RADOVI MATEMATIČKI
Vol. 12 (2004), 233–240

Arcwise quasi–monotone mappings onto fans

Janusz J. Charatonik (México)

Abstract. A dendroid X is said to be *weakly arcwise open* if for each point p of X each arc component of $X \setminus \{p\}$ either is open or has empty interior. Let f be an arcwise quasi-monotone mapping from a dendroid X onto a fan Y. Conditions are studied (related either to the structure of the domain space X or to the mapping f) under which f preserves the property of being weakly arcwise open.

A dendroid is an arcwise connected and hereditarily unicoherent metric continuum. A dendroid having exactly one ramification point is called a fan. Investigation of shore points (special kinds of noncut points) in dendroids, see [7] and [9], leads in a natural way to establish a class of dendroids called weakly arcwise open (WAO); for dendroids, WAO is a generalization of the property of Kelley, see [8]. The class of WAO dendroids was studied in [1], [2], [8], [9, Section 3] and [10]. The aim of this paper is to continue this study to get a further progress in this area, especially with respect to mapping properties of WAO fans.

In [10, Proposition 2.5, p. 120] it is shown that surjective quasi–monotone mappings between fans preserve WAO property provided that the top of the domain is mapped to the top of the range. We extend this result to a wider class of mappings, namely to arcwise quasi–monotone ones (see the definition below) defined on dendroids, under some additional assumptions.

A *continuum* is a compact connected metric space. A continuum is said to be *hereditarily unicoherent* provided that the intersection of any two of its subcontinua is connected. A *dendroid* is an arcwise connected and hereditarily unicoherent continuum. Recall that each subcontinuum

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2000 Mathematics Subject Classification: 54E40, 54F15, 54F50, 54F55.
Key words and phrases: Arc component, continuum, dendroid, fan, mapping, monotone.

of a dendroid is a dendroid, and that for every two points a and b of a dendroid X there is a unique arc $ab \subset X$ joining a and b.

Given a subset S of a dendroid X, we denote by $\mathfrak{A}(S)$ the family of all arc components of S in X. A point p of a dendroid X is called a *ramification point* (an *end point*) of X provided that $\operatorname{card}(\mathfrak{A}(X \setminus \{p\})) \geq 3$ (if $\operatorname{card}(\mathfrak{A}(X \setminus \{p\})) = 1$, respectively). We denote by $E(X)$ the set of all end points of a dendroid X. If a dendroid X has exactly one ramification point, then it is called a *fan*, and the ramification point is named the *top* of the fan.

A dendroid X is said to be *weakly arcwise open at a point* $p \in X$ provided that each element of $\mathfrak{A}(X \setminus \{p\})$ either is open or has empty interior. If the mentioned condition holds at each point $p \in X$, then X is said to be *weakly arcwise open* (shortly WAO).

More generally, if K is a nonempty closed subset of a dendroid X, we say that *X is WAO at K* provided that each element of $\mathfrak{A}(X \setminus K)$ either is open or has empty interior. Connections between being WAO at K and being WAO at each point of K were studied in [10]. In particular, the following result is shown [10, Proposition 1.7, p. 117].

Proposition 1. *Let K be a closed subset of a dendroid X such that K has a finite number of components. Then the following implication holds:*

$$X \text{ is } WAO \text{ at } x \text{ for each } x \in K \quad \Longrightarrow \quad X \text{ is } WAO \text{ at } K. \tag{1.1}$$

The following notation will be used in the paper. $\mathbb{N}$ stands for the set of all positive integers. $\mathcal{C}$ means the Cantor ternary set in the closed unit interval $[0, 1]$. In the Euclidean plane we denote by $\overline{uv}$ the straight line segment with end points u and v.

