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New method for the study of solutions of the difference equation $u_{n+2} = \dfrac{u_{n+1} + a}{u_n + b}$

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Abstract. We prove two general results of convergence for solutions of difference equations of the type $x_{n+2} = x_{n+1} f(x_{n+1}, x_n)$. The first one is a particular case of a known result (see [9], [12]), but the method is different and allows us to prove the second one, which is new.

We use them for studying solutions of the difference equation $u_{n+2} = \frac{u_{n+1}+a}{u_n+b}$. Our method gives again a result of [6]: for $(a, b) \in \mathbb{R}_*^{+2} \setminus D$ the sequence (u_n) always converges, where D is the region $D = \{0 < b < 1\} \cap \left\{a > \frac{2b(1+b^2)}{(1-b)^2}\right\}$.

The case $b = 0$ gives the so-called Lyness' difference equation. We give some results about Lyness' cubic related to such a sequence.

1. Introduction

We study the behavior of the solutions of the difference equation

$$u_{n+2} = \frac{u_{n+1} + a}{u_n + b}, \quad a \geq 0, \quad b \geq 0, \tag{1}$$

with initial conditions $u_0 > 0$ and $u_1 > 0$. This sequence is a particular case of rational sequences studied by some authors (see [1], [7], [9], [10], [12], [13]), and some previous results are known about this sequence, see in particular [6] and [11], but some of our methods are different and sometimes simpler. Kocic and Ladas conjectured in [9] that the sequence converges for $a, b > 0$, but we prove only some part of this conjecture, as in [6]: it is true for $(a, b) \in \mathbb{R}_*^{+2} \setminus D$, where

$$D = \{0 < b < 1\} \cap \left\{a > \frac{2b(1 + b^2)}{(1 - b)^2}\right\}. \tag{2}$$

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For $a = 0$, the case $0 < b < 1$, already known, follows also from our general results, the cases $b = 1$ and $b > 1$ being solved by particular methods.

For $b = 0$ this sequence is the "Lyness' sequence". Some authors have studied this sequence, and in part III we synthesize the already known results on Lyness' sequence and give some preliminary results about the "Lyness' cubic". We solved in another paper the conjecture about Lyness' difference equation given in [2], and gave in fact the complete study of Lyness' sequence (determination of periods, density of periodic points and of points with dense orbits in "Lyness' cubic", sensitivity to initial conditions, rational periodic solutions..., see [3]), and apply the same methods for the study of other algebraic difference equations of the form $u_{n+2}\,u_n = \psi(u_{n+1})$ (see [4]).

We start with general results about convergence of the solutions of the difference equation

$$x_{n+2} = x_{n+1} f(x_{n+1}, x_n), \tag{3}$$

which was previously studied in [9]. As in this paper, the key to the method was to use the property of partial monotonicity of the function f, but we do not study the semi-cycles of the solutions: we iterate some inequalities about the sequence (x_n) or lim inf and lim sup of it.

2. Two result of convergence for sequences $x_{n+2} = x_{n+1} f(x_{n+1}, x_n)$ in $\mathbb{R}_*^+$

Let $f : \mathbb{R}_*^{+2} \to \mathbb{R}_*^+$ be continuous. We define the maps :

$f_u : v \mapsto f(u,v)$; $f_v : u \mapsto f(u,v)$; $\tilde{f} : x \mapsto f(x,x)$; $h(u,v) := uf(u,v)$;
$g(u) := h(u,u) = uf(u,u)$; $p_y(x) := h(h(x,y),x)$; $\phi(y) := p_y(y)$.

For the function f, we list some possible properties :

(0) the equation $f(t,t) = 1$ has a solution ℓ;
(0') the equation $f(t,t) = 1$ has a unique solution ℓ;
(1) the functions f_u and f_v are nonincreasing;
(2) the function $\tilde{f}$ is decreasing;
(3) the function g is increasing;
(4) the function $u \to h(u,v)$ is nondecreasing for every v;
(4') the function $v \to h(u,v)$ is majorized for every u;
(5) the functions p_y are nonincreasing;
(6) the function ϕ is nonincreasing;
(6') the function ϕ is majorized;
(7) the function ϕ has no 2-periodic point;
(8) if $x \neq \ell$, then $\Big(\ell f(\ell f(x,x), \ell f(x,x)) - x\Big)(\ell - x) > 0$.

We introduce now the sequence (x_n) defined by

$$x_{n+2} = x_{n+1} f(x_{n+1}, x_n), \quad x_0 > 0, \quad x_1 > 0. \tag{4}$$

We list three more possible properties; the first one concerns the map f, the second and the third ones the sequence (x_n):

(M) f satisfies (1) and $\tilde{f}(0) < +\infty$;
(H$_-$) $\exists m > 0$ such that $\forall n \; x_n \geq m$;
(H$^+$) $\exists M < +\infty$ such that $\forall n \; x_n \leq M$.

Our first result is the following:

Theorem 1. *If in addition to the properties* (0'), (1) *and* (3) *one of the conditions* (M) *or* (H$_+$) *or* (H$_-$) *is satisfied, then the sequence* (x_n) *converges to* ℓ.

This result is weaker than the Theorem 2.1.1 of [9], which supposes none of the three properties (M), (H$_-$), (H$^+$), and which asserts also the asymptotic stability of the fixed point ℓ. But our method of proof is different and gives also the second result that we shall use for the study of the difference equation (1).

Theorem 2. *If the properties* (0), (1), (2), (4), (4'), (5), (6) *and* (7) *are satisfied along with one of the conditions* (H$_-$) *or* (H$^+$), *then the sequence* (x_n) *converges vers* ℓ.

This result is different from Theorem 2.3.2 of [9], but it is of an analogous type. Moreover our Theorem 2 has to be compared with Theorem 2.1 of [6]: it is different, but the applications of these two results to the equation $u_{n+2} = \frac{u_{n+1}+a}{u_n+b}$, which do not work in the same manner, lead to the same calculations on this sequence, and so will give exactly the same domain of convergence.

Before the proofs, we state some easy relations between properties (0) to (8).

