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Homology and orthology with chordal triangles

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Abstract. In a paper "Equilateral Chordal Triangles" Floor van Lamoen considered a configuration when a circle intersects each of the sidelines of a triangle ABC in such a way that three chords not along the sidelines bound the chordal triangle. In this paper we show that if the center of the circle is the circumcenter O of ABC then all eight chordal triangles are homologic to the complementary triangle $A_g B_g C_g$ and that if the center of the circle is the symmedian point K then all four pairs of the related chordal triangles are orthologic with ABC and to each other.

1. Introduction

Recently in the reference [5] Floor van Lamoen studied chordal triangles that one gets from intersections of a circle k with sidelines BC, CA, and AB of the base triangle $T = ABC$. More precisely, let k intersect these sidelines in points B_a, C_a, C_b, A_b, A_c, and B_c. The intersections $A_1 = B_c B_a \cap C_a C_b$, $B_1 = C_a C_b \cap A_b A_c$, $C_1 = A_b A_c \cap B_c B_a$, $A_2 = A_c C_a \cap A_b B_a$, $B_2 = A_b B_a \cap C_b B_c$, $C_2 = C_b B_c \cap A_c C_a$, are the vertices of the two chordal triangles of k with respect to ABC.

The chordal triangles $T_1 = A_1 B_1 C_1$ and $T_2 = A_2 B_2 C_2$ are closely related as was noticed in [5] that their corresponding sides are antiparallel with respect to triangle T and both are homologic to it. In other words, the lines AA_1, BB_1, and CC_1 and the lines AA_2, BB_2, and CC_2 are concurrent. The pairs (ABC, $A_1 B_1 C_1$) and (ABC, $A_2 B_2 C_2$) have the same homology centers (or centers of perspectivity) because it is easy to check that $\{A, A_1, A_2\}$, $\{B, B_1, B_2\}$, and $\{C, C_1, C_2\}$ are triples of collinear points.

Of course, there are three more related pairs of chordal triangles that we denote (T_3, T_4), (T_5, T_6), and (T_7, T_8). Their first vertices A_3, A_4, A_5, A_6, A_7, and A_8 are the following intersections: $A_c C_a \cap B_a C_b$, $B_c B_a \cap C_a A_b$, $A_c B_a \cap C_a A_b$, $B_c C_a \cap B_a C_b$, $B_c C_a \cap B_a A_b$, and $A_c B_a \cap C_a C_b$.

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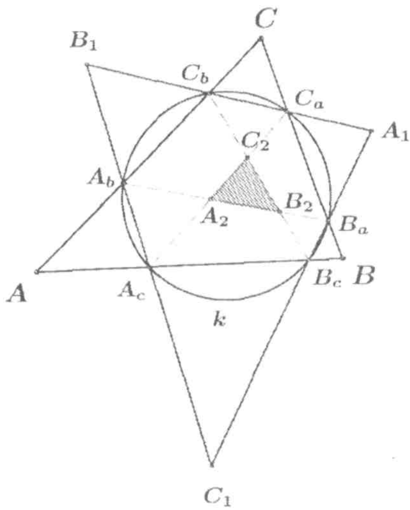

Figure 1.The pair of related chordal triangles $A_1B_1C_1$ and $A_2B_2C_2$ from the intersections of the circle k with the sidelines of the triangle ABC.

The purpose of this paper is to show some new results about chordal triangles. The first is that when the circle k has its center in the circumcenter O of T (i.e., when k is concentric to the circumcircle of T) then the eight chordal triangles $T_1, \dots, T_8$ and the complementary triangle $T_g = A_gB_gC_g$ of T with vertices at the midpoints A_g, B_g, C_g of sides BC, CA, AB are homologic.

The second result shows that when the circle k has its center in the symmedian or Grebe-Lemoine point K of T then the pairs of related chordal triangles (T_1, T_2), (T_3, T_4), (T_5, T_6), and (T_7, T_8) and the triangle T are orthologic. Moreover, the chordal triangles T_{2m-1} and T_{2m} (for $m = 1, 2, 3, 4$) are also orthologic.