A surjective mapping $f : X \to Y$ between continua is said to be:
– *monotone* provided that for each point $y \in Y$ the set $f^{-1}(y)$ is connected; equivalently, if for each subcontinuum Q of Y, the inverse image $f^{-1}(Q)$ is connected;
– *quasi-monotone* provided that for each subcontinuum Q of Y with nonempty interior, the inverse image $f^{-1}(Q)$ has finitely many components, each of which is mapped onto Q under f;
– *arcwise quasi–monotone* (abbreviated AQM) provided that for each subarc Q of Y with nonempty interior, the inverse image $f^{-1}(Q)$ has finitely many components, each of which is mapped onto Q under f.

Obviously each monotone mapping is quasi–monotone, and each quasi-monotone is arcwise quasi–monotone. Quasi–monotone mappings preserve the property of being a dendroid, [6, (7.31), p. 67; compare Table IV,

p. 69], while it is not so for arcwise quasi–monotone ones. The following example shows this.

Example 2. There exists an arcwise quasi–monotone mapping from the Cantor fan (i.e., the cone over the Cantor set) onto a 2–cell.

Proof. For a space X define $\mathrm{Cone}\,(X) = (X \times [0,1])/(X \times \{1\})$. Points of $\mathrm{Cone}\,(X)$ are denoted by (x,t) for $x \in X$ and $t \in [0,1]$. In particular, $(x_1, 1) = (x_2, 1)$ for every $x_1, x_2 \in X$. For a mapping $f : X \to Y$ the induced mapping $\mathrm{Cone}\,(f) : \mathrm{Cone}\,(X) \to \mathrm{Cone}\,(Y)$ is defined by $\mathrm{Cone}\,(f)(x,t) = (f(x), t)$.

Consider the well known Cantor–Lebesgue step function $\varphi : \mathcal{C} \to [0,1]$ (see e.g. [5, §16, II, (8), p. 150]; compare [11, Chapter II, §4, p. 35]). Then

$$f = \mathrm{Cone}\,(\varphi) : X = \mathrm{Cone}\,(\mathcal{C}) \to Y = \mathrm{Cone}\,([0,1])$$

maps the Cantor fan X onto the two–cell Y and it is AQM, since the condition of the definition of an AQM mapping is satisfied vacuously: there is no arc with nonempty interior in Y.

Remark 3. Note that the mapping f defined in Example 2 is AQM while not quasi–monotone. Thus Example 2 exhibits the difference between the two classes of mappings. □

It is known that monotone mappings preserve WAO property for dendroids, see [10, Theorem 2.2, p. 120]. For quasi–monotone mappings such a result is not known, see [10, Problem 2, p. 121].

We will use the following lemma (see [10, Lemma 1.5 a), p. 116]).

Lemma 4. *Let $f : X \to Y$ be a mapping between dendroids X and Y, and W be a closed subset of Y. If an arc component $B \in \mathfrak{A}(Y \setminus W)$ is such that $B \setminus \mathrm{int}\,(B) \neq \emptyset$, then there exists an arc component $A \in \mathfrak{A}(X \setminus f^{-1}(W))$ such that*

$$(A \setminus \mathrm{int}\,(A)) \cap f^{-1}(B \setminus \mathrm{int}\,(B)) \neq \emptyset.$$

Moreover, for each point $v \in B \setminus \mathrm{int}\,(B)$ there is a point $u \in f^{-1}(v)$ such that $u \in A(u)$, where $A(u) \in \mathfrak{A}(X \setminus f^{-1}(W))$.

The proof of the next lemma is modeled after the one of Lemma 1.5 c) in [10, p. 117].

