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Slant submanifolds of a Kenmotsu manifold

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Abstract. In this paper we have obtained some results on slant submanifolds of a Kenmotsu manifold. We have given a necessary and sufficient condition for a 3–dimensional submanifold of a 5–dimensional Kenmotsu manifold to be a minimal proper slant submanifold.

1. Introduction

The notion of a slant submanifold of an almost Hermitian manifold was introduced and studied by B.Y. Chen [3, 4]. Examples of slant submanifolds of C^2 and C^4 were given by Chen and Tazawa [4, 5, 6], while that of slant submanifolds of a Kaehler manifold were given by Maeda, Ohnita and Udagawa [11]. On the other hand A. Lotta [1] has defined and studied slant submanifolds of an almost contact metric manifold. He has also studied the intrinsic geometry of 3–dimensional non–antiinvariant slant submanifolds of K–contact manifolds [2].

Later, L. Cabrerizo and others have investigated slant submanifolds of a Sasakian manifold and obtained many interesting results [8, 9].

Let $\overline{M}$ be a $(2m+1)$–dimensional almost contact metric manifold with structure tensors (φ, ξ, η, g), where φ is a $(1,1)$ tensor field, ξ a vector field, η a 1–form and g is the Riemannian metric on $\overline{M}$. These tensors satisfy [7]

$$\varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\varphi X) = 0 \tag{1.1}$$

and $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$, $\eta(X) = g(X,\xi)$ for any $X, Y \in T\overline{M}$, where $T\overline{M}$ denotes the tangent bundle of $\overline{M}$. An almost contact metric manifold is called Kenmotsu manifold if [10]

$$(\overline{\nabla}_X \varphi)(Y) = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad \overline{\nabla}_X \xi = X - \eta(X)\xi \tag{1.2}$$

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where $\overline{\nabla}$ denotes the Levi–Civita connection on $\overline{M}$.

Let M be an m–dimensional Riemannian manifold with induced metric g isometrically immersed in $\overline{M}$. We denote by TM and $T^{\perp}M$ the tangent and the normal bundles of M respectively.

For any $X \in TM$ and $N \in T^{\perp}M$, we write

$$\varphi X = PX + FX \text{ and } \varphi N = tN + fN \tag{1.3}$$

where PX (resp. FX) denotes the tangential (resp. normal) component of φX, and tN (resp. fN) denotes the tangential (resp. normal) component of φN.

In what follows, we suppose that the structure vector field ξ is tangent to M. Hence, if we denote by D the orthogonal distribution to ξ in TM, we can consider the orthogonal direct decomposition $TM = D \oplus \{\xi\}$.

For each non zero X tangent to M at x such that X is not proportional to ξ_x, we denote by $\theta(X)$ the Wirtinger angle of X, that is, the angle between φX and T_xM.

The submanifold M is called slant if the Wirtinger angle $\theta(X)$ is a constant, which is independent of the choice of $x \in M$ and $X \in T_xM - \{\xi_x\}$[1]. The Wirtinger angle θ of a slant immersion is called the slant angle of the immersion. Invariant and anti–invariant immersions are slant immersions with slant angle θ equal to 0 and $\pi/2$, respectively. A slant immersion which is neither invariant nor anti–invariant is called a proper slant immersion.

Let ∇ be the Riemannian connection on M. Then the Gauss and Weingarten formulae are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{1.4}$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{1.5}$$

for $X, Y \in TM$, $N \in T^{\perp}M$; and where h and A_N are the second fundamental forms related by

$$g(A_N X, Y) = g\big(h(X,Y), N\big) \tag{1.6}$$

and $\nabla^{\perp}$ is the connection in the normal bundle $T^{\perp}M$ of M.

The mean curvature vector H is defined by $H = (1/m)$ trace h. We say that M is minimal if H vanishes identically.