Recall that triangles ABC and XYZ are *orthologic* provided the perpendiculars at vertices of ABC onto sides YZ, ZX, and XY of XYZ are concurrent. The point of concurrence of these perpendiculars is denoted by $[ABC, XYZ]$. It is well-known that the relation of orthology for triangles is reflexive and symmetric. Hence, the perpendiculars at vertices of XYZ onto sides BC, CA, and AB of ABC are concurrent at the point $[XYZ, ABC]$.

Both theorems are proved in two ways. The first method uses complex numbers and avoids the appearance of square roots. The second method is with trilinear coordinates. Some steps and verifications are quite demanding if done by hand without the use of a computer.

2. Chordal triangles and the complementary triangle

In this section we shall consider chordal triangles of circles concentric to the circumcircle. In our first theorem we shall also need the complementary triangle T_g of the base triangle T whose vertices are the midpoints of its sides.

Theorem 1. *Let k be a circle whose center is the circumcenter O of ABC. The chordal triangles $T_1, \ldots, T_8$ are homologic to the complementary triangle $A_gB_gC_g$.*

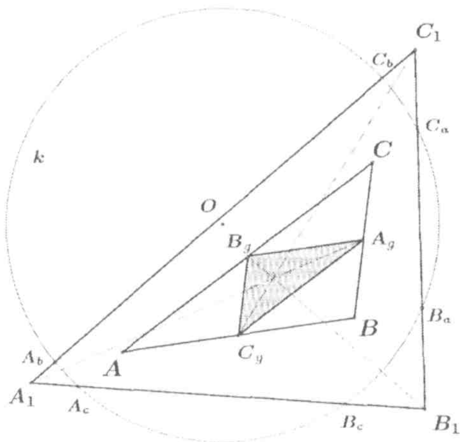

Figure 2. The triangles T_1 and T_g are homologic.

First proof using complex numbers. We shall only prove that the triangles T_1 and T_g are homologic. Our method applies with only minor changes to show that the remaining chordal triangles $T_2, \ldots, T_8$ are also homologic with the complementary triangle T_g.

Without loss of generality, we shall assume that the circle k is the unit circle in the complex plane and that the points A_b, A_c, B_c, B_a, C_a, and C_b correspond to six different complex numbers $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$, and 1 (*one*) for some unimodular complex numbers a, b, c, d, and e, where $\bar{z}$ denotes the complex conjugate $u - i\,v$ of the complex number $z = u + i\,v$ and $i = \sqrt{-1}$ is the imaginary unit.

The vertices of T are intersections of lines B_aC_a, C_bA_b, and A_cB_c while the vertices of T_1 are intersections of lines A_bA_c, B_cB_a, and C_aC_b. Using the techniques of the conjugate coordinate system for plane Euclidean geometry by W. B. Carver (see [1]) it is easy to check that lines C_bA_b and A_cB_c have map equations

$$a\,z + \bar{z} = a + 1 \qquad \text{and} \qquad bc\,z + \bar{z} = b + c,$$

so that their intersection A has the affix $\frac{a-b-c+1}{a-bc}$. In the same way we find that the points B, C, A_1, B_1, and C_1 are $\frac{b+c-d-e}{bc-de}$, $\frac{a-d-e+1}{a-de}$, $\frac{c+d-e-1}{cd-e}$, $\frac{a+b-e-1}{ab-e}$, and $\frac{a+b-c-d}{ab-cd}$. Finally, the midpoints A_g, B_g, and C_g of sides of T are $\frac{B+C}{2}$, $\frac{C+A}{2}$, and $\frac{A+B}{2}$. Let $p = b + c$, $q = bc$, $u = d + e$, and $v = de$.