Lemma 1. *We have the following implications:*
$\{(0)$ *and* $(2)\} \Rightarrow (0')$; $\{(0), (1)$ *and* $(3)\} \Rightarrow (8)$;
$\{(1), (4), (4')$ *and* $(6)\} \Rightarrow (6')$.

Proof of the Lemma 1. The first implication is obvious, the second one is Lemma 2.1.1 of [9]. So we prove only the third.

By (1) and (4'), $h(u,0) := \lim_{v \to 0} h(u,v) < +\infty$, and $u \mapsto h(u,0)$ is positive, and nondecreasing by (4). Because ϕ is nonincreasing by (6), it is sufficient to prove that ϕ is majorized on $]0,1]$.

One has $\phi(x) = xf(x,x)f(xf(x,x),x)$. Let $u := xf(x,x)$; then $u \leq h(1,x) = f(1,x)$ if $x \leq 1$ by (4), and $h(1,x) \leq h(1,0)$. Thus $\phi(x) = uf(u,x) \leq h(u,0) \leq h(h(1,0),0) < +\infty$. □

Proof of the Theorem 1.

(a) We prove here the following assertion (A):

(A) for every $N \geq 0$ and for every $\gamma \geq 0$, one has: if $f(\gamma,\gamma)$ is defined and $f(\gamma,\gamma) \geq 1$, and if $\forall n \geq N$ $x_n \geq \gamma$, then $\limsup_{n\to+\infty} x_n \leq \ell f(\gamma,\gamma)$.

So, we start from the hypothesis of (A). We prove then the relation

$$\frac{x_{n+1}}{x_n} \leq f\left(x_n, \frac{x_n}{f(\gamma,\gamma)}\right), \tag{5}$$

if $n \geq N+2$. Under this hypothesis one has $x_n = x_{n-1}f(x_{n-1},x_{n-2}) \leq x_{n-1}f(\gamma,\gamma)$, by property (1). Thus we have $x_{n-1} \geq \frac{x_n}{f(\gamma,\gamma)}$, and, by (1), $f(x_n,x_{n-1}) \leq f\left(x_n, \frac{x_n}{f(\gamma,\gamma)}\right)$, which gives inequality (5).

Then, because $f(\gamma,\gamma) \geq 1$, and with property (1), we have $\frac{x_{n+1}}{x_n} \leq f\left(\frac{x_n}{f(\gamma,\gamma)}, \frac{x_n}{f(\gamma,\gamma)}\right)$.

Now we set $u_n = \frac{x_n}{f(\gamma,\gamma)}$. Thus the sequence (u_n) satisfies the inequality $u_{n+1} \leq u_n f(u_n,u_n) = g(u_n)$ for $n \geq N+2$.

We define the sequence (v_n) by: $v_{N+2} = u_{N+2}$, and if $n \geq N+2$, $v_{n+1} = g(v_n)$.

Property (3) of g implies that $u_n \leq v_n$ for $n \geq N+2$. Since g is increasing, the equation $g(\ell) = \ell$ is equivalent to the relation $f(\ell,\ell) = 1$, which has the unique solution ℓ by (0'), and $g(x) - x = x(f(x,x)-1)$ has the same sign as $\ell - x$ and is not 0 if $x \neq \ell$. Thus $v_n \to \ell$. So we have $\limsup_{n\to\infty} u_n \leq \ell$, that is $\limsup_{n\to\infty} x_n \leq \ell f(\gamma,\gamma)$, and this last number is greater than ℓ because $f(\gamma,\gamma) \geq 1$. So assertion (A) is proved.

(b) We prove now the following assertion (B) :

(B) for every $N \geq 0$ and for every $\delta > 0$, one has: if $f(\delta,\delta) \leq 1$, and if $\forall n \geq N$, $x_n \leq \delta$, then $\liminf_{n\to+\infty} x_n \geq \ell f(\delta,\delta)$.

With the hypothesis of (B), we prove first the inequality

$$\frac{x_{n+1}}{x_n} \geq f\left(x_n, \frac{x_n}{f(\delta,\delta)}\right), \tag{6}$$

if $n \geq N+2$, by exactly the same method as this one used for inequality (5). Because $f(\delta,\delta) \leq 1$, we have $\frac{x_{n+1}}{x_n} \geq f\left(\frac{x_n}{f(\delta,\delta)}, \frac{x_n}{f(\delta,\delta)}\right)$. With the sequences

$u_n = \frac{x_n}{f(\delta,\delta)}$ and $v_{N+2} = u_{N+2}$, and, for $n \geq N+2$, $v_{n+1} = g(v_n)$, we prove assertion (B) in the same manner as assertion (A).

(c) Proof of the convergence of x_n.

1. If one of the hypothesis (M) or (H_-) holds, we start with the hypothesis of (A) for $N=0$ and: $\gamma = 0$ if (M) holds (observe that in this case $\tilde{f}(0) \geq 1$), or $0 < \gamma < \min(m, \ell)$ if (H_-) holds. Thus we can apply assertion (A), and we obtain $\limsup x_n \leq \ell f(\gamma,\gamma)$. Next, we apply assertion (B) with $\delta = \ell f(\gamma,\gamma)+\varepsilon$ and $N = N_\varepsilon$. We obtain then $\liminf x_n \geq \ell f\big(\ell f(\gamma,\gamma)+\varepsilon, \ell f(\gamma,\gamma)+\varepsilon\big)$. But this is true for every $\varepsilon > 0$, thus the result is: $\liminf x_n \geq \ell f\big(\ell f(\gamma,\gamma), \ell f(\gamma,\gamma)\big) := \psi(\gamma)$. Now $\psi(\gamma) > 0$, and we can iterate the method of using alternately assertions (A) and (B). Thus we have, for every integer $p \geq 1$:

$$\liminf_{n\to\infty} x_n \geq \psi^{(p)}(\gamma) \quad \text{and} \quad \limsup_{n\to\infty} x_n \leq \ell f\big(\psi^{(p)}(\gamma), \psi^{(p)}(\gamma)\big). \tag{7}$$

But we know by property (1) that ψ is nondecreasing, that $\psi(\ell) = \ell$, and Lemma 1 gives us the inequalities: $\psi(x) > x$ on $]0,\ell[$ and $\psi(x) < x$ on $]\ell,+\infty[$. It follows from these conditions that the sequence $\psi^{(p)}(\gamma)$ is convergent to ℓ. But then $\ell f\big(\psi^{(p)}(\gamma), \psi^{(p)}(\gamma)\big) \to \ell f(\ell,\ell) = \ell$. Thus we have the equalities $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \ell$, and $x_n \to \ell$.