If P is the endomorphism defined by (1.3), then

$$g(PX, Y) + g(X, PY) = 0 \tag{1.7}$$

Thus P^2, which is denoted by Q, is self–adjoint. We define the covariant derivatives of Q, P and F as

$$(\nabla_X Q)Y = \nabla_X(QY) - Q(\nabla_X Y) \tag{1.8}$$
$$(\nabla_X P)Y = \nabla_X(PY) - P(\nabla_X Y) \tag{1.9}$$

and

$$(\nabla_X F)Y = \nabla^{\perp}_X FY - F(\nabla_X Y) \tag{1.10}$$

for $X, Y \in TM$.
If

$$(\nabla_X P)Y = -\eta(Y)PX + g(Y, PX)\,\xi \tag{1.11}$$

then

$$(\nabla_X Q)Y = -\eta(Y)QX - g(QX, Y)\,\xi \tag{1.12}$$

for $X, Y \in TM$.
On the other hand, Gauss and Weingarten formulae together with (1.2) and (1.3) imply

$$(\nabla_X P)Y = A_{FY}X + th(X, Y) + g(Y, PX)\,\xi - \eta(Y)PX \tag{1.13}$$
$$(\nabla_X F)Y = fh(X, Y) - h(X, PY) - \eta(Y)FX \tag{1.14}$$

for any $X, Y \in TM$. It can be easily verified that (1.11) exists if and only if

$$A_{FY}X = A_{FX}Y \tag{1.15}$$

Using (1.14), a similar calculation shows that

$$(\nabla_X F)Y = -\eta(Y)FX \ \text{ if and only if } \ A_N PY = -A_{fN}Y \tag{1.16}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$.

2. Slant submanifolds of Kenmotsu manifolds

In the present section we prove a characterization theorem for slant submanifolds of a Kenmotsu manifold. We mention the following results for later use.

Theorem A. [8] *Let M be a submanifold of an almost contact metric manifold $\overline{M}$ such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$P^2 = -\lambda(I - \eta \otimes \xi). \tag{2.1}$$

Further more, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

Corollary B. [8] *Let M be a slant submanifold of an almost contact metric manifold $\overline{M}$ with slant angle θ. Then, for any $X, Y \in TM$, we have*

$$g(PX, PY) = \cos^2\theta\big(g(X,Y) - \eta(X)\eta(Y)\big) \tag{2.2}$$
$$g(FX, FY) = \sin^2\theta\big(g(X,Y) - \eta(X)\eta(Y)\big). \tag{2.3}$$

Lemma C. [1] *Let M be a slant submanifold of an almost contact metric manifold $\overline{M}$ with slant angle θ. Then, at each point x of $M, Q\mid_D$ has only one eigenvalue $\lambda_1 = -\cos^2\theta$.*

We now prove

Theorem 2.1. *Let M be a slant submanifold of a Kenmotsu manifold $\overline{M}$ such that $\xi \in TM$. Then, $\nabla Q = 0$ if and only if M is an anti–invariant submanifold.*

Proof. Denote by θ the slant angle of M. Then from (2.1), we get

$$Q(\nabla_X Y) = -\cos^2\theta(\nabla_X Y) + \cos^2\theta\,\eta(\nabla_X Y)\,\xi \tag{2.4}$$

for any $X, Y \in TM$. On the other hand, by taking the covariant derivative of (2.1), we get

$$\begin{aligned}\nabla_X QY = &-\cos^2\theta(\nabla_X Y) + \cos^2\theta\,\eta(\nabla_X Y)\,\xi\\ &+\cos^2\theta\,g(Y, \nabla_X\xi)\,\xi + \cos^2\theta\,\eta(Y)\nabla_X\,\xi\end{aligned} \tag{2.5}$$

Now, since M is a submanifold of Kenmotsu manifold $\overline{M}$, from (1.2), we have $\nabla_X\,\xi = X - \eta(X)\,\xi$ for any $X \in TM$. Putting the value of $\nabla_X\,\xi$ in (2.5), we obtain

$$\begin{aligned}\nabla_X QY = &-\cos^2\theta(\nabla_X Y) + \cos^2\theta\eta(\nabla_X Y)\,\xi + \cos^2\theta\big(g(Y,X)\,\xi - \eta(X)\eta(Y)\,\xi\big)\\ &+\cos^2\theta\big(\eta(Y)X - \eta(X)\eta(Y)\,\xi\big)\end{aligned} \tag{2.6}$$