Lemma 5. *Let X be a dendroid and Y be a fan with the top q. Assume that a surjective mapping $f : X \to Y$ is such that for each arc $M \subset Y$ with $\mathrm{int}\,(M) \neq \emptyset$ the inverse image $f^{-1}(M)$ has finitely many components. If $q \notin M$, then*

$$A \cap f^{-1}(\mathrm{int}\,(M)) \subset \mathrm{int}\,(A) \quad \textit{for each} \quad A \in \mathfrak{A}(X \setminus f^{-1}(q)). \tag{5.1}$$

Proof. Let M be a subarc of Y, and let $A \in \mathfrak{A}(X \setminus f^{-1}(q))$. We may assume $A \cap f^{-1}(\operatorname{int}(M)) \neq \emptyset$, so $\operatorname{int}(M) \neq \emptyset$, whence $f^{-1}(M)$ has finitely many components. Suppose on the contrary that there is a point $u \in (A \setminus \operatorname{int}(A)) \cap f^{-1}(\operatorname{int}(M))$. Since u is not in $\operatorname{int}(A)$ and since $f^{-1}(\operatorname{int}(M))$ is a neighborhood of u, it contains an infinite sequence of points $\{u_n\} \subset X \setminus A$ converging to u. Further, since $f^{-1}(M)$ has finitely many components, at least one of them, say L, contains an infinite subsequence of the sequence $\{u_n\}$, whence it follows by closedness of L that $u \in L$. Since L is a subdendroid of X that contains points of A (the point u, for example), and points out of A (namely points u_n), it follows that $L \cap f^{-1}(q) \neq \emptyset$. Let $x \in L \cap f^{-1}(q)$. Since $L \subset f^{-1}(M)$, we get $q = f(x) \in f(L) \subset M$, a contradiction with the assumption $q \notin M$. This finishes the proof.

Lemma 6. *Let a surjective mapping $f : X \to Y$ be defined between a dendroid X and a fan Y, and let q be the top of Y. Then*

(6.1) *for each arc component $B \in \mathfrak{A}(Y \setminus \{q\})$ with $\operatorname{int}(B) \neq \emptyset$ there exists an arc K such that $\emptyset \neq \operatorname{int}(K) \subset K \subset B$.*

Proof. In fact, B is of the form $qe \setminus \{q\}$, where $e \in E(Y)$. If $\operatorname{int}(B) \neq \emptyset$, let $y \in \operatorname{int}(B)$. Then there are points $y_1, y_2 \in \operatorname{int}(B)$ such that $y \in y_1y_2 \subset \operatorname{int}(B) \subset B$. Thus $K = y_1y_2$ is the needed arc.

Proposition 7. *Let an AQM surjective mapping $f : X \to Y$ be defined between a dendroid X and a fan Y, and let q be the top of Y. Then the following implication holds.*

$$\textit{If } X \textit{ is WAO at } f^{-1}(q), \textit{ then } Y \textit{ is WAO at } q. \tag{7.1}$$

Proof. Let $B \in \mathfrak{A}(Y \setminus \{q\})$. Suppose on the contrary that $B \setminus \operatorname{int}(B) \neq \emptyset \neq \operatorname{int}(B)$. Take $v \in B \setminus \operatorname{int}(B)$. Then by Lemma 4 there is a point $u \in f^{-1}(v)$ such that $u \in A \setminus \operatorname{int}(A)$ for some $A \in \mathfrak{A}(X \setminus f^{-1}(q))$. Hence A is not open. Since X is WAO at $f^{-1}(q)$, we infer that

$$\operatorname{int}(A) = \emptyset. \tag{7.2}$$

On the other hand, by Lemma 6, there is an arc $K \subset B$ such that $\operatorname{int}(K) \neq \emptyset$. Let $v^* \in \operatorname{int}(K)$, and put $M = K \cup vv^*$. Then M is a subcontinuum of Y with $\operatorname{int}(M) \subset M \subset B$. Since $B \in \mathfrak{A}(Y \setminus \{q\})$, each subcontinuum of B is an arc. Thus M is an arc, and $\emptyset \neq \operatorname{int}(K) \subset \operatorname{int}(M)$. Further, since f is AQM, the component L of $f^{-1}(M)$ containing u contains a point u^* such that $f(u^*) = v^*$. Note that $L \subset A$. Thus $u^* \in A \cap f^{-1}(\mathrm{int}\,(M))$. But since f is AQM, the set $f^{-1}(M)$ has finitely many components, whence $u^* \in \mathrm{int}\,(A)$ by (5.1) of Lemma 5, a contradiction with (7.2). The proof is complete. □

To prove the main result of the paper we will use the following proposition, [10, Proposition 1.9, p. 118].