2. If hypothesis (H^+) holds, we start with $\delta > \max(\ell, M)$ and apply assertion (B). We obtain $\liminf_{n\to\infty} x_n \geq \ell f(\delta,\delta) := \gamma$, and $0 < \gamma \leq \ell$. We can then apply assertion (A) with $\gamma - \varepsilon$, and make $\varepsilon \to 0$. Then we iterate the method as in 1., and obtain the same result. □

Proof of Theorem 2.

(a) We prove here the following assertion (A'):

(A') for every $N \geq 0$ and for every $\beta > 0$, one has : if $\forall n \geq N,\ x_n \geq \beta$, then $\forall n \geq N+3,\ x_n \leq \phi(\beta)$.

Let be $\beta > 0$ such that $x_n \geq \beta$ for every $n \geq N$. One has

$$x_{n+1} = x_n f(x_n, x_{n-1}) \leq x_n f(x_n, \beta)$$

if $n \geq N+1$, by property (1). So, by property (4) one has

$$x_{n+2} = x_{n+1} f(x_{n+1}, x_n) \leq x_n f(x_n,\beta) f\big(x_n f(x_n,\beta), x_n\big) = p_\beta(x_n).$$

Thus we have $x_{n+2} \leq p_\beta(x_n)$ which in view of $x_n \geq \beta$ and property (5) gives $x_{n+2} \leq p_\beta(\beta) = \phi(\beta)$ for every $n \geq N+1$, that is assertion (A').

(b) We have now the following assertion (B'):

(B') for every $N \geq 0$ and for every $\delta > 0$, one has : if $\forall n \geq N,\ x_n \leq \delta$, then $\forall n \geq N+3\ x_n \geq \phi(\delta)$.

The proof is exactly the same as for assertion (A'), so we omit it.

(c) Proof of the convergence of x_n.

We prove first the convergence of x_n if hypothesis (H_-) holds. We start with a number β, so that $0 < \beta < \ell$ and $x_n \geq \beta$ for all $n \geq 0$. We iterate assertions (A') and (B') and obtain $\forall p \geq 0$ (observing that $\phi(x) = xf(x,x)f\big(xf(x,x),x\big) > 0$)

$$x_n \geq \phi^{(2p)}(\beta) \ \text{ and } \ \forall n \geq 6p+3,\ x_n \leq \phi^{(2p+1)}(\beta) \ \text{ if } \ n \geq 6p. \qquad (8)$$

Thus we have to study the sequence $\gamma_n = \phi^{(n)}(\beta)$ for $0 \leq \beta < \ell$. By property (6) and Lemma 1, the function $\phi > 0$ is nonincreasing and majorized, so we can define $\phi(0) := \lim_{x\to 0} \phi(x) < +\infty$. So it is obvious that the sequence γ_n lies in the interval $[\phi\circ\phi(0), \phi(0)]$. By property (7), $\phi\circ\phi(\beta) \neq \beta$. There are two cases:

($*$) If $\phi\circ\phi(\beta) < \beta$, then the sequence γ_{2p} is nonincreasing and it converges to a number u, with $0 < \phi\circ\phi(0) \leq u < \ell$, and similarly γ_{2p+1} is nondecreasing and it converges to a number v, with $\ell < v \leq \phi(0)$. So we have $\phi\circ\phi(u) = u$ and $\phi\circ\phi(v) = v$ which is impossible by property (7).

($**$) If $\phi\circ\phi(\beta) > \beta$, then the sequence γ_{2p} is nondecreasing and it converges to a number u, with $u \leq \ell$, and similarly γ_{2p+1} is nonincreasing and it converges to a number v, with $\ell \leq v$. If $u \neq \ell$ (and $v \neq \ell$), then $\phi\circ\phi(u) = u$ and $\phi\circ\phi(v) = v$, which is impossible by (7). So the sequence γ_n converges to ℓ. The two relations (8) imply then that $\lim_{n\to+\infty} x_n = \ell$.

The proof of the convergence of x_n under the hypothesis (H_+) is exactly the same, but with $\gamma_{2p+1} \leq \ell \leq \gamma_{2p}$. □

3. Three applications to the difference equation $u_{n+2} = \frac{u_{n+1}+a}{u_n+b}$

We prove convergence results about the solutions of this equation. The most important of them is already in [6], with another method of proof. We summarize the known results before this paper and our one.

(a) *The previous known results about its solutions*

We can synthesize these results by the Fig. 1. This figure shows in the (a,b)–plane the behavior of the nonconstant solutions of the equation. The possible fixed point ℓ is given by equation $\ell^2 + (b-1)\ell - a = 0$ if $a \neq 0$, by $\ell = 0$ if $a = 0$ and $b > 1$, and by $\ell = 1 - b$ if $a = 0$ and $0 < b \leq 1$.

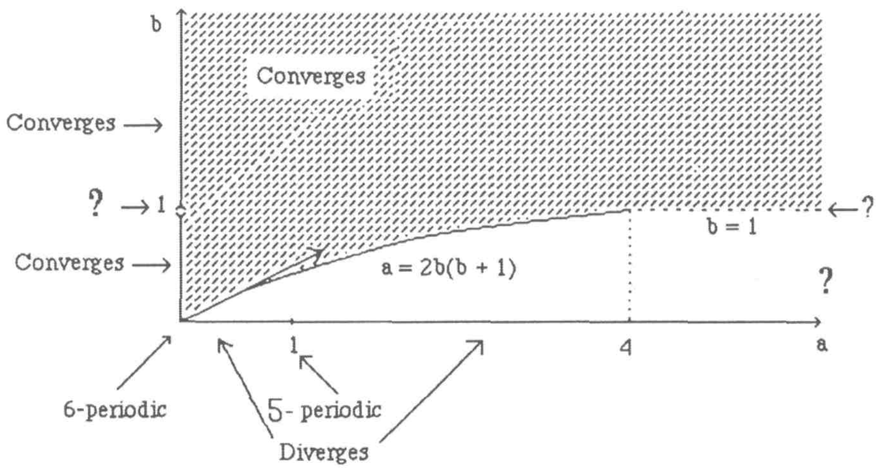

Figure 1.

