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Common fixed points for compatible mappings in metric spaces

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Abstract. In the present paper, a common fixed point theorem for two pairs of compatible mappings is proved under a new contractive condition, which is independent of the known contractive definitions.

1. Introduction

The study of common fixed point of mappings satisfying contractive type conditions has been a very active field of research activity during the last two decades. The most general of the common fixed point theorems pertain to four mappings, say A, B, S and T of a metric space (X, d