The assumption of the theorem is that the circumcenter O_T of T agrees with the center of the circle k (i. e., with the origin). We conclude that O_T and $\overline{O_T}$ are both zero so that their numerators N_1 and N_2 must both be equal to zero, where

$$N_1 = N_1(a) = P\,a^2 + Q\,a + R, \qquad N_2 = N_2(a) = U\,a^2 + V\,a + W,$$

$$P = q - v, \qquad Q = v(u-1) - q(p-1), \qquad R = qv(p-u),$$

$$U = qu - pv, \qquad V = v^2(p-q) - q^2(u-v), \qquad W = qv(q-v).$$

On the other hand, we can easily find equations of lines A_1A_g, B_1B_g, and C_1C_g and conclude (since the determinant of the matrix of their coefficients is zero) that the triangles T_1 and T_g are homologic if and only if $M_1\,M_2\,M_3^2/M_4 = 0$ where $M_1 = (a+b-e-1)(e-cd) - (c+d-e-1)(e-ab)$, $M_2 = \sum_{i,j=0}^{3} k_{i,j}\,d^i e^j$, with only the following coefficients $k_{i,j}$ different from zero

$$k_{3,3} = 2((a-b-ab)c+ab),\ k_{3,2} = (2b^2+ab-a)c^2 + ac(b-1)(a-b-1) - 2a^2,$$
$$k_{2,3} = (2b-1)bc^2 + (b+1)bc + a(ab-2a-b^2-b),\ k_{3,1} = ac(b+1)(a-bc),$$
$$k_{2,2} = b(1-2b)c^3 - (2b^3+a-b)c^2 + ac(b^3+ab^2-1)c + a^2(2+2a-b^2-ab),$$
$$k_{1,3} = b(a+c)(a-bc),\ k_{2,0} = abc(a+c)(bc-a),\ k_{0,0} = 2a^2b^2c^2(b+c-a-1),$$
$$k_{1,2} = (c-ab)(b^2c(c-1)+a(a-b)),\ k_{2,1} = (c-ab)(b(a-b)c^2+a^2(c-1)),$$
$$k_{1,1} = bc^3(2b^2(a+1)-a-b) - ac^2(ab^3-a-b) + a^2c(ab^2-b^3-2a) + a^3b(b-2),$$
$$k_{0,2} = abc(b+1)(bc-a),\ k_{1,0} = abc[(b-a-ab-2b^2)c^2 + a^2(b+1)c - a^2(b-2)],$$
$$k_{0,1} = abc[a(2a+b-b^2)+(b-1)(ab+a-b)c-2b^2c^2],$$

$$M_3 = (a-b-c+1)(a-de) + (d+e-a-1)(a-bc), \text{ and}$$
$$M_4 = 8(a-bc)^2(a-de)^2(bc-de)^2(ab-cd)(ab-e)(cd-e).$$

In order to eliminate a^2 from equations $N_1 = 0$ and $N_2 = 0$, let us multiply the first by U and the second by P and subtract these products. We get

$$(QU - PV)\,a - bcde(b-d)(b-e)(c-d)(c-e) = 0.$$

Notice that $QU-PV \neq 0$ because the product $bcde(b-d)(b-e)(c-d)(c-e) \neq 0$ for six different points A_c, B_c, B_a, C_a, C_b, and A_b on the unit circle.

The statement of the theorem follows from the fact that substitutions of the value $a = a_0 = \frac{bcde(b-d)(b-e)(c-d)(c-e)}{QU-PV}$ into polynomials N_1 and M_2 have a common nontrivial factor (which is a cubic polynomial in b with coefficients polynomials in c, d, and e). More precisely, $N_1(a_0) = \frac{bcde(b-1)(c-1)(d-1)(e-1)(bc-de)^3 P_3}{(QU-PV)^2}$, where P_3 denotes the following cubic polynomial in b :

