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MEASURE THEORETIC GENERALIZATION OF PEČARIĆ, MERCER AND WU-SRIVASTAVA RESULTS

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Abstract. In this paper generalizations of Steffensen's inequality obtained by Pečarić, Mercer and Wu-Srivastava are further extended in a measure theoretic sense. Motivated by Wu and Srivastava's refined and sharpened version of Mercer's result related inequalities for positive Borel measures are obtained.

1. Introduction

Well-known Steffensen's inequality is firstly published in 1918 (see [9]):

Theorem 1.1. *Suppose that f is nonincreasing and g is integrable on $[a,b]$ with $0 \le g \le 1$ and*

$$\lambda = \int_a^b g(t)dt. \tag{1.1}$$

Then we have

$$\int_{b-\lambda}^b f(t)dt \le \int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)dt. \tag{1.2}$$

Since its appearance, Steffensen's inequality has been generalized in many ways. A comprehensive survey on its generalizations and refinements can be found in [8]. We recall the following generalizations of Steffensen's inequality proved by Pečarić in [6]. Here, explicit form (1.1) for λ is replaced with an implicit equation.

Theorem 1.2. *Let h be a positive integrable function on $[a,b]$ and f be an integrable function such that f/h is nondecreasing on $[a,b]$. If g is a*

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real-valued integrable function such that $0 \leq g \leq 1$, *then*

$$\int_a^b f(t)g(t)dt \geq \int_a^{a+\lambda} f(t)dt$$

holds, where λ *is the solution of the equation*

$$\int_a^{a+\lambda} h(t)dt = \int_a^b h(t)g(t)dt.$$

Theorem 1.3. *Let the conditions of Theorem 1.2 be fulfilled. Then*

$$\int_a^b f(t)g(t)dt \leq \int_{b-\lambda}^b f(t)dt$$

where λ *is the solution of the equation*

$$\int_{b-\lambda}^b h(t)dt = \int_a^b h(t)g(t)dt.$$

In the following theorem we recall weaker conditions for function g in Steffensen's inequality given by Milovanović and Pečarić in [5].

Theorem 1.4. *Let* f *and* g *be integrable functions on* $[a, b]$ *and let* $\lambda = \int_a^b g(t)dt$.

(a) *The second inequality in* (1.2) *holds for every nonincreasing function* f *if and only if*

$$\int_a^x g(t)dt \leq x - a \quad \text{and} \quad \int_x^b g(t)dt \geq 0, \quad \text{for every } x \in [a, b].$$

(b) *The first inequality in* (1.2) *holds for every nonincreasing function* f *if and only if*

$$\int_x^b g(t)dt \leq b - x \text{ and } \int_a^x g(t)dt \geq 0, \quad \text{for every } x \in [a, b].$$

The aim of this paper is to extend generalizations of Steffensen's inequality obtained by Pečarić, Mercer, Wu and Srivastava in [4, 6, 10] to analogous results with measures on Borel σ-algebra. Motivated by refined and sharpened versions given by Wu and Srivastava in [10] and by Pečarić, Perušić and Smoljak in [7] we obtain related inequalities for positive measures on Borel σ-algebra. Further, we give weaker conditions for function g in obtained generalizations and refinements. We conclude the paper with applications related to linear functionals defined as the difference between the left-hand and the right-hand side of obtained inequalities.

2. Main results

In the sequel by $\mathcal{B}([a,b])$ we denote Borel σ-algebra on $[a,b]$. We first give measure theoretic version of Theorems 1.2 and 1.3.

Theorem 2.1. *Let μ be a positive finite measure on $\mathcal{B}([a,b])$, let f,g and h be measurable functions on $[a,b]$ such that h is positive, f/h is nonincreasing and $0 \le g \le 1$.*

(a) *If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\int_{[a,a+\lambda]} h(t)d\mu(t) = \int_{[a,b]} h(t)g(t)d\mu(t), \tag{2.1}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \le \int_{[a,a+\lambda]} f(t)d\mu(t). \tag{2.2}$$