Most of these results are due to Kocic and Ladas in [9], except for $b = 0$, the Lyness' case. For the case of the Lyness' equation, the results are in [5], [7], [10] and [14]. We will synthesize these last results in part 4. Our goal is to apply the Theorems of part 2. for giving greater domain in the (a, b)–plane where the solutions of the equation converge.

(b) *Main result on the convergence of the solutions of the difference equation*

We collect old and new results in the following assertion, and in particular the main result of [6].

Theorem 3. *The behavior of the nonconstant solutions of the difference equation*

$$u_{n+2} = \frac{u_{n+1} + a}{u_n + b} \tag{9}$$

is given in Fig. 2 below. Moreover, for $ab > 0$ and $(a, b) \notin D$, for $a = 0$, $b > 1$, and for $a = 0$, $0 < b < 1$ the fixed point ℓ is attractive, and thus globally asymptotically stable. If $a = 0$ and $b = 1$, then u_n decreases and one has $u_n \underset{n\to\infty}{\sim} \frac{1}{n}$.

We recall that in [9] Kocic and Ladas made the conjecture that u_n converges for every $(a, b) \in \mathbb{R}_*^{+2}$. So the domain where this conjecture is open is now

$$D = \left\{ (a,b) \,|\, 0 < b < 1 \text{ and } a > \frac{2b(1+b^2)}{(1-b)^2} \right\}.$$

We begin with easy results about the sequence $u_{n+2} = \frac{u_{n+1}+a}{u_n+b}$.

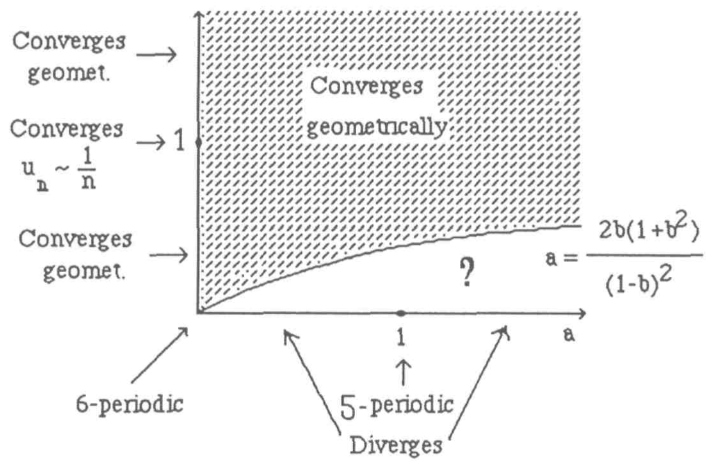

Figure 2.

Lemma 2. *If $a > 0$ and $b > 0$, there exist m and M which depend only on a and b, such that $0 < m \leq M$, and such that one has $m \leq u_n \leq M$ for every $n \geq 5$ (the sequence is "permanent"). If $a = 0$ and $b > 0$, the sequence is majorized, for $n \geq 3$, by a constant M which depends only on b.*

The proof is easy and gives: if $ab > 0$, $M = \frac{1}{b}\max\left(1, \frac{a}{b}\right) + \frac{a}{b}$ and $m = \frac{a}{M+b}$; if $a = 0$ and $b > 0$, $M = \frac{1}{b}$.

Lemma 3. *Let be $\Omega = \mathbb{R}_*^{+2}$, and define $F : \Omega \to \Omega$ by*

$$F(x,y) = \begin{pmatrix} \dfrac{x+a}{y+b} \\ x \end{pmatrix}.$$

Then the fixed point of F is (ℓ,ℓ), and if $ab > 0$ the two singular values of the differential $\mathrm{d}F(\ell,\ell)$ have modulus < 1. So the fixed point is attractive. The same thing happens if $a = 0$ and $0 < b < 1$.

Proof. We have

$$\mathrm{d}F(x,y) = \begin{pmatrix} \dfrac{1}{y+b} & -\dfrac{x+a}{(y+b)^2} \\ 1 & 0 \end{pmatrix}.$$

So the singular values of the differential of F at the point (ℓ,ℓ) are the roots of equation $\lambda^2 - \frac{\lambda}{\ell+b} + \frac{\ell+a}{(\ell+b)^2} = 0$, whose discriminant is $\frac{1-4(\ell+a)}{(\ell+b)^2}$.
(1) If $1-4(\ell+a) \geq 0$, the two roots λ_1 et λ_2 are positive :

$$0 < \lambda_1 = \frac{1-\sqrt{1-4(\ell+a)}}{2(\ell+b)} \leq \frac{1+\sqrt{1-4(\ell+a)}}{2(\ell+b)} = \lambda_2.$$

But $1+\sqrt{1-4(\ell+a)} < 2$, thus $0 < \lambda_1 \leq \lambda_2 < \frac{1}{\ell+b} = \frac{\ell}{\ell+a} < 1$;
(2) If $1-4(\ell+a) < 0$, the two roots are conjugated complex numbers, with the same modulus given by $|\lambda_i|^2 = \frac{\ell+a}{(\ell+b)^2} = \frac{\ell}{\ell+b} < 1$.
(3) If $a = 0$ and $0 < b < 1$, then the singular values are the roots of the equation $\lambda^2 - \lambda + 1 - b = 0$, because we shall see in the following (c) that the fixed point is $\ell = 1-b$, and so the result is obvious. □

In the sequel, we put

$$f(u,v) = \frac{1+\frac{a}{u}}{v+b}. \tag{10}$$

So the difference equation (1) becomes $u_{n+2} = u_{n+1} f(u_{n+1}, u_n)$ and we can apply Theorems 1 and 2, as it was done in [9] for Theorem 2.1.1.