$$(p-u)\,(q+v)\,(pv-qu) - (q-v)\,[\,qu\,(v-1) - pv\,(q-1)\,]\,.$$

On the other hand, when we put $a = a_0$ into the polynomial M_2 we shall obtain

$$\frac{bcde(1-b)(c-1)(d-1)(e-1)(bc-de)^2\, P_3\, Q_3}{(QU-PV)^3},$$

where Q_3 is a polynomial of order 5 with respect to b. □

Second proof using trilinear coordinates. Recall that the *actual trilinear coordinates* of a point P with respect to the triangle ABC are signed distances f, g, and h of P from the lines BC, CA, and AB. We shall regard P as lying on the positive side of BC if P lies on the same side of BC as A. Similarly, we shall regard P as lying on the positive side of CA if it lies on the same side of CA as B, and similarly with regard to the side AB. Ordered triples $x : y : z$ of real numbers proportional to $(f,\, g,\, h)$ (that is such that $x = \mu f$, $y = \mu g$, and $z = \mu h$, for some real number μ different from zero) are called *trilinear coordinates* of P. The advantage of their use is that a high degree of symmetry is present so that it usually suffices to describe part of the information and the rest is self evident. For example, $X_2(\frac{1}{a})$ or $G(\frac{1}{a})$ say that the centroid has has trilinears $\frac{1}{a} : \frac{1}{b} : \frac{1}{c}$, where a, b, c are lengths of sides of ABC. The expressions in terms of sides a, b, and c can be shortened using the following notation.

$$d_a = b-c,\;\; d_b = c-a,\;\; d_c = a-b,\;\; z_a = b+c,\;\; z_b = c+a,\;\; z_c = a+b,$$
$$s = a+b+c, \quad s_a = b+c-a, \quad s_b = c+a-b, \quad s_c = a+b-c,$$
$$m = abc, \quad m_a = bc, \quad m_b = ca, \quad m_c = ab, \quad S = \tfrac{1}{4}\sqrt{s s_a s_b s_c},$$

For an integer n and any symbol Z, let $s_n = a^n + b^n + c^n$ and $d_{na} = b^n - c^n$ and $m(Z)$ and $s_a(Z)$ mean $Z_a Z_b Z_b$ and $Z_b + Z_c - Z_a$ and similarly for other cases.

In order to achieve even greater economy in our presentation, we shall describe coordinates or equations of only one object from triples of related objects and use cyclic permutations φ and ψ to obtain the rest. For example, the first vertex A_g of the complementary triangle $A_gB_gC_g$ of ABC has trilinears $0 : c : b$. Then the trilinears of B_g and C_g need not be described because they are easily figured out and memorized by relations $B_g = \varphi(A_g)$ and $C_g = \psi(A_g)$. One must remember always that transformations φ and ψ are not only permutations of letters but also of positions, i. e., $\varphi(a, b, c, 1, 2, 3 \to b, c, a, 2, 3, 1)$ and $\psi(a, b, c, 1, 2, 3 \to c, a, b, 3, 1, 2)$. Therefore, the trilinears of B_g and C_g are $c : 0 : a$ and $b : a : 0$.

The equation of a circle with the center at the circumcenter $O(as_{2a})$ of ABC and with the radius T is $m[\sum a(a^2 m_a x^2 + w_a yz)] - 16S^2T^2(\sum ax)^2 = 0$, with $w_a = a^4 - 2z_{2a}a^2 + z_{4a}$. This circle intersects the line BC in the points

$$B_a(0 : c(v_a - aK_a) : 2bW) \qquad \text{and} \qquad C_a(0 : c(v_a + aK_a) : 2bW),$$

with $v_a = a^2 w_a - 32S^2T^2$, $K_a = 4S\sqrt{64S^2T^2 - a^2 s_{2a}^2}$, and $W = 16S^2T^2 - m^2$. Moreover, $C_b = \varphi(B_a)$, $A_b = \varphi(C_a)$, $A_c = \psi(B_a)$ and $B_c = \psi(C_a)$. It follows that the first vertex A_1 of the first chordal triangle T_1 has the trilinear coordinates