(b) *If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\int_{(b-\lambda,b]} h(t)d\mu(t) = \int_{[a,b]} h(t)g(t)d\mu(t), \tag{2.3}$$

then

$$\int_{(b-\lambda,b]} f(t)d\mu(t) \le \int_{[a,b]} f(t)g(t)d\mu(t). \tag{2.4}$$

Proof. Let us prove the (a)-part. Transformation of the difference between the right-hand side and the left-hand side of inequality (2.2) gives

$$\begin{aligned}
&\int_{[a,a+\lambda]} f(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\
&= \int_{[a,a+\lambda]} (1-g(t))f(t)d\mu(t) - \int_{(a+\lambda,b]} f(t)g(t)d\mu(t) \\
&\ge \frac{f(a+\lambda)}{h(a+\lambda)} \int_{[a,a+\lambda]} h(t)(1-g(t))d\mu(t) - \int_{(a+\lambda,b]} f(t)g(t)d\mu(t) \\
&= \frac{f(a+\lambda)}{h(a+\lambda)} \Big(\int_{[a,b]} h(t)g(t)d\mu(t) - \int_{[a,a+\lambda]} h(t)g(t)d\mu(t) \Big) \\
&- \int_{(a+\lambda,b]} f(t)g(t)d\mu(t) = \int_{(a+\lambda,b]} g(t)h(t) \left(\frac{f(a+\lambda)}{h(a+\lambda)} - \frac{f(t)}{h(t)} \right) d\mu(t) \ge 0,
\end{aligned}$$

where we use (2.1). Proof of the (b)-part is similar so we omit the details. □

Theorem 2.2. *Let μ be a positive finite measure on $\mathcal{B}([a,b])$, let f,g and h be measurable functions on $[a,b]$ such that h is positive, f is nonnegative, $0 \le g \le 1$ and f/h is nonincreasing.*

(a) *If there exists* $\lambda \in \mathbb{R}_+$ *such that*

$$\int_{[a,a+\lambda]} h(t)d\mu(t) \geq \int_{[a,b]} h(t)g(t)d\mu(t), \tag{2.5}$$

then (2.2) *holds.*

(b) *If there exists* $\lambda \in \mathbb{R}_+$ *such that*

$$\int_{(b-\lambda,b]} h(t)d\mu(t) \leq \int_{[a,b]} h(t)g(t)d\mu(t),$$

then (2.4) *holds.*

Proof. From the conditions of theorem we have that f/h is nonnegative. Hence, condition (2.5) together with $f(a+\lambda)/h(a+\lambda) \geq 0$ enables us to re-adjust the proof of Theorem 2.1 (a) to prove the (a)-part. Similarly we obtain the (b)-part. □

Remark 2.1. Taking $h \equiv 1$ in Theorems 2.1 and 2.2 we have Steffensen's inequality for positive measures obtained by Jakšetić and Pečarić in [2].

In [4] Mercer proved a generalization of Steffensen's inequality. Wu and Srivastava noted that his generalization is incorrect as stated (see [10]). Pečarić, Perušić and Smoljak proved in [7] that corrected version of Mercer's results follows from Theorems 1.2 and 1.3. Using approach from [7] in the following theorems we obtain corrected version of Mercer's generalization in measure theory settings.

Theorem 2.3. *Let* μ *be a positive finite measure on* $\mathcal{B}([a,b])$, *let* f, g *and* h *be measurable functions on* $[a,b]$ *such that* h *is positive,* f *is nonincreasing and* $0 \leq g \leq h$.

(a) *If there exists* $\lambda \in \mathbb{R}_+$ *such that*

$$\int_{[a,a+\lambda]} h(t)d\mu(t) = \int_{[a,b]} g(t)d\mu(t), \tag{2.6}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda]} f(t)h(t)d\mu(t). \tag{2.7}$$

(b) *If there exists* $\lambda \in \mathbb{R}_+$ *such that*

$$\int_{(b-\lambda,b]} h(t)d\mu(t) = \int_{[a,b]} g(t)d\mu(t), \tag{2.8}$$

then

$$\int_{(b-\lambda,b]} f(t)h(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t). \tag{2.9}$$

Proof. Putting substitutions $g \mapsto g/h$ and $f \mapsto fh$ in Theorem 2.1 we obtain statements of this theorem. □

Theorem 2.4. *Let μ be a positive finite measure on $\mathcal{B}([a,b])$, let f, g and h be measurable functions on $[a,b]$ such that h is positive, $0 \leq g \leq h$, and f is nonnegative and nonincreasing.*

(a) *If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\int_{[a,a+\lambda]} h(t)d\mu(t) \geq \int_{[a,b]} g(t)d\mu(t), \tag{2.10}$$

then (2.7) holds.

(b) *If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\int_{(b-\lambda,b]} h(t)d\mu(t) \leq \int_{[a,b]} g(t)d\mu(t), \tag{2.11}$$

then (2.9) holds.

Proof. Putting substitutions $g \mapsto g/h$ and $f \mapsto fh$ in Theorem 2.2 we obtain statements of this theorem. □

In [4] Mercer also proved a generalization of Steffensen's inequality which is equaivalent to Theorem 1.2. This equivalence was showed in [7]. In the following theorem we obtain its generalization in measure theory settings.

