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A note on the characteristic properties of Möbius transformations

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Abstract. In this paper, we give a new characterization of Möbius transformations. To do this, we use the notion of $\lambda-$Apollonius points of a $(2n-1)$–gon.

1. Introduction

Recently, several new characteristics of Möbius transformations were given (see [3], [4], [5], [6], [8], [10], [1] and [2]). The purpose of this paper is to give a new characterization of Möbius transformations. To do this, we introduce the notion of $\lambda-$Apollonius point of a $(2n-1)-$gon in the next section where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2n-2})$ and $\lambda_i \in \mathbb{R}^+$ $(i = 1, 2, \ldots, 2n-2)$. This definition gives us the definition of $(k, l)-$Apollonius points of triangles for $n = 2$ (see [8]) and the definition of $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)-$Apollonius points of pentagons for $n = 3$ (see [2]).

Throughout the paper, unless otherwise stated, let $w = f(z)$ be a non-constant meromorphic function on the complex plane $\mathbb{C}$.

2. $\lambda-$Apollonius points of a $(2n-1)-$gon

Definition 2.1. Let $Z = Z_1 Z_2 \ldots Z_{2n-1}$ be an arbitrary $(2n-1)-$gon (not necessarily simple) and L be a point on $\mathbb{C}$. If the following equality holds for $2 \leq k \leq 2n-1$

$$|L - Z_1| \prod_{i=1}^{n-1} |Z_{2i} - Z_{2i+1}| = \lambda_{k-1} |L - Z_k| \prod_{i=1}^{n-1} |Z_{2i+k-1} - Z_{2i+k}|$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{2n-2} \in \mathbb{R}^+$, then L is said to be a $(\lambda_1, \lambda_2, \ldots, \lambda_{2n-2})-$Apollonius point of Z. Here if the values of $2i+k-1$ and $2i+k$ are different

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from $2n-1$, then we consider these values in $mod(2n-1)$. We denote this as (λ, Z, L). $\square$

For example, let $Z = Z_1Z_2\dots Z_{2n-1}$ be an arbitrary regular $(2n-1)-$gon. Then, the center of circumscribed circle of Z is its only $(1,1,\dots,1)-$Apollonius point.

Note that for $\lambda_1 = \lambda_2 = \cdots = \lambda_{2n-2} = 1$, this definition coincides with the definition of an Apollonius point of an arbitrary $(2n-1)-$gon which is given in [1]. Let $Z = Z_1Z_2\dots Z_{2n-1}$ be an arbitrary $(2n-1)-$gon on the complex plane and let the positive real numbers $\lambda_1, \lambda_2, \dots, \lambda_{2n-2}$ be fixed. Then the number of $\lambda-$Apollonius points of Z is at most 2. This follows from the Theorem of Apollonius, [4], and from the fact that if two circles meet, including straight lines among circles, then there are at most two points of intersection.

For the proof of the main theorem, we need the following definition and theorem given in [10].

Definition 2.2. A $2n-$gon (not necessarily simple) in the complex plane is called Apollonius if for the consecutive vertices $z_1, z_2, \dots, z_{2n} \in \mathbb{C}$, the following condition holds

$$A(z_1, z_2, \dots, z_{2n}) = 1,$$

where

$$A(z_1, z_2, \dots, z_{2n}) = \frac{|(z_1 - z_2)(z_3 - z_4)\cdots(z_{2n-1} - z_{2n})|}{|(z_2 - z_3)(z_4 - z_5)\cdots(z_{2n-2} - z_{2n-1})(z_{2n} - z_1)|}.$$

Theorem 2.3. *If f is an analytic univalent function on an open region Δ, then the following propositions are equivalent:*

(i) *f is a Möbius Transformation.*

(ii) *There is $c > 0$ such that $A(f(z_1), f(z_2), \dots f(z_{2n})) = c$, for every $z_1, z_2, \dots, z_{2n} \in \Delta$ with $A(z_1, z_2, \dots, z_{2n}) = c$.*

Now we can give the our main theorem.

Theorem 2.4. *If $w = f(z)$ is analytic and univalent in a nonempty domain R, then the following propositions are equivalent:*

(i) *$f(z)$ is a Möbius transformation.*

(ii) *For every (λ, Z, L): L is a $\lambda-$Apollonius point of the $(2n-1)-$gon $Z = Z_1Z_2\dots Z_{2n-1}$ contained in R, then $f(L)$ is a $\lambda-$Apollonius point of the $(2n-1)-$gon $Z' = Z'_1Z'_2\dots Z'_{2n-1}$ where $Z'_i = f(Z_i), 1 \le i \le 2n-1$.*

Proof. Let $w = f(z)$ is a Möbius transformation. Suppose that $Z = Z_1 Z_2 \dots Z_{2n-1}$ be an arbitrary $(2n-1)-$gon contained in R and let its λ-Apollonius point L be a point of R. If we set $Z_i' = f(Z_i)$ for $1 \leq i \leq 2n-1$, then by the univalency of $w = f(z)$, the $2n-1$ points Z_i' $(1 \leq i \leq 2n-1)$ are different.