(c) *Case* $a = 0$, $0 < b < 1$

If $a = 0$, then $f(u,v) = \frac{1}{v+b}$. Properties (0'), (1) and (3) are easy to verify (with $\ell = 1-b$) and Lemma 2 gives property (H^+). Thus by Theorem 1 the sequence converges to ℓ and the limit is attractive by Lemma 3.

(d) *Case* $b > a > 0$

In this case, $f(u,v) = \frac{1+\frac{a}{u}}{v+b}$. Properties (0') and (1) are obvious. Property (3) is : $x \mapsto \frac{x+a}{x+b}$ is increasing and this is obvious with the inequalities $b > a > 0$. With hypotheses (H^+) and (H^-), true by Lemma 2, Theorem 1 gives the convergence of the sequence, and the fixed point is attractive by Lemma 3.

It is to be noticed that these cases (c) and (d) are exactly the same as in [9]. Only the proof of Theorem 1 is different.

(e) *Case* $a \geq \frac{b}{b+1}$ *and* $a \leq \frac{2b(1+b^2)}{(1-b)^2}$ *with* $0 < b < 1$

We shall apply here Theorem 2. Properties (0), (1), (2), (4) and (4') are obvious. We have $p_y(x) = h(h(x,y),x) = \frac{a}{x+b} + \frac{x+a}{(x+b)(y+b)}$. This function of x is nonincreasing if $\forall y > 0$ one has $b - a - ab \leq ay$, that is $b \leq a(b+1)$, which is true by hypothesis in this case; so property (5) is true. For property(6), we have to prove that the function

$$\phi(x) = \frac{a}{x+b} + \frac{x+a}{(x+b)^2} \tag{11}$$

is nonincreasing. A calculation analogous to the preceding one gives condition $a \geq \frac{b}{b+2}$, which is true by the hypothesis. It remains to verify property (7). We remark by Lemma 1 that we can define $\phi(0)$, but here it is obvious: $\phi(0) = \frac{ab+a}{b^2}$.

Lemma 4. *The function ϕ has no 2-periodic point on* $[0, +\infty[$.

So we can apply Theorem 2, and have the convergence of the sequence u_n in case (e).

Proof of Lemma 4. We put $x + b = X$. The equation $\varphi \circ \varphi(x) = x$ has the form $P(X) = 0$, where P is a polynomial which has as a factor the polynomial Q such that $Q(X) = 0$ means the equality $\varphi(x) = x$. One has $Q(X) = X^3 - bX^2 - (a+1)X + b - a = (X+1)[X^2 - (b+1)X + b - a]$, and one find $P(X) = b^2X^5 + (ab - 2b^3 + 2b - a)X^4 + [2b(a-b) - 2b^2(a+1)]X^3 + [(a+1)(a-b) - b(a+1)^2 - 2b(a-b)]X^2 + [(a-b)^2 - 2b(a-b)(a+1)]X - b(a-b)^2$.

A calculation gives $P = QR$, where $R(X) = b^2X^2 + (ab+2b-a)X + b(a-b)$. We put $S(x) = R(x+b)$, and obtain $S(x) = b^2x^2 + [2b(b^2+1) - a(1-b)]x + b^2(1+a+b^2)$.

We wish to prove that $S(x) > 0$ for every $x \geq 0$, $x \neq \ell$. We distinguish two cases :

(i) $S'(0) \geq 0$. Then we have $S(x) > 0$ for $x \geq 0$. This condition is written $a(1-b) \leq 2b(1+b^2)$; it is satisfied if $b \geq 1$. Otherwise, we must have $a \leq \frac{2b(1+b^2)}{1-b} := u(b)$. In the Fig. 3, this means that (a,b) does not belong to the region Z_1.

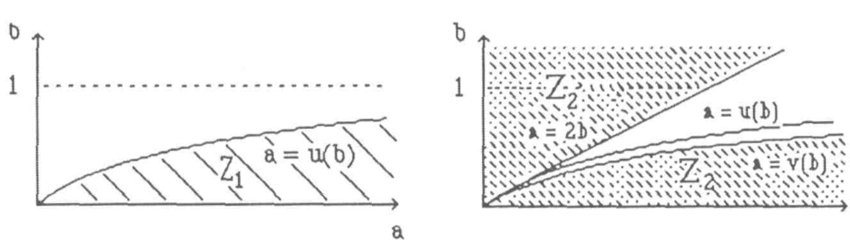

Figure 3.

(ii) $S'(0) < 0$. This inequality implies first that $(a, b) \in Z_1$. Then the condition $S(x) > 0$ for $x \geq 0$ is that the discriminant Δ of S is negative. We have

$$\Delta = [2b(b^2+1) - a(1-b)]^2 - 4b^4(1+a+b^2) = (a-2b)[a(1-b)^2 - 2b(b^2+1)].$$

Then the relation $\Delta < 0$ means : $2b < a < \frac{2b(1+b^2)}{(1-b)^2}$. Put $v(b) = \frac{2b(1+b^2)}{(1-b)^2} = \frac{u(b)}{1-b}$. Then we have $(a, b) \notin Z_2$ (figure 6). Thus one or the other of the cases *(i)* or *(ii)* implies the nonexistence of points of order 2 distinct of ℓ. This happens as soon as $a \geq \dfrac{b}{b+2}$ and $(a, b) \notin Z_1 \cap Z_2$. So if hypotheses of case (e) are satisfied, $S'(0) < 0$ and $\Delta < 0$, then ϕ has no 2–periodic point.

It remains the case where $S'(0) < 0$ and $\Delta = (a-2b)[a(1-b)^2 - 2b(b^2+1)] = 0$; we have to prove that in this case the root of S is ℓ. Then S has a double root $X_0 = \frac{a(1-b)-2b(1+b^2)}{2b^2}$. If $a = 2b$, one has $X_0 = -1-b$, which is impossible, thus $a = v(b)$ and $X_0 = \frac{1+b^2}{1-b} = \ell$. So, in this case also ϕ has no 2–periodic point. □

The cases (d) and (e) give the convergence of u_n for (a, b) in the domains of Fig. 4.