$$8m_a K_a W^2(v_c + cK_c) : c(v_a + aK_a)Z_1(c) : -2bWZ_2(b),$$

where

$$Z_1(c) = m(K)m + m(v) - s_c(v\,m(v)\,m + aK\,m(v)) + 8W^3,$$

and

$$Z_2(b) = m(K)m - m(v) + s_b(v\,m(K)\,m - aK\,m(v)) - 8W^3.$$

Of course, $B_1 = \varphi(A_1)$ and $C_1 = \psi(A_1)$. The condition for the lines A_gA_1, B_gB_1, and C_gC_1 to be concurrent is $m^5 F_1 F_2 = 0$, where F_1 is

$$64W^6 - 32s(v + aK)W^5 + 16s(m(K)m + aK\,z(v) + m(v))W^4 + 4m(aK + v)\\ s(aK - v)W^2 + 2m(aK + v)s(m(K)m - aK\,z(v) + m(v))W + m(a^2K^2 - v^2),$$

and F_2 is $64W^6 - 16(2m(aK) - m(v) + s(v\,m(K)\,m))W^3 - m(a^2K^2 - v^2)$. If we substitute the above values into F_1 we shall discover that it is indeed equal to zero so that the triangles T_1 and T_g are homologic. □

3. Circles with centers at the symmedian point K

Recall that the symmedian or the Grebe–Lemoine point K of the triangle T is the intersection of the symmedian lines that are isogonal to median lines which join vertices with midpoints of opposite sides. Hence, it is the isogonal conjugate of the centroid G and in the Kimberling list (see [3] and [4]) it is X_6.

Theorem 2. *Let k be a circle whose center is the symmedian point K of T. Then for every pair $(U, V) \in \{(T_1, T_2), (T_3, T_4), (T_5, T_6), (T_7, T_8)\}$ of the related chordal triangles, the triangles U and V are orthologic to the triangle T and to each other.*

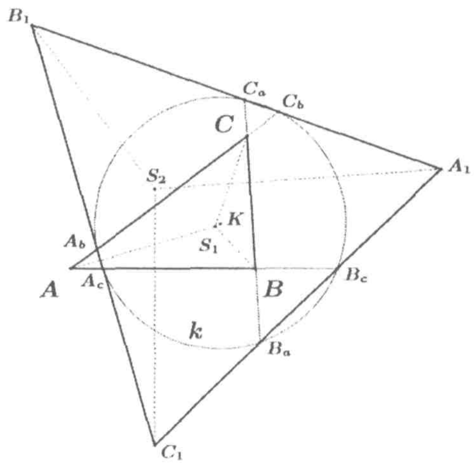

Figure 3. The first chordal triangle $T_1 = A_1B_1C_1$ of the circle k with the center at the symmedian point K is orthologic with ABC and the points S_1 and S_2 are their orthology centers.

First proof using complex numbers. We shall only prove that the triangles T_1 and T_2 are orthologic to the base triangle T and to each other. Our method applies with only minor changes to show that the remaining pairs of related chordal triangles have analogous properties.

Without loss of generality, we can retain the assumptions about the circle k and the points B_a, C_a, C_b, A_b, A_c, and B_c that we made in the proof of Theorem 1.

It follows that the vertices of triangles T and T_1 are the ones found earlier. In the same way we also find that the vertices of T_2 are $\frac{a+d-b-e}{ad-be}$, $\frac{a+d-c-1}{ad-c}$ and $\frac{b+e-c-1}{be-c}$.

This time the assumption of the theorem is that the symmedian point K_T of T agrees with the center of the circle k (i. e., with the origin). We conclude that K_T and $\overline{K_T}$ are both zero so that their numerators K_1 and K_2 must both be equal to zero, where $K_1 = K_1(a) = P\,a^2 + Q\,a + R$,

$K_2 = K_2(a) = U\,a^2 - V\,a + W,$

$$P = q + v, \quad Q = p(q - 2v) - (2u + 2v - 1)q + v(u + 1), \quad R = qv(p + u - 2),$$
$$U = pv + qu - 2qv,\ V = pv(2q - v) - (u + v)q^2 + qv(2u - v + 2),\ W = qv(q + v).$$