Combining (2.4) and (2.6), we have

$$(\nabla_X Q)Y = \cos^2\theta\big(g(Y,X)\,\xi - 2\eta(X)\eta(Y)\,\xi + \eta(Y)X\big) \tag{2.7}$$

for any $X, Y \in TM$. Here, we note that

$$g(Y,X)\,\xi - 2\eta(X)\eta(Y)\,\xi + \eta(Y)X \neq 0$$

Hence $\nabla Q = 0$ if and only if $\theta = \pi/2$, which proves our assertion. □

Now, we state the main result of this section.

Theorem 2.2. *Let M be a submanifold of a Kenmotsu manifold $\overline{M}$ such that $\xi \in TM$. Then M is slant if and only if*

(a) *The endomorphism $Q\mid_D$ has only one eigenvalue at each point of M.*

(b) *There exists a function $\lambda : M \to [0,1]$ such that*

$$(\nabla_X Q)Y = \lambda\big(g(Y,X)\,\xi - 2\eta(X)\eta(Y)\,\xi + \eta(Y)X\big) \tag{2.8}$$

for any $X, Y \in TM$.

Moreover, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

Proof. Suppose that M is slant. Then, statements (a) and (b) follow directly from Lemma C and equation (2.7), respectively. Conversely, let $\lambda_1(x)$ be the eigenvalue of $Q\mid_D$ at each point x of M and $Y \in D$ be a unit eigenvector associated with λ_1, i.e., $QY = \lambda_1 Y$. Then, from (b), we have

$$X(\lambda_1)Y + \lambda_1\nabla_X Y = \nabla_X(QY) = Q(\nabla_X Y) + \lambda\, g\,(Y,X)\,\xi \tag{2.9}$$

for any $X \in TM$. Since both $\nabla_X Y$ and $Q(\nabla_X Y)$ are perpendicular to Y, we conclude that λ_1 is constant on M.

Now we prove that M is slant. In view of (2.1), it is enough to show that there exists a constant μ such that $Q = -\mu\,(I - \eta\otimes\xi)$. Let X be in TM. Then $\widetilde{X} = X - \eta(X)\,\xi \in D$. Hence, $QX = Q\widetilde{X}$. Since $Q\mid_D = \lambda_1 I$ we have, $Q\widetilde{X} = \lambda_1\widetilde{X}$ and so $QX = \lambda_1\widetilde{X} = \lambda_1(X - \eta(X)\,\xi)$. By taking $\mu = -\lambda_1$, we get that M is slant. Moreover, if M is slant, it follows from (2.9) and (2.1) that $\lambda = -\lambda_1 = \mu = \cos^2\theta$, where θ denotes the slant angle of M.

3. Slant submanifolds of dimension three

In this section we study 3–dimensional slant submanifolds of Kenmotsu manifolds. To characterize these submanifolds we need the following lemma.

Lemma 3.1. *Let M be a 3–dimensional submanifold of an almost contact metric manifold $\overline{M}$ such that $\xi \in TM$. Then, in a neighborhood of a point $p \in M$, there exist vector fields e_1, e_2 tangent to M such that the basis $\{e_1, e_2, \xi\}$ form a local orthonormal frame satisfying*

$$Pe_1 = \lambda_1 e_2, \quad Pe_2 = -\lambda_1 e_1,$$

where λ_1 is a locally defined function on M. If M is a slant with slant angle θ then we may take $\lambda_1 = \cos\theta$.