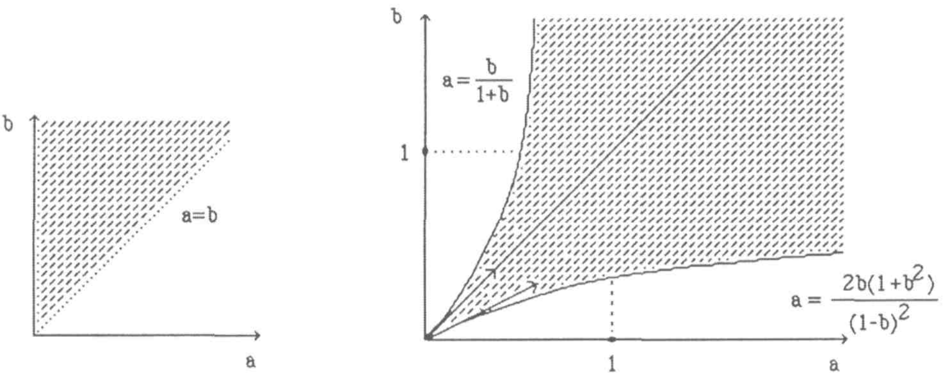

Figure 4.

Remark 1. We obtain the condition $a \leq \frac{2b(1+b^2)}{(1-b)^2}$ by direct calculations on the equation $\phi \circ \phi(x) = x$, whereas in [6] this condition is obtained by general assertion which uses results about Schwarzian derivative. It may seem strange that with quite different methods [6] and our paper obtain the same technical sufficient condition on (a, b) for convergence of solutions of (1). The reason why this phenomenon appears is the following : in the two papers, this condition is equivalent to the absence of 2–periodic points for the function ϕ, here, and for the function G of [6], and one has $\phi = G$!

(f) *Other cases of convergence*

We remark first that Fig. 4 shows that u_n converges if $b > 1$ and $a > 0$; but this result has also several simple proofs (one is in [9]).

The two following results are straightforward:

Lemma 5. *If $a = b = 0$, the sequence $u_{n+2} = \frac{u_{n+1}}{u_n}$ is 6–periodic.*

Lemma 6. *If $a = 0$ and $b > 1$, then $u_n \leq \frac{u_0}{b^n}$, and so $u_n \to 0$.*

For proving case $b > 0$ of Theorem 3, it is now enough to look at the case $a = 0$, $b = 1$.

Lemma 7. *If $a = 0$ and $b = 1$, then $u_n \underset{n\to\infty}{\sim} \frac{1}{n}$.*

Proof. One has $u_{n+2} = \frac{u_{n+1}}{u_n+1}$ if $n \geq 0$, thus u_n decreases, and converges, necessarily to 0. We have

$$\frac{1}{u_k} - \frac{1}{u_{k-1}} = \frac{u_{k-2}}{u_{k-1}} = 1 + u_{k-3}.$$

By summation of these relations from $k = 3$ to $k = n$, we obtain for $n \geq 3$

$$\frac{1}{u_n} = \frac{1}{u_2} + n - 2 + u_0 + u_1 + \ldots + u_{n-3} = n\left(1 + \frac{\frac{1}{u_2} - 2}{n} + \frac{n-2}{n}\,\frac{u_0 + \ldots + u_{n-3}}{n-2}\right),$$

and thus, by Cesaro's theorem, $\frac{1}{u_n} \underset{n\to\infty}{\sim} n$. □

Remark 2. If $(a, b) \in D$ we have no general result of convergence for solutions of equation (1) with u_1, $u_2 > 0$. But for given (u_1, u_0) we have a numerical criterion for *proving* the convergence: let $\mathcal{E}$ be the ellipse defined by

$$(y - \ell)^2 + \frac{[2(\ell + b)(x - \ell) - (y - \ell)]^2}{4(\ell + a) - 1} \leq 9.10^{-4}(\ell + b)^2; \tag{12}$$

if there exists an integer n_0 such that $(u_{n_0+1}, u_{n_0}) \in \mathcal{E}$, then the solution u_n converges. So, we can prove with a computer some individual results of convergence. For example, if $b = \frac{1}{2}$, $a = 14$, and $u_1 = 2$, $u_0 = 4$, then $n_0 = 53$ works, and the solution converges. If $b = 10^{-1}$, $a = \frac{99.1}{4}$, and $u_1 = u_0 = 3$, then $n_0 = 213$ works, and the solution converges. If $b = 10^{-4}$, $a = \frac{99.00019999}{4}$, and $u_1 = 10^4$, $u_0 = 10^{-2}$, then $n_0 = 479\,764$ works, and the sequence u_n converges.

4. Case $b = 0$: divergence of the solutions of Lyness' difference equation

The first result is the starting point of all the works on Lyness' sequences.

Lemma 8. (Lyness [14]) *If $a = 1$ and $b = 0$, and if $(u_1, u_0) \neq (\ell, \ell)$ the sequence is 5–periodic. Conversely, if $b = 0$ and if some point $(u_1, u_0) \neq (\ell, \ell)$ is 5–periodic, then $a = 1$.*

The first assertion is an obvious calculation. The converse becomes clear from the form of the relation $(u_4, u_3) = (u_{-1}, u_{-2})$: it is equivalent to $(a-1)(xy - x - a) = 0$ and $(a-1)(y-x) = 0$, if we put $(u_1, u_0) = (x, y)$.

Lyness' sequences $u_{n+2} = \frac{u_{n+1}+a}{u_n}$ were for a long time mysterious, from their discovery in [14]. The most interesting conjecture about them is formulated in [2], and we proved it in [3]. In the present paper we synthesize some results about Lyness' sequence and the so–called Lyness' cubic, which is the main tool of [3].

Proposition 1. ([13]) *When $b = 0$ and $a > 0$, the sequence diverges as soon as $(u_1, u_0) \neq (\ell, \ell)$.*

The proof follows from the following classical invariance lemma (see [11]).