Theorem 2.5. *Let μ be a positive finite measure on $\mathcal{B}([a,b])$, let f, g, h and k be measurable functions on $[a,b]$ such that k is positive, $0 \leq g \leq h$ and f/k is nonincreasing.*

(a) *If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\int_{[a,a+\lambda]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t), \tag{2.12}$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda]} f(t)h(t)d\mu(t). \tag{2.13}$$

(b) *If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\int_{(b-\lambda,b]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t), \tag{2.14}$$

then

$$\int_{(b-\lambda,b]} f(t)h(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t). \tag{2.15}$$

Proof. Take $h \mapsto kh$, $g \mapsto g/h$ and $f \mapsto fh$ in Theorem 2.1. □

Theorem 2.6. *Let μ be a positive finite measure on $\mathcal{B}([a,b])$, let f, g, h and k be measurable functions on $[a,b]$ such that k is positive, f is nonnegative, $0 \le g \le h$ and f/k is nonincreasing.*

(a) *If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\int_{[a,a+\lambda]} h(t)k(t)d\mu(t) \ge \int_{[a,b]} g(t)k(t)d\mu(t), \tag{2.16}$$

then (2.13) holds.

(b) *If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\int_{(b-\lambda,b]} h(t)k(t)d\mu(t) \le \int_{[a,b]} g(t)k(t)d\mu(t), \tag{2.17}$$

then (2.15) holds.

Proof. Take $h \mapsto kh$, $g \mapsto g/h$ and $f \mapsto fh$ in Theorem 2.2. □

Remark 2.2. Taking $k \equiv 1$ in Theorems 2.5 and 2.6 we have results given in Theorems 2.3 and 2.4.

Motivated by corrected and refined version of Mercer's results given by Wu and Srivastava in [10] we obtain the following results for positive measure.

Theorem 2.7. *Let μ be a positive finite measure on $\mathcal{B}([a,b])$, let f, g and h be measurable functions on $[a,b]$ such that $0 \le g \le h$ and f is nonincreasing.*

(a) *If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\int_{[a,a+\lambda]} h(t)d\mu(t) = \int_{[a,b]} g(t)d\mu(t),$$

then

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) &\le \int_{[a,a+\lambda]} \left(f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)]\right) d\mu(t)\\ &\le \int_{[a,a+\lambda]} f(t)h(t)d\mu(t).\end{aligned} \tag{2.18}$$

(b) *If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\int_{(b-\lambda,b]} h(t)d\mu(t) = \int_{[a,b]} g(t)d\mu(t),$$

then

$$\begin{aligned}\int_{(b-\lambda,b]} f(t)h(t)d\mu(t) &\le \int_{(b-\lambda,b]} \left(f(t)h(t) - [f(t) - f(b-\lambda)][h(t) - g(t)]\right) d\mu(t)\\ &\le \int_{[a,b]} f(t)g(t)d\mu(t).\end{aligned} \tag{2.19}$$

Proof. The proof is based on the following identities:

$$\int_{[a,b]} f(t)g(t)d\mu(t) = \int_{[a,a+\lambda]} (f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)])\, d\mu(t) + \int_{(a+\lambda,b]} [f(t) - f(a+\lambda)]g(t)d\mu(t) \tag{2.20}$$

and

$$\int_{[a,b]} f(t)g(t)d\mu(t) = \int_{(b-\lambda,b]} (f(t)h(t) - [f(t) - f(b-\lambda)][h(t) - g(t)])\, d\mu(t) + \int_{[a,b-\lambda]} [f(t) - f(b-\lambda)]g(t)d\mu(t). \tag{2.21}$$

Let us prove the first one. Transformation of the right-hand side of identity (2.20) gives the following

$$\begin{aligned}
&\int_{[a,a+\lambda]} (f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)])\, d\mu(t) \\
&+ \int_{(a+\lambda,b]} [f(t) - f(a+\lambda)]g(t)d\mu(t) = \int_{(a+\lambda,b]} f(t)g(t)d\mu(t) \\
&+ \int_{[a,a+\lambda]} [f(t)g(t) + f(a+\lambda)(h(t) - g(t))]\, d\mu(t) - f(a+\lambda) \int_{(a+\lambda,b]} g(t)d\mu(t) \\
&= \int_{[a,b]} f(t)g(t)d\mu(t) + f(a+\lambda) \left[\int_{[a,a+\lambda]} (h(t) - g(t))d\mu(t) - \int_{(a+\lambda,b]} g(t)d\mu(t) \right] \\
&= \int_{[a,b]} f(t)g(t)d\mu(t) + f(a+\lambda) \left[\int_{[a,a+\lambda]} h(t)d\mu(t) - \int_{[a,b]} g(t)d\mu(t) \right] \\
&= \int_{[a,b]} f(t)g(t)d\mu(t)
\end{aligned}$$

where in the last equality we use a definition of λ i.e.

$$\int_{[a,a+\lambda]} h(t)d\mu(t) = \int_{[a,b]} g(t)d\mu(t).$$

The second identity can be proved in a similar manner.