Proposition 8. *A fan is WAO if and only if it is WAO at its top.*

Theorem 9. *Let an AQM surjective mapping $f : X \to Y$ be defined between a dendroid X and a fan Y, and let q be the top of Y. Assume that*

(9.1) *$f^{-1}(q)$ has finitely many components.*

Then f preserves the WAO property, that is, the following implication holds.

(9.2) *If X is WAO, then Y is WAO.*

Proof. Let the dendroid X be WAO. Thus X is WAO at each point of $f^{-1}(q)$, which is a closed subset of X having finitely many components by (9.1). Thus X is WAO at $f^{-1}(q)$ according to Proposition 1, and consequently Y is WAO at its top q by implication (7.1) of Proposition 7. Therefore Y is WAO by Proposition 8. The proof is complete. □

Question 10. Let an AQM surjective mapping $f : X \to Y$ be defined between a dendroid X and a fan Y, and let q be the top of Y. Must then condition (9.1) be satisfied?

Question 11. Is (9.1) an essential assumption in Theorem 9?

The following problem was suggested to the author by the referee. Its solution seems to be an important step for a future research in the area.

Problem 12. Let X and Y be dendroids and let an AQM surjective mapping $f : X \to Y$ be such that

(12.1) $f^{-1}(q)$ has finitely many components for each ramification point q of Y.

Does f preserve the WAO property, that is, does implication (9.2) hold?

A dendroid X is said to be *of Type* 1 provided that there are in X a point p and a sequence of points $\{a_n\}$ converging to a point a such that $Y = \mathrm{cl}\,(\bigcup\{pa_n : n \in \mathbb{N}\})$, the sequence of arcs $\{pa_n\}$ converges to a subcontinuum L of X, and there are an end point s of L and an open neighborhood U of s in X such that $s \neq a$ and, if C is the component of $L \cap U$ containing s, then $C \cap (\bigcup\{pa_n : n \in \mathbb{N}\}) = \emptyset$ (see [3, p. 192-193]). The following result is is shown in [10, Proposition 1.7, p. 117].

Proposition 13. *If a dendroid X does not contain any subdendroid of Type* 1*, then for each closed subset K of X the implication holds*

$$X \text{ is } WAO \text{ at } p \text{ for each } p \in K \quad \Longrightarrow \quad X \text{ is } WAO \text{ at } K.$$

An example is presented in [10, p. 118] showing that containing no subdendroid of Type 1 is an essential assumption in this result.

Theorem 14. *Let an AQM surjective mapping $f : X \to Y$ be defined between a dendroid X that contains no copy of Type* 1 *subdendroid and a fan Y. Then f preserves the WAO property, that is,* (9.2) *holds.*

Proof. Let the dendroid X be WAO and let q be the top of the fan Y. Hence X is WAO at each point of $f^{-1}(q)$, which is a closed subset of X. Thus X is WAO at $f^{-1}(q)$ according to Proposition 13, and consequently Y is WAO at its top q by implication (7.1) of Proposition 7. Therefore Y is WAO by Proposition 8. The proof is complete. □

A dendroid X is said to be *smooth at a point* $p \in X$ provided that for each point $a \in X$ and for each sequence of points $\{a_n\}$ converging to a in X the sequence of the arcs pa_n converges to the arc pa (with respect to the Hausdorff metric); it is said to be *smooth* provided that there exists a point $p \in X$ at which X is smooth (see e.g. [4, p. 194]). It is known that if a dendroid is smooth, then it does not contain any subdendroid of Type 1 (see [3, Theorem 1, p. 194]). Hence Theorem 14 implies a corollary.