Lemma 9. *If $b = 0$ and $a > 0$, then $\left(1 + \frac{1}{u_n}\right)\left(1 + \frac{1}{u_{n+1}}\right)(a + u_n + u_{n+1})$ is constant.*

This lemma was already given by Lyness in [11] when $a = 1$.

Proof of Proposition 1.

* *Analytical proof.*

Such a proof is in [11] or [13], but we give a simpler one (without the use of Hessian).

Let Ω be the open set $\mathbb{R}_*^{+2}$, and G defined in Ω by

$$G(X) = \left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)\left(a + x + y\right)$$

where $X = (x, y)$. The function G is C^1 and tends to infinity when X tends to the point at infinity of the locally compact space Ω. Thus G achieves its minimum in Ω. Then there exists a critical point of G in this open set, where $\mathrm{d}G(X) = 0$. It is immediate that a critical point in Ω is given by the equations : $x^2 = a + y$ and $y^2 = a + x$. This gets $x = y = \ell$, the only positive solution of $x^2 = a + x$, i.e. the fixed point (equilibrium) of the sequence $(u_n)_{n \geq 0}$. Thus G has a unique critical point in Ω, and the minimum of G

is thus achieved at this point $L = (\ell, \ell)$ and only at this point. Its value is $G(L) = \dfrac{(\ell+1)^3}{\ell}$.

Suppose then that u_n converges : the limit must be ℓ, thus by continuity of G, $G(u_{n+1}, u_n) \to G(L)$. But $n \longmapsto G(u_{n+1}, u_n)$ is constant, by Lemma 9, this constant value is thus the minimum, and for every $n, u_n = \ell$.

∗ *Geometrical proof.*

We give such a proof because it uses the so–called "Lyness' cubic" $\Gamma_a(K)$ of $\mathbb{R}^2$ (which appeared in [14] for $a = 1$); its equation is $(x+1)(y+1)(x+y+a) - Kxy = 0$.

The point $X_n = (u_{n+1}, u_n)$ belongs to the part of the cubic $\Gamma_a(K)$ which is in Ω, if $K = K(u_0, u_1, a)$ is the constant of Lemma 9 (see Fig. 5 and 6).

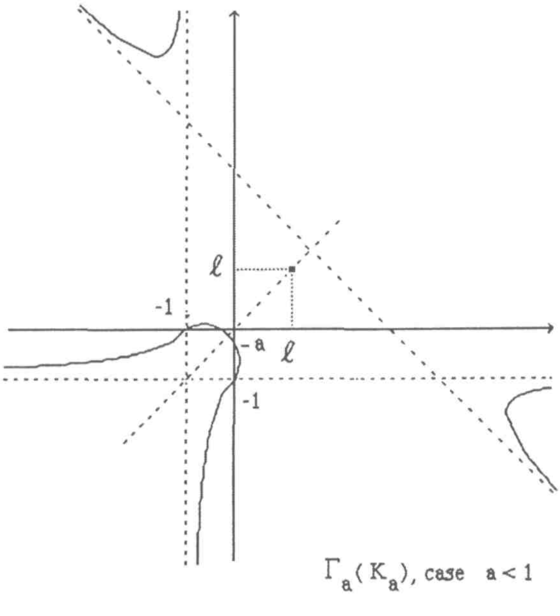

Figure 5.

Then put $K_a = K(\ell, \ell, a)$. It happens that the point $L = (\ell, \ell)$ is a singular isolated point of the particular cubic $\Gamma_a(K_a)$. We have $K_a = \frac{(\ell+1)^3}{\ell}$. Then, with the change of coordinates $x = \ell + X, y = \ell + Y$, the equation of the curve becomes : $(\ell+1)(X^2 - \frac{XY}{\ell} + Y^2) + XY(X+Y) = 0$. The quadratic form $X^2 - \frac{XY}{\ell} + Y^2$ is positive definite (because $\dfrac{1}{\ell^2} - 4 < 0$). This shows that the point $L = (0,0)$ is isolated on the curve $\Gamma_a(K_a)$.

If the sequence would converge (to ℓ), then Lemma 9 would show that $K(u_0, u_1, a) = K_a$; thus one would have $(u_{n+1}, u_n) \in \Gamma_a(K_a)$, and $(u_{n+1}, u_n) \to (\ell, \ell)$; this would imply the equality $(u_{n+1}, u_n) = (\ell, \ell)$ for n sufficiently large. But we would have then $u_n = \ell$ for every n. □

Remark. The analog of Lemma 9 appears in [5] and [10] for the more general sequences $u_{n+k} = \frac{a+u_{n+1}+\ldots+u_{n+k-1}}{u_n}$ (for $k = 3$ it is Todd's difference equation). The proof is the same as for $k = 2$, and these sequences satisfy also proposition 1 (with the same easy analytic proof).

In the general case $(u_1, u_0) \neq (\ell, \ell)$, the point (u_{n+1}, u_n) moves on the bounded connected component $\Gamma_a^0(K)$ of the cubic $\Gamma_a(K)$ (see [2] et [3] about this phenomenon).

Proposition 2. *If $(u_1, u_0) \neq (\ell, \ell)$, the cubic $\Gamma_a(K)$ has a compact connected component $\Gamma_a^0(K)$, included in $\mathbb{R}_*^{+2}$. This component $\Gamma_a^0(K)$ contains the initial point (u_1, u_0) of the sequence, and $X_n := (u_{n+1}, u_n)$ moves on $\Gamma_a^0(K)$. The map F preserves $\Gamma_a^0(K)$, and has the following geometric interpretation : $M' = F(M)$ is the symmetric (with respect to the diagonal) of the second point M_1 of $\Gamma_a^0(K)$ with the same x-coordinate as M ; or, one takes the image $M_2 \in \Gamma_a^0(K)$ of the point M by the symmetry with respect to the diagonal, and the point M' is the other point of $\Gamma_a^0(K)$ with the same y-coordinate as M_2.*