Proof. By (1.1), (1.3) and (1.7), we have $Pe_1 = \lambda_1 e_2$, where $\lambda_1 = g(Pe_1, e_2)$. In the same way, we have $Pe_2 = \mu_1 e_1$, where $\mu_1 = g(Pe_2, e_1)$. However, it follows from (1.7) that $\mu_1 = g(Pe_2, e_1) = -g(e_2, Pe_1) = -\lambda_1$. $\square$

Now, we prove

Theorem 3.1. *Let M be a 3–dimensional submanifold of a Kenmotsu manifold $\overline{M}$ such that $\xi \in TM$. Then M is slant if and only if P satisfies*

$$(\nabla_X P)Y = -\eta(Y)PX + g(Y, PX)\,\xi \tag{3.1}$$

for any $X, Y \in TM$.

Proof. By Lemma 3.1, $Q\mid_D$ has only one eigenvalue $-\lambda_1^2$. Moreover, $QX = -\lambda_1^2(X - \eta(X)]\,\xi)$ holds. It follows from (1.11) and (1.12) that Q satisfies (2.8) with $\lambda = -\lambda_1^2$. Thus, if P satisfies (3.1) then Theorem 2.2 implies that M is slant. Conversely, assume that M is slant. Let $p \in M$ and $\{e_1, e_2, \xi\}$ be the orthonormal frame in a neighborhood U of p as in Lemma 3.1. Let ω_i^j be the structural 1–forms defined by

$$\nabla_X e_i = \sum_{j=1}^{3} \omega_i^j\,(X)e_j$$

for each vector field X tangent to M.
Notice that by (1.2),

$$(\nabla_X P)e_3 = \nabla_X Pe_3 - P(\nabla_X e_3) = -PX.$$

Similarly, we get

$$(\nabla_X P)e_1 = \cos\theta\,\omega_2^3(X)e_3$$

and

$$(\nabla_X P)e_2 = -\cos\theta\,\omega_1^3(X)e_3.$$

On the other hand, writing

$$Y = \eta(Y)e_3 + g\,(Y, e_1)e_1 + g\,(Y, e_2)e_2,$$

for all $Y \in TM$ and using the above formulae it follows that

$$(\nabla_X P)Y = -\eta(Y)PX + g\,(Y, PX)\,\xi$$

and we have (3.1). $\square$

From Theorem 3.1. and the equivalence between (1.11) and (1.15), we obtain the following characterization of 3–dimensional slant submanifolds of Kenmotsu manifolds in terms of the Weingarten map.

Corollary 3.1. *Let M be a 3–dimensional submanifold of a Kenmotsu manifold $\overline{M}$. Then, M is slant if and only if*

$$A_{FY}X = A_{FX}Y$$

for any $X, Y \in TM$.

If M is an invariant submanifold of a Kenmotsu manifold $\overline{M}$, then (3.1) also holds and $\nabla F = 0$ is satisfied. On the other hand, if M is an anti–invariant submanifold, it is obvious that $\nabla P = 0$, i.e. (3.1) holds. We also know that

$$(\nabla_X F)Y = fh(X,Y) - \eta(Y)FX, \text{ for any } X, Y \in TM.$$

The following result gives us a sufficient condition for $\nabla F = 0$.

Proposition 3.1. *Let M be an anti–invariant submanifold of a Kenmotsu manifold $\overline{M}$ such that $\xi \in TM$. Suppose that* $\dim M = 3$, $\dim \overline{M} = 5$ *and $TM = D \oplus \{\xi\}$. Then, $\nabla_X F \mid_D = 0$ for all $X \in TM$.*

Proof. From (2.3), if we choose a local orthonormal frame $\{e_1, e_2, \xi\}$ of TM, then $\{Fe_1, Fe_2\}$ is a local orthonormal frame of $T^{\perp}M$ and so $f = 0$ and for all $Y \in D$, $\eta(Y) = 0$. Consequently, we obtain $(\nabla_X F)Y = 0$ for all $Y \in D$ and for any $X \in TM$. □

We now calculate the value of ∇F for a three–dimensional slant submanifold M of a Kenmotsu manifold $\overline{M}$ with dim = 5.