Corollary 15. *Let an AQM surjective mapping $f : X \to Y$ be defined between a smooth dendroid X and a fan Y. Then f preserves the WAO property, that is,* (9.2) *holds.*

A surjective mapping $f : X \to Y$ between continua is said to be *monotone relative to a point* $x \in X$ provided that for each subcontinuum Q of Y with $f(x) \in Q$, the inverse image $f^{-1}(Q)$ is connected. The next example shows that being an AQM mapping is an essential assumption in Theorems 9 and 14 and in Corollary 15.

Example 16. There are fans X and Y and a surjective mapping $f : X \to Y$ such that

(16.1) X is smooth (so, it does not contain any subdendroid of Type 1);
(16.2) X is WAO;
(16.3) Y is smooth;
(16.4) Y is not WAO;
(16.5) f is monotone relative to the top of X;
(16.6) f is not AQM.

Proof. In the Cartesian coordinates in the plane put $s = (0,0)$, $a = (1,0)$, $p = (2,0)$ and $a_n = (1, \frac{1}{n})$ for each $n \in \mathbb{N}$. Define $X = \overline{sa} \cup \bigcup\{\overline{sa_n} : n \in \mathbb{N}\}$. Thus X is a smooth WAO fan with the top s. Define $Y = \overline{sp} \cup \bigcup\{\overline{sa_n} : n \in \mathbb{N} \setminus \{1\}\}$. Thus Y is a smooth fan that is not WAO. Finally define $f : X \to Y$ as follows. Let $f|\overline{sa_1} : \overline{sa_1} \to \overline{sp}$ be the linear homeomorphism with $f(s) = s$, and let $f|\overline{sa}$ and $f|\overline{sa_n}$ for $n \geq 2$ be the identities. Then (16.5) and (16.6) hold by the definition. $\square$

In connection with Theorem 14 and Corollary 15 one can ask if the properties of not containing a Type 1 dendroid or being smooth are invariant under AQM mappings between dendroids and fans. An answer is negative by the following example.

Example 17. There exists an AQM mapping $f : X \to Y$ from the Cantor fan X onto a fan Y which contains a Type 1 subfan (so, Y is not smooth).

Proof. In the Cartesian coordinates in the plane put $v = (0,1)$ and for each real number $c \in \mathcal{C} \subset [0,1]$ let $e(c) = (c,0)$. Then

$$X = \bigcup\{\overline{ve(c)} : c \in \mathcal{C}\}$$

is the Cantor fan with the top v. Define an equivalence relation $\sim$ on X as follows. If $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ are points of X, then

$$p_1 \sim p_2 \qquad \Longleftrightarrow \qquad (p_1 = p_2) \text{ or } (x_1 = x_2 = 0 \text{ and } y_1 + y_2 = 1).$$

In other words, the only nondegenerate abstract classes of $\sim$ are two point sets in the segment $\overline{ve(0)}$ whose elements are symmetric with respect to the mid point $(0, \frac{1}{2})$ of the segment. Then the quotient space $Y = X/\sim$ is a fan that contains a Type 1 subfan, and the quotient mapping $f : X \to Y$ is AQM. Note that both fans X and Y are WAO. $\square$

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(Received: July 14, 2003)
(Revised: February 9, 2004)

Instituto de Matemáticas
UNAM, Circuito Exterior
Ciudad Universitaria
04510 México, D. F.
México
E–mail: jjc@math.unam.mx

Kvazimonotona preslikavanja po lukovima na fanove

Janusz J. Charatonik

Sadržaj

Za denroidu X kažemo da je *slabo otvorena po luku* ako je za svaku tačku p iz X svaka lučna komponenta skupa $X \setminus \{p\}$ ili otvorena ili ima praznu unutrašnjost. Neka je f kvazimonotono preslikavanje po luku iz dentroide X na fan Y. Proučavaju se uvjeti (vezani za strukturu domena prostora X ili za preslikavanje f) pod kojim će f očuvati osobinu da bude slabo otvorena po luku.