On the other hand, using the Theorem 9 in [2], we conclude that the triangles T_1 and T are orthologic if and only if $M_1\,L_2\,M_3/L_4 = 0$, where M_1 and M_3 have been defined earlier,

$$L_4 = (a - bc)(a - de)(bc - de)(ab - cd)(ab - e)(cd - e)$$

and

$$L_2 = bd(c + e)(a^2 + ce) - ace(d + 1)(b^2 + d) + ab(c - de)(cd - e).$$

The method of proof of Theorem 1 in which we eliminate a^2 from equations $K_1 = 0$ and $K_2 = 0$ could be applied here except that a rather difficult special case must be considered separately. This is the reason for the following alternative method.

Let us substitute $b = \frac{p-K}{2}$ and $c = \frac{p+K}{2}$ with $K = \sqrt{p^2 - 4q}$ (i. e., $p = b + c$ and $q = bc$) into K_1, K_2, $2L_2$, and M_3 to obtain

$$K_1^* = [(a + v)q - 2av]\,p + [a^2 + (1 - 2(u + v))\,a + v(u - 2)]\,q + av(a + u + 1),$$
$$K_2^* = av(a + v - 2q)p + q[(v + a(u + v))q + (u - 2v)a^2 + av(v - 2(u + 1))a + v^2],$$
$$L_2^* = [(v + a(u + v))\,q - av(a + u + 1)]\,K +$$
$$(v + a(d - e - v))\,pq + av(a - d + e - 1)\,p + 2(ad - e)(a - v)q,$$
$$M_3^* = (v - a)\,p + (a - u + 1)\,q + (u - v)\,a - v.$$

Suppose first that $(a + v)q - 2av \neq 0$. Then we can solve $K_1^* = 0$ for p to get

$$p = \frac{[a^2 + (1 - 2(u + v))\,a + v(u - 2)]\,q + av(a + u + 1)}{2av - (a + v)\,q}.$$

Substituting this value into K_2^* we obtain $K_2^{**} = \frac{K_3\,K_4}{2av - (a+v)q}$, where

$$K_3 = (v + a(u + v))\,q - av(a + u + 1)$$

and $K_4 = (a + v)(q^2 + av) + (a^2 - 6\,av + v^2)\,q$. Since $K_2 = 0$, we infer that $K_3 = 0$ or $K_4 = 0$. For $K_3 = 0$, we have either $v + a(u + v) = 0$ and $av(a + u + 1) = 0$ (which is possible only for $d = -a$ and $e = -1$ or for $d = -1$ and $e = -a$ when the point C is not well-defined) or $v + a(u + v) \neq 0$ (when $q = q_1 = \frac{av(a+u+1)}{v+a(u+v)}$). The corresponding value of p is $p = p_1 = -(a + u + 1)$. The statement of the theorem is now consequence of the fact that substituting $p = p_1$ and $q = q_1$ into L_2^* gives the value zero.

For $K_4 = 0$, we have either $a + v = 0$ and $a^2 - 6\,av + v^2 = 0$ (which holds only for $d = 0$ or for $e = 0$ and therefore is really impossible) or $a + v \neq 0$ when

$$q = q_2 = \frac{6\,av - a^2 - v^2 - \sqrt{(a-v)^2(a^2 - 14av + v^2)}}{2(a+v)}$$

or

$$q = q_3 = \frac{6\,av - a^2 - v^2 + \sqrt{(a-v)^2(a^2 - 14av + v^2)}}{2(a+v)}.$$

Then we find the corresponding values p_2 and p_3 of p. This time the statement of the theorem is the consequence of the fact that substituting both $p = p_2$ and $q = q_2$ and $p = p_3$ and $q = q_3$ into M_3^* gives the value zero.

