (b) part of Theorem 5.5 is satisfied, so the operator in that example is supraposinormal, although it is neither posinormal nor coposinormal.

Corollary 5.8. *If $\{a_n\}$ is a positive, strictly decreasing sequence and $M :\equiv [a_i \cdot 1] \in B(\ell^2)$, then the terraced matrix M is supraposinormal.*

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