We now prove that if any triple of Z_i' $(1 \leq i \leq 2n-1)$ are not collinear, then the point $L' = f(L)$ is also a $\lambda-$Apollonius point of $Z' = Z_1' Z_2' \dots Z_{2n-1}'$. Since L is a $\lambda-$Apollonius point of Z, by the Definition 2.1, for $k = 2n-1$, we have

$$|L - Z_1| \prod_{i=1}^{n-1} |Z_{2i} - Z_{2i+1}| = \lambda_{2n-2} |L - Z_{2n-1}| \prod_{i=1}^{n-1} |Z_{2i+2n-2} - Z_{2i+2n-1}|$$

and so

$$\begin{aligned} &|L - Z_1| \cdot |Z_2 - Z_3| \cdots |Z_{2n-2} - Z_{2n-1}| \\ &= \lambda_{2n-2} |L - Z_{2n-1}| \cdot |Z_1 - Z_2| \cdots |Z_{2n-3} - Z_{2n-2}| . \end{aligned}$$

Therefore by Definition 2.2, $LZ_1 Z_2 \dots Z_{2n-1}$ is a $\lambda_{2n-2}-$Apollonius $2n-$gon. By Theorem 2.2, $L' Z_1' Z_2' \dots Z_{2n-1}'$ is a $\lambda_{2n-2}-$Apollonius $2n-$gon. Hence we obtain

$$|L' - Z_1'| \prod_{i=1}^{n-1} \left|Z_{2i}' - Z_{2i+1}'\right| = \lambda_{2n-2} \left|L' - Z_{2n-1}'\right| \prod_{i=1}^{n-1} \left|Z_{2i+2n-2}' - Z_{2i+2n-1}'\right| . \quad (1)$$

Similarly, we have

$$\lambda_{2n-2} \left|L' - Z_{2n-1}'\right| \prod_{i=1}^{n-1} \left|Z_{2i+2n-2}' - Z_{2i+2n-1}'\right| \quad (2)$$

$$= \lambda_{2n-3} \left|L' - Z_{2n-2}'\right| \prod_{i=1}^{n-1} \left|Z_{2i+2n-3}' - Z_{2i+2n-2}'\right| \quad (3)$$

$$= \cdots = \lambda_2 |L' - Z_3'| \prod_{i=1}^{n-1} \left|Z_{2i+2}' - Z_{2i+3}'\right| = \lambda_1 |L' - Z_2'| \prod_{i=1}^{n-1} \left|Z_{2i+1}' - Z_{2i+2}'\right| . \quad (4)$$

By (1) – (4), we obtain that the following products are equal for every $2 \leq k \leq 2n-1$:

$$|L' - Z_1'| \prod_{i=1}^{n-1} \left|Z_{2i}' - Z_{2i+1}'\right| = \lambda_{k-1} |L' - Z_k'| \prod_{i=1}^{n-1} \left|Z_{2i+k-1}' - Z_{2i+k}'\right| .$$

By definition, we obtain that $L' = f(L)$ is also a $(\lambda_1, \lambda_2, \dots, \lambda_{2n-2})-$Apollonius point of Z'. Consequently, $w = f(z)$ satisfies (ii).

Now assume that the function $w = f(z)$ satisfies (ii). Since $w = f(z)$ is analytic and univalent in the domain R, by a well-known lemma

$$f'(z) \neq 0 \tag{5}$$

holds in R.

If x is an arbitrarily fixed point of R, then by (5), we obtain

$$f'(x) \neq 0. \tag{6}$$

Let L be the point represented by x. Since $L \in R$, there exists a positive real number r such that the r closed circular neighborhood of L is contained in R. We denote this closed circular neighborhood by V.

Then by choosing an arbitrary regular $(2n-1)-$gon which is contained in V and whose center is at L, the proof follows easily as in [1]. □

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Neke primjedbe o karakterističnim svojstvima Möbiusovih transformacija

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Sadržaj

U radu se daje nova karakterizacija Möbiusovih transformacija. U tu svrhu se koristi pojam $\lambda-$Apolloniusovih tačaka $(2n-1)$–gona.