Since f is nonincreasing on $[a,b]$ we have $f(t) \geq f(a+\lambda)$ for all $t \in [a, a+\lambda]$ and $f(t) \leq f(a+\lambda)$ for all $t \in [a+\lambda, b]$. Then

$$\int_{(a+\lambda,b]} [f(t) - f(a+\lambda)]g(t)d\mu(t) \leq 0$$

and

$$\int_{[a,a+\lambda]} [f(t) - f(a+\lambda)][h(t) - g(t)]d\mu(t) \geq 0.$$

Using (2.20) and above inequalities we obtain

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) &\leq \int_{[a,a+\lambda]} (f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)])\, d\mu(t) \\ &\leq \int_{[a,a+\lambda]} f(t)h(t)d\mu(t).\end{aligned}$$

Similarly, we obtain (2.19) using identity (2.21). □

Theorem 2.8. *Let μ be a positive finite measure on $\mathcal{B}([a,b])$, let f, g and h be measurable functions on $[a,b]$ such that $0 \leq g \leq h$ and f is nonnegative and nonincreasing.*

(a) *If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\int_{[a,a+\lambda]} h(t)d\mu(t) \geq \int_{[a,b]} g(t)d\mu(t),$$

then (2.18) *holds.*

(b) *If there exists $\lambda \in \mathbb{R}_+$ such that*

$$\int_{(b-\lambda,b]} h(t)d\mu(t) \leq \int_{[a,b]} g(t)d\mu(t),$$

then (2.19) *holds.*

Proof. Re-adjusting proof of Theorem 2.7 we have that the proof is based on the following inequalities:

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) \leq &\int_{[a,a+\lambda]} (f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)])\, d\mu(t) \\ &+ \int_{(a+\lambda,b]} [f(t) - f(a+\lambda)]g(t)d\mu(t)\end{aligned}$$

and

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) \geq &\int_{(b-\lambda,b]} (f(t)h(t) - [f(t) - f(b-\lambda)][h(t) - g(t)])\, d\mu(t) \\ &+ \int_{[a,b-\lambda]} [f(t) - f(b-\lambda)]g(t)d\mu(t).\end{aligned}$$

□

In the following theorems we obtain a refined version of results given in Theorems 2.5 and 2.6.

Theorem 2.9. *Let μ be a positive finite measure on $\mathcal{B}([a,b])$, let f, g, h and k be measurable functions on $[a,b]$ such that $0 \leq g \leq h$ and f/k is nonincreasing.*

(a) *If there exists* $\lambda \in \mathbb{R}_+$ *such that* (2.12) *holds, then*

$$\begin{aligned}\int_{[a,b]} f(t)g(t)d\mu(t) \\ &\leq \int_{[a,a+\lambda]} \left(f(t)h(t) - \left[\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)}\right] k(t)[h(t)-g(t)] \right) d\mu(t) \\ &\leq \int_{[a,a+\lambda]} f(t)h(t)d\mu(t). \end{aligned} \tag{2.22}$$

(b) *If there exists* $\lambda \in \mathbb{R}_+$ *such that* (2.14) *holds, then*

$$\begin{aligned}\int_{(b-\lambda,b]} f(t)h(t)d\mu(t) \\ &\leq \int_{(b-\lambda,b]} \left(f(t)h(t) - \left[\frac{f(t)}{k(t)} - \frac{f(b-\lambda)}{k(b-\lambda)}\right] k(t)[h(t)-g(t)] \right) d\mu(t) \\ &\leq \int_{[a,b]} f(t)g(t)d\mu(t). \end{aligned} \tag{2.23}$$

Proof. Take $h \mapsto kh$, $g \mapsto kg$ and $f \mapsto f/k$ in Theorem 2.7. □

Theorem 2.10. *Let* μ *be a positive finite measure on* $\mathcal{B}([a,b])$, *let* f, g, h *and* k *be measurable functions on* $[a,b]$ *such that* f *is nonnegative,* $0 \leq g \leq h$ *and* f/k *is nonincreasing.*

(a) *If there exists* $\lambda \in \mathbb{R}_+$ *such that* (2.16) *holds, then* (2.22) *holds.*
(b) *If there exists* $\lambda \in \mathbb{R}_+$ *such that* (2.17) *holds, then* (2.23) *holds.*

Wu and Srivastava also proved a new sharpened and generalized version of Mercer's result (see [10]). We extend their result to Borel σ-algebra.