Suppose that M is proper slant with slant angle θ. Then, for a unit tangent vector field e_1 of M perpendicular to ξ, we put

$$e_2 = (\sec\theta)Pe_1, \quad e_3 = \xi, \quad e_4 = (\csc\theta)Fe_1, \quad e_5 = (\csc\theta)Fe_2.$$

Then $e_1 = -(\sec\theta)Pe_2$ and by virtue of (2.2) and (2.3), e_1, e_2, e_3, e_4, e_5 form an orthonormal frame such that e_1, e_2, e_3 are tangent to M and e_4, e_5 are normal to M. We call such an orthonormal frame an adapted slant frame. We also have

$$te_4 = -\sin\theta e_1, \quad te_5 = -\sin\theta e_2, \quad fe_4 = -\cos\theta e_5, \quad fe_5 = \cos\theta e_4.$$

If we put $h_{ij}^r = g\big(h(e_i,e_j),e_r\big)$, $i,j = 1,2,3$, $r = 4,5$, then we have the following

Lemma 3.2. *Under the above conditions, we have*

$$h_{12}^4 = h_{11}^5, \qquad h_{22}^4 = h_{12}^5 \tag{3.2}$$

$$h_{13}^4 = h_{32}^4 = h_{33}^4 = h_{13}^5 = h_{23}^5 = h_{33}^5 = 0. \tag{3.3}$$

Proof. We obtain (3.2) by virtue of Corollary 3.1, while (3.3) holds because $\overline{M}$ is a Kenmotsu manifold.

We prove

Theorem 3.2. *Let M be a 3–dimensional submanifold of a 5–dimensional Kenmotsu manifold $\overline{M}$ such that $\xi \in TM$. Then M is a minimal proper slant submanifold of $\overline{M}$ if and only if*

$$(\nabla_X F)Y = -\eta(Y)FX \tag{3.4}$$

for any $X,Y \in TM$.

Proof. To prove (3.4), we choose e_1,e_2,e_3,e_4,e_5 as an adapted slant frame. Then, (3.4) follows from (1.14), (3.2) and (3.3). Now let (3.4) hold. We choose a unit local vector field e_1 perpendicular to ξ such that

$$P^2 e_1 = -\cos^2\theta_1 e_1,$$

where $\theta_1 = \theta(e_1) \in (0,\pi/2)$ denotes the Wirtinger angle of e_1. Define a local orthonormal frame formed by e_1,e_2,e_3,e_4,e_5 such that

$$e_2 = (\sec\theta_1)Pe_1, \quad e_3 = \xi, \quad e_4 = (\csc\theta_1)Fe_1, \quad e_5 = (\csc\theta_1)Fe_2$$

and

$$te_4 = -\sin\theta_1 e_1, \quad te_5 = -\sin\theta_1 e_2, \quad fe_4 = -\cos\theta_1 e_5, \quad fe_5 = \cos\theta_1 e_4.$$

Since $(\nabla_X F)Y = -\eta(Y)FX$, from (1.16) we have

$$A_N PY = -A_{fN}Y \text{ for any } X,Y \in TM \text{ and } N \in T^\perp M.$$

Therefore

$$A_{Fe_1}e_2 = A_{Fe_2}e_1 \text{ and } A_{Fe_1}e_3 = A_{Fe_2}e_3 = 0,$$

which gives $A_{FX}Y = A_{FY}X$, for any $X,Y \in TM$.

Thus, from Corollary 3.1, we can say that M is a slant submanifold with $\theta_1 = \theta(e_1) \in (0, \pi/2)$. It is easy to show that M is also minimal. □

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“Slant” podmnogostrukosti Kenmotsu mnogostrukosti

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Sadržaj

Autori su u ovom radu dobili neke rezultate o “slant” podmnogostrukostima Kenmotsu mnogostrukosti. Dat je potreban i dovoljan uvjet da bi 3–dimenzionalna podmnogostrukost 5–dimenzionalne Kenmotsu mnogostrukosti bila minimalna prava “slant” podmnogostrukost.