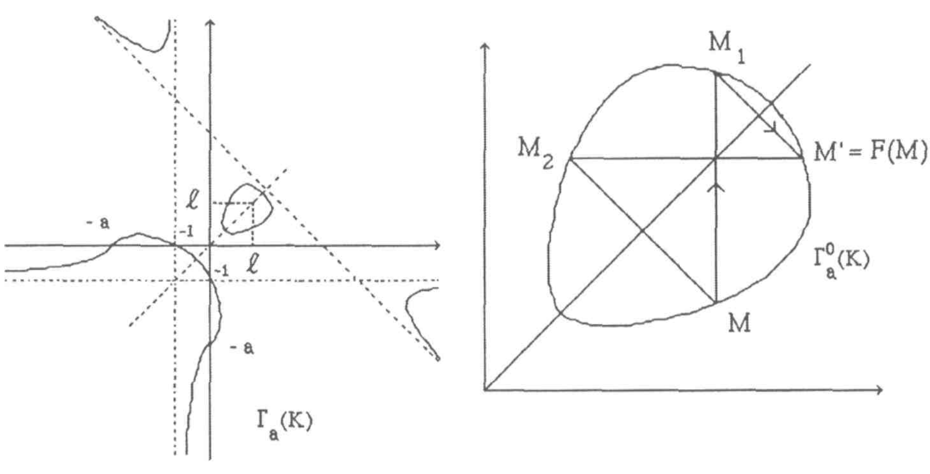

Figure 6.

Proof. The equation of the cubic $\Gamma_a(K)$ is also : $(1+\frac{1}{x})(1+\frac{1}{y})(x+y+a) = K$. But $(u_1, u_0) \in \Gamma_a(K)$, so $K = (1 + \frac{1}{u_1})(1 + \frac{1}{u_0})(u_1 + u_0 + a)$. One has

$\Gamma_a^0(K) = \Gamma_a(K) \cap \Omega$. It is easy to see that $\Gamma_a^0(K)$ is included in the triangle of Ω defined by the inequalities $x \geq aK^{-1}$, $y \geq aK^{-1}$ and $x+y \leq K-a$; this triangle is not empty, because it contains the point (u_1, u_0).

Obviously, X_n moves on $\Gamma_a^0(K)$, and more generally, if $X \in \Gamma_a^0(K)$, $F(X) \in \Gamma_a^0(K)$. Because $F(x,y) = (\frac{x+a}{y}, x)$, the image $M' = F(X)$ of the point $M = X = (x,y)$ is the point of $\Gamma_a^0(K)$ symmetric of M_1 with respect to the diagonal. □

Proposition 3. *The equilibrium point $L = (\ell, \ell)$ of the difference equation $u_{n+2}\,u_n = u_{n+1} + a$ is locally stable : $\forall \varepsilon > 0\ \exists \eta > 0$ such that if $||M_0 - L|| < \eta$ then $\forall n \geq 0$ one has $||M_n - L|| < \varepsilon$.*

Proof. First, we remark that $\Gamma_a^0(K_a)$ reduces to the point L, because a line through L intersects $\Gamma_a(K_a)$ only in points which are at infinity (in horizontal or vertical direction, or in direction $x+y=0$) or in the second, third or fourth quadrant in $\mathbb{R}^2$.

It follows from this fact that the loops $\Gamma_a^0(K)$ collapse uniformly to $\{L\}$ when $K \to K_a$, that is : $\forall \varepsilon > 0\ \exists \theta > 0$ such that if $K_a \leq K < K_a + \theta$ then $\forall N \in \Gamma_a^0(K)$, $||L - N|| < \varepsilon$: if not, $\exists \varepsilon > 0$, $K_a \leq K_p < K_a + \frac{1}{p}$, and $M_p \in \Gamma_a^0(K_p)$ such that $||M_p - L|| \geq \varepsilon$. But the loops $\Gamma_a^0(K_p)$ are included in a fixed compact for $K_a \leq K \leq K_a + 1$, by proposition 2; thus there exists a subsequence M_{p_q} which converges to a point $M \in \Gamma_a^0(K_a)$, and $||M_{p_q} - L|| \geq \varepsilon$, thus $L \neq M$; and this contradicts the fact that $\Gamma_a^0(K_a) = \{L\}$.

Now, if $||M_0 - L|| < \eta$, then $K_a \leq K = G(u_1, u_0) \leq K_a + \theta$, and then $\forall N \in \Gamma_a^0(K)$, $||L - N|| < \varepsilon$. But $\forall n \geq 0\ M_n = F^{(n)}(M_0) \in \Gamma_a^0(K)$, thus $||L - M_n|| < \varepsilon$: L is locally stable. □

In [3] we gave the complete study of the behavior of X_n on $\Gamma_a^0(K)$. The starting point of the methods of [3] is the last assertion of present Proposition 2, which allows us to use the group law on Lyness' cubic, and parametrize this cubic by Weierstrass' elliptic function. In the paper [4], we generalize [3] to the study of algebraic difference equations of the form $u_{n+2}\,u_n = \psi(u_{n+1})$, related to conics, cubics or quartics.

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Nova metoda proučavanja rješenja diferentne jednadžbe $u_{n+2} = \dfrac{u_{n+1} + a}{u_n + b}$

G. Bastien i M. Rogalski

Sadržaj

U radu se dokazuju dva glavna rezultata konvergencije za rješenja diferentnih jednadžbi tipa $x_{n+2} = x_{n+1} f(x_{n+1}, x_n)$. Prvi je poseban slučaj poznatog rezultata (vidi [9], [12]) ali je metoda različita i dozvoljava da se dokaže drugi, koji je nov. Oni se koriste za proučavanje rješenja diferentne jednadžbe $u_{n+2} = \frac{u_{n+1}+a}{u_n+b}$. Korištena metoda daje opet rezultat kao u [6]: za $(a,b) \in \mathbb{R}_*^{+2} \setminus D$ niz (u_n) uvijek konvergira, gdje je $D = \{0 < b < 1\} \cap \{a > \frac{2b(1+b^2)}{(1-b)^2}\}$.

Slučaj $b = 0$ daje takozvanu Lyness–ovu diferentnu jednadžbu.