Theorem 2.11. *Let* μ *be a positive finite measure on* $\mathcal{B}([a,b])$, *let* f, g, h *and* ψ *be measurable functions on* $[a,b]$ *such that* $0 \leq \psi \leq g \leq h - \psi$ *and* f *is nonincreasing.*

(a) *If there exists* $\lambda \in \mathbb{R}_+$ *such that* (2.6) *holds, then*

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda]} f(t)h(t)d\mu(t) - \int_{[a,b]} |f(t)-f(a+\lambda)|\, \psi(t)d\mu(t). \tag{2.24}$$

(b) *If there exists* $\lambda \in \mathbb{R}_+$ *such that* (2.8) *holds, then*

$$\int_{(b-\lambda,b]} f(t)h(t)d\mu(t) + \int_{[a,b]} |f(t)-f(b-\lambda)|\, \psi(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t). \tag{2.25}$$

Proof. Since f is nonincreasing on $[a,b]$ we have $f(t) \geq f(a+\lambda)$ for all $t \in [a, a+\lambda]$ and $f(t) \leq f(a+\lambda)$ for all $t \in [a+\lambda, b]$. Now using identity (2.20) we get

$$\begin{aligned}
&\int_{[a,a+\lambda]} f(t)h(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\
&= \int_{[a,a+\lambda]} [f(t) - f(a+\lambda)][h(t) - g(t)]d\mu(t) \\
&\qquad\qquad - \int_{(a+\lambda,b]} [f(t) - f(a+\lambda)]g(t)d\mu(t) \\
&\geq \int_{[a,a+\lambda]} |f(t) - f(a+\lambda)|\psi(t)d\mu(t) + \int_{(a+\lambda,b]} |f(a+\lambda) - f(t)|\psi(t)d\mu(t) \\
&= \int_{[a,b]} |f(t) - f(a+\lambda)|\psi(t)d\mu(t)
\end{aligned}$$

and the proof of the first statement is established.

The second statement can be proved in a similar manner using identity (2.21). □

Theorem 2.12. *Let μ be a positive finite measure on $\mathcal{B}([a,b])$, let f, g, h and ψ be measurable functions on $[a,b]$ such that $0 \leq \psi \leq g \leq h - \psi$ and f is nonnegative and nonincreasing.*

(a) *If there exists $\lambda \in \mathbb{R}_+$ such that* (2.10) *holds, then* (2.24) *holds.*
(b) *If there exists $\lambda \in \mathbb{R}_+$ such that* (2.11) *holds, then* (2.25) *holds.*

In the following theorems we obtain sharpening of results given in Theorems 2.5 and 2.6.

Theorem 2.13. *Let μ be a positive finite measure on $\mathcal{B}([a,b])$, let f, g, h, k and ψ be measurable functions on $[a,b]$ such that k is positive, $0 \leq \psi \leq g \leq h - \psi$ and f/k is nonincreasing.*

(a) *If there exists $\lambda \in \mathbb{R}_+$ such that* (2.12) *holds, then*

$$\begin{aligned}
\int_{[a,b]} f(t)g(t)d\mu(t) \leq &\int_{[a,a+\lambda]} f(t)h(t)d\mu(t) \\
&- \int_{[a,b]} \left| \left(\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)} \right) \right| k(t)\psi(t)d\mu(t). \quad (2.26)
\end{aligned}$$

(b) *If there exists* $\lambda \in \mathbb{R}_+$ *such that* (2.14) *holds, then*

$$\int_{(b-\lambda,b]} f(t)h(t)d\mu(t) + \int_{[a,b]} \left| \left(\frac{f(t)}{k(t)} - \frac{f(b-\lambda)}{k(b-\lambda)} \right) \right| k(t)\psi(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t). \quad (2.27)$$

Proof. Take $g \mapsto kg$, $f \mapsto f/k$, $h \mapsto kh$ and $\psi \mapsto k\psi$ in Theorem 2.11. □