Thus it only remains to discuss the case when $(a + v)q - 2av = 0$. From $K_1^* = 0$ we see that

$$[a^2 + (1 - 2(u+v))\,a + v(u-2)]\,q + av(a + u + 1) = 0. \qquad (*)$$

Notice that $a + v \neq 0$ because otherwise either a, d, or e must be zero which is not possible for unimodular complex numbers. Hence, $q = \frac{2av}{a+v}$ and (*) is

$$\frac{3av(a-v)(a-u+1)}{a+v} = 0.$$

Therefore, either $a = v$ (which is ruled out because the affix of the vertex C has $a - v$ in the denominator) or $a = u - 1$. This complex number is unimodular only when either $d = 1$, $e = 1$, or $d = -e$. Since we made the assumption that A_c, B_c, B_a, C_a, C_b, and A_b are six different points the first two cases do not happen. For $d = -e$ we have $a = -1$ and K_2 is the product $-e^2\,p(2\,q + e^2 + 1)$. From this we infer that either $b = -c$ or $2\,q + e^2 + 1 = 0$. In the first case (i. e., when $a = -1$, $d = -e$ and $b = -c$) both L_2 and M_3 are zero so that the statement of the theorem holds. The proof will be completed once we show that $2\,q + e^2 + 1 = 0$ does not happen in our configuration. Then $2\,q + e^2 + 1 = 0$ and $\frac{2}{q} + \frac{1}{e^2} + 1 = 0$ (by taking complex conjugate of the first equation). Eliminating q we conclude that $e = 1$ or $e = -1$. However, both these values are forbidden because the affix of the point B_c is 1 and $a = -1$ (the affix of the point B_a is -1). □

Second proof using trilinear coordinates. The equation of a circle with the center at the symmedian point $K(a)$ of ABC and with the radius T is

$$m\left(\sum m(a^2 + 2s_{2a})\,x^2 + w_a\,yz\right) - s_2^2\,T^2\left(\sum a\,x\right)^2 = 0,$$

with $w_a = a(s_{4a} - 4\,m_{2a})$. This circle intersects the line BC in the points

$$B_a(0 : c(v_a - a\,K_a) : 2\,b\,u_b) \qquad \text{and} \qquad C_a(0 : c(v_a + a\,K_a) : 2\,b\,u_b),$$

(here u_b, K_a, v_a are $s_2^2\,T^2 - m_{2b}(b^2 + 2\,s_{2b})$, $2\,s_2\sqrt{s_2^2\,T^2 - 4\,a^2\,S^2}$, $a\,w_a - 2\,s_2^2\,T^2$). Moreover, $C_b = \varphi(B_a)$, $A_b = \varphi(C_a)$, $A_c = \psi(B_a)$, and $B_c = \psi(C_a)$. It follows that the first vertex A_1 of the first chordal triangle T_1 has the trilinear coordinates

$$8\,m_a\,K_a\,u_b\,u_c(v_c + cK_c) : c(v_a + aK_a)Z_1(c) : -2\,b\,u_b\,Z_2(b),$$

where

$$Z_1(c) = m(aK) + m(v) + m(2\,u) - s_c(v\,m(K)\,m + a\,K\,m(v)),$$

and

$$Z_2(b) = m(aK) - m(v) - m(2\,u) + s_b(v\,m(K)\,m - a\,K\,m(v)).$$

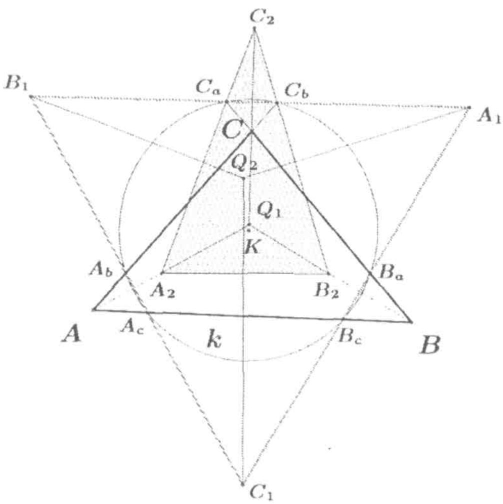

Figure 4. The chordal triangles $A_1B_1C_1$ and $A_2B_2C_2$ of the circle k with the center at the symmedian point k are orthologic and the points Q_1 and Q_2 are their orthology centers.