Theorem 2.14. *Let* μ *be a positive finite measure on* $\mathcal{B}([a,b])$, *let* f, g, h, k *and* ψ *be measurable functions on* $[a,b]$ *such that* k *is positive,* $0 \leq \psi \leq g \leq h - \psi$, f *is nonnegative and* f/k *is nonincreasing.*

(a) *If there exists* $\lambda \in \mathbb{R}_+$ *such that* (2.16) *holds, then* (2.26) *holds.*
(b) *If there exists* $\lambda \in \mathbb{R}_+$ *such that* (2.17) *holds, then* (2.27) *holds.*

Proof. Take $g \mapsto kg$, $f \mapsto f/k$, $h \mapsto kh$ and $\psi \mapsto k\psi$ in Theorem 2.12. □

3. Weaker conditions

Motivated by weaker conditions given in Theorem 1.4 we obtain weaker conditions for some generalizations and refinements given in the previous section. In the following theorem we obtain weaker conditions for Theorem 2.5 and more general version of analog theorem from [3].

Theorem 3.1. *Let* μ *be a positive finite measure on* $\mathcal{B}([a,b])$, *let* g, h *and* k *be* $\mu-$*integrable functions on* $[a,b]$ *such that* k *is positive and* h *is nonnegative.*

(a) *Let consider* λ *be a positive constant such that* $\int_{[a,a+\lambda]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. *The inequality*

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda]} f(t)h(t)d\mu(t) \quad (3.1)$$

holds for every nonincreasing, right-continuous function $f/k : [a,b] \to \mathbb{R}$ *if and only if*

$$\int_{[a,x)} k(t)g(t)d\mu(t) \leq \int_{[a,x)} k(t)h(t)d\mu(t) \quad \text{and} \quad \int_{[x,b]} k(t)g(t)d\mu(t) \geq 0, \quad (3.2)$$

for every $x \in [a,b]$.

(b) *Let consider* λ *be a positive constant such that* $\int_{(b-\lambda,b]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. *The inequality*

$$\int_{(b-\lambda,b]} f(t)h(t)d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t) \quad (3.3)$$

holds for every nonincreasing, right-continuous function $f/k : [a,b] \to \mathbb{R}$ if and only if

$$\int_{[x,b]} k(t)g(t)d\mu(t) \le \int_{[x,b]} k(t)h(t)d\mu(t) \quad \text{and} \quad \int_{[a,x)} k(t)g(t)d\mu(t) \ge 0,$$

for every $x \in [a,b]$.

Proof.
(a) For the sufficiency part we use the identity

$$\begin{aligned}
&\int_{[a,a+\lambda]} f(t)h(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\
&= \int_{[a,a+\lambda]} \left[\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)}\right] k(t)[h(t)-g(t)]d\mu(t) \\
&+ \int_{(a+\lambda,b]} \left[\frac{f(a+\lambda)}{k(a+\lambda)} - \frac{f(t)}{k(t)}\right] k(t)g(t)d\mu(t). \qquad (3.4)
\end{aligned}$$

We define a new measure ν on $\sigma-$algebra $\mathcal{B}((a,b])$ such that, on an algebra of finite disjoint unions of half open intervals, we set

$$\nu((c,d]) = \frac{f(c)}{k(c)} - \frac{f(d)}{k(d)}, \qquad \text{for } a < c < d \le b,$$

and then we pass to $\mathcal{B}((a,b])$ in a unique way (for details see, for example, [1, p. 21]).

Now, using Fubini, we have

$$\begin{aligned}
&\int_{[a,a+\lambda]} \left[\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)}\right] k(t)[h(t)-g(t)]d\mu(t) \\
&= \int_{[a,a+\lambda]} \left[\int_{(t,a+\lambda]} d\nu(x)\right] k(t)[h(t)-g(t)]d\mu(t) \\
&= \int_{(a,a+\lambda]} \left[\int_{[a,x)} k(t)[h(t)-g(t)]d\mu(t)\right] d\nu(x). \qquad (3.5)
\end{aligned}$$

Similarly,

$$\int_{(a+\lambda,b]} \left[\frac{f(a+\lambda)}{k(a+\lambda)} - \frac{f(t)}{k(t)}\right] k(t)g(t)\mu(t) = \int_{(a+\lambda,b]} \left[\int_{[x,b]} k(t)g(t)d\mu(t)\right] d\nu(x). \qquad (3.6)$$

Now using (3.5) and (3.6) we have that (3.4) is in fact

$$\begin{aligned}\int_{[a,a+\lambda]} f(t)h(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\ = \int_{(a,a+\lambda]} \left[\int_{[a,x)} k(t)(h(t)-g(t))d\mu(t) \right] d\nu(x) \\ + \int_{(a+\lambda,b]} \left[\int_{[x,b]} k(t)g(t)d\mu(t) \right] d\nu(x),\end{aligned}$$

concluding (3.1) under assumptions (3.2).

The previous conditions are also necessary. In fact, if x is any element of $[a,b]$, then let f be the function defined by

$$f(t) = \begin{cases} k(t), & t < x; \\ 0, & t \geq x. \end{cases}$$

We have that f/k is a nonincreasing function. Using inequality (3.1) we obtain

$$\begin{aligned}\int_{[a,x)} k(t)g(t)d\mu(t) = \int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda]} f(t)h(t)d\mu(t) \\ = \begin{cases} \int_{[a,x)} k(t)h(t)d\mu(t), & x \in [a,a+\lambda]; \\ \int_{[a,a+\lambda]} k(t)h(t)d\mu(t), & x \in (a+\lambda,b]. \end{cases}\end{aligned} \tag{3.7}$$

If $x \in (a+\lambda,b]$ then $\int_{[a,x)} k(t)h(t)d\mu(t) \geq \int_{[a,a+\lambda]} k(t)h(t)d\mu(t)$, so from (3.7), we have

$$\int_{[a,x)} k(t)g(t)d\mu(t) \leq \int_{[a,x)} k(t)h(t)d\mu(t), \quad \text{for every } x \in [a,b].$$

Also, if $x \in (a+\lambda,b]$, from (3.7) and definition of λ we have $\int_{[a,x)} k(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda]} k(t)h(t)d\mu(t) = \int_{[a,b]} k(t)g(t)d\mu(t)$, concluding

$$\int_{[x,b]} k(t)g(t)d\mu(t) \geq 0, \quad \text{for every } x \in (a+\lambda,b].$$

Finally, if $x \in [a,a+\lambda]$, then

$$\begin{aligned}&\int_{[x,b]} k(t)g(t)d\mu(t) = \int_{[a,b]} k(t)g(t)d\mu(t) - \int_{[a,x)} k(t)g(t)d\mu(t) \\ &\geq \int_{[a,a+\lambda]} k(t)h(t)d\mu(t) - \int_{[a,x)} k(t)h(t)d\mu(t) = \int_{[x,a+\lambda]} k(t)h(t)d\mu(t) \geq 0,\end{aligned}$$

concluding

$$\int_{[x,b]} k(t)g(t)d\mu(t) \geq 0, \quad \text{for every } x \in [a,b].$$

(b) The proof of this part is similar to the proof of (a)-part so we omit the details. □

Remark 3.1. Taking $h \equiv 1$ in Theorem 3.1 we obtain weaker conditions for Theorem 2.1.

We continue with weaker conditions for Theorem 2.9.

Theorem 3.2. *Let μ be a positive finite measure on $\mathcal{B}([a,b])$, let g, h and k be $\mu-$integrable functions on $[a,b]$ such that k is positive and h is nonnegative.*

(a) *Let consider λ be a positive constant such that $\int_{[a,a+\lambda]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. If conditions* (3.2) *hold for every $x \in [a,b]$, then* (2.22) *holds for every nonincreasing, right-continuous function $f/k : [a,b] \to \mathbb{R}$.*

(b) *Let consider λ be a positive constant such that $\int_{(b-\lambda,b]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. If conditions* (3.3) *hold for every $x \in [a,b]$, then* (2.23) *holds for every nonincreasing, right-continuous function $f/k : [a,b] \to \mathbb{R}$.*

Proof. Let us prove (a)-part. Using identity (3.4) and a measure ν on $\sigma-$algebra $\mathcal{B}((a,b])$ as in the proof of Theorem 3.1 we have

$$\begin{aligned}
&\int_{[a,a+\lambda]} f(t)h(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t) \\
&\qquad\qquad - \int_{[a,a+\lambda]} \left[\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)}\right] k(t)[h(t)-g(t)]d\mu(t) \\
&= \int_{(a+\lambda,b]} \left[\frac{f(a+\lambda)}{k(a+\lambda)} - \frac{f(t)}{k(t)}\right] k(t)g(t)d\mu(t) \\
&= \int_{(a+\lambda,b]} \left[\int_{[x,b]} k(t)g(t)d\mu(t)\right] d\nu(x).
\end{aligned}$$

From here we conclude that the left-hand side inequality in (2.22) holds when conditions (3.2) hold.