Of course, $B_1 = \varphi(A_1)$ and $C_1 = \psi(A_1)$. The left hand side of the condition for the triangles ABC and $A_1B_1C_1$ to be orthologic contains as a factor the following expression

$$m(4\,u^2) + 32\left(\sum s_{2b}(b\,K_b + v_b)u_a^2\,u_b^2\,u_c\right) + 16\left(\sum (b(2\,a^2 + s_{2b})K_b - v_b\,s_{2c})(a\,K_a + v_a)u_a^2\,u_b\,u_c\right)$$

$$+\,32\left(\sum d_{2a}^2(b\,c\,v_a\,K_b\,K_c - a\,v_b\,v_c\,K_a)\right)u_a\,u_b\,u_c$$
$$+\,4\left(\sum (b^2\,K_b^2 - v_b^2)(a\,K_a + v_a)(c(3\,a^2 - c^2 + b^2)K_c - v_c\,s_{2b})u_a\,u_b\right)$$
$$-\,2\left(\sum s_{2c}(b^2\,K_b^2 - v_b^2)(a^2\,K_a^2 - v_a^2)(c\,K_c + v_c)\,u_a\right) + s_2\,m(a^2\,K^2 - v^2).$$

If we substitute the above values into this expression we shall discover that it is equal to zero so that the triangles T_1 and T are orthologic (because then the condition is true). In an analogous way we can prove that the triangles T and T_2 and also that the triangles T_1 and T_2 are orthologic.

This could also be seen as follows. By [5], the sides of T_2 are antiparallel to the sides of T_1. So the lines from A perpendicular to B_1C_1 and B_2C_2 are isogonal conjugates, etc. So the orthology of T and T_2 follows immediately. Noting that the alternative chordal triangle of T when T_1 is used as the reference triangle is again T_2 the orthology of T_1 and T_2 follows in the same way.

4. Concluding remarks

I am thankful to the referee for the following remarks.

(1) All other chordal triangles $T_3, \ldots, T_8$ result from T_1, T_2 by interchanging C_a, B_a or A_c, B_c or A_b, C_b. From this it is obvious that the proofs of Theorems 1 and 2 can be restricted to the comparisons of T_1 and T_g or T_1, T_2 and T, respectively.

(2) One might wonder if there are some converses of Theorems 1 and 2 that characterize the circumcenter O or the symmedian point K. More precisely, some experiments in a sketchpad suggest that the following statement is true. However, I have no proof for it.

Let m and n be two different circles with a common center S. If triangles $T_1(m)$ (the T_1 triangle with respect to the circle m) and $T_1(n)$ (the T_1 triangle with respect to the circle n) are both homologic to the triangle T_g then S is the circumcenter O of T.

(3) One might wonder if in Theorems 1 and 2 we can replace circles with conics. Whenever the six points $A_b, \ldots, C_b$ are located on a conic, then due to standard results of Projective Geometry (Pappus-Pascal Theorem), the triangles T, T_1 and T_2 are perspective with respect to the same center. So, this (projective) part of the theory would be standard. However, the addressed problems are of Euclidean nature and therefore much more difficult.

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Homologija i ortologija s tetivnim trokutima

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Sadržaj

U svom radu "Jednakostranični tetivni trokuti" Floor van Lamoen je razmatrao konfiguraciju u kojoj kružnica presjeca svaki od pravaca stranica trokuta ABC na takav način da tri tetive, ne uz pravce stranica, odredjuju tzv. tetivni trokut.

U ovom radu pokazujemo da ako se središte kružnice podudara sa središtem O kružnice opisane trokutu ABC, onda je svih osam tetivnih trokuta homologno s komplementarnim trokutom $A_g B_g C_g$ i ako je središte kružnice baš simedijalna (ili Grebe-Lemoineova) točka K trokuta ABC onda su sva četiri para odgovarajućih tetivnih trokuta ortologna trokutu ABC i izmedju sebe.