Further, we have

$$\begin{aligned}
&\int_{[a,a+\lambda]} \left[\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)}\right] k(t)[h(t)-g(t)]d\mu(t) \\
&\qquad = \int_{(a,a+\lambda]} \left[\int_{[a,x)} k(t)[h(t)-g(t)]d\mu(t)\right] d\nu(x) \geq 0
\end{aligned}$$

if the first condition in (3.2) is satisfied. Hence, the right-hand side inequality in (2.22) holds.

Proof of (b)-part is similar so we omit the details. □

Theorem 3.3. *Let μ be a positive finite measure on $\mathcal{B}([a,b])$, let g,h and k be $\mu-$integrable functions on $[a,b]$ such that k is positive, h is nonnegative and f/k is nonincreasing, right-continuous function.*

(a) *Let consider λ be a positive constant such that $\int_{[a,a+\lambda]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. If*

$$\int_{[x,b]} k(t)g(t)d\mu(t) \geq 0, \qquad \textit{for } x \in (a+\lambda, b],$$

then

$$\int_{[a,b]} f(t)g(t)d\mu(t) \leq \int_{[a,a+\lambda]} f(t)h(t)d\mu(t) - \int_{[a,a+\lambda]} \left[\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)}\right] k(t)[h(t)-g(t)]d\mu(t).$$

If we additionally have

$$\int_{[a,x)} k(t)g(t)d\mu(t) \leq \int_{[a,x)} k(t)h(t)d\mu(t), \quad \textit{for } x \in [a, a+\lambda],$$

then (2.22) *holds.*

(b) *Let consider λ be a positive constant such that $\int_{(b-\lambda,b]} h(t)k(t)d\mu(t) = \int_{[a,b]} g(t)k(t)d\mu(t)$. If*

$$\int_{[a,x)} k(t)g(t)d\mu(t) \geq 0, \qquad \textit{for } x \in [a, b-\lambda],$$

then

$$\int_{(b-\lambda,b]} f(t)h(t)d\mu(t) - \int_{(b-\lambda,b]} \left[\frac{f(t)}{k(t)} - \frac{f(b-\lambda)}{k(b-\lambda)}\right] k(t)[h(t)-g(t)]d\mu(t) \leq \int_{[a,b]} f(t)g(t)d\mu(t).$$

If we additionally have

$$\int_{[x,b]} k(t)g(t)d\mu(t) \leq \int_{[x,b]} k(t)h(t)d\mu(t), \quad \textit{for } x \in (b-\lambda, b],$$

then (2.23) *holds.*

Proof. Similar to the proof of Theorem 3.2. □

4. Concluding remarks and some applications

Motivated by inequalities (2.13), (2.15), (2.22) and (2.23) we can define the following linear functionals:

$$L_1(f) = \int_{[a,b]} f(t)g(t)d\mu(t) - \int_{[a,a+\lambda]} f(t)h(t)d\mu(t), \tag{4.1}$$

$$L_2(f) = \int_{[b-\lambda,b]} f(t)h(t)d\mu(t) - \int_{[a,b]} f(t)g(t)d\mu(t), \tag{4.2}$$

$$\begin{aligned} L_3(f) = &\int_{[a,b]} f(t)g(t)d\mu(t) \\ &- \int_{[a,a+\lambda]} \left(f(t)h(t) - \left[\frac{f(t)}{k(t)} - \frac{f(a+\lambda)}{k(a+\lambda)} \right] k(t)[h(t)-g(t)] \right) d\mu(t), \end{aligned} \tag{4.3}$$

$$\begin{aligned} L_4(f) = &\int_{[b-\lambda,b]} \left(f(t)h(t) - \left[\frac{f(t)}{k(t)} - \frac{f(b-\lambda)}{k(b-\lambda)} \right] k(t)[h(t)-g(t)] \right) d\mu(t) \\ &- \int_{[a,b]} f(t)g(t)d\mu(t). \end{aligned} \tag{4.4}$$

We can consider functionals L_1 and L_2 under assumptions of Theorems 2.5, 2.6 or 3.1. Further, taking $h \equiv 1$ these functionals can be considered under assumptions of Theorems 2.1 and 2.2.
Functionals L_3 and L_4 can be considered under conditions of Theorems 2.9, 2.10, 3.2 or 3.3. Further, taking $k \equiv 1$ these functionals can also be considered under assumptions of Theorems 2.7 and 2.8.

Remark 4.1. $L_i(f) \geq 0$, $i = 1, \ldots, 4$ for all nondecreasing functions f/k.

Remark 4.2. Using similar construction as in [2] we could produce exponentially convex functions and new Cauchy means related to functionals L_i, $i = 1, \ldots, 4$ but here we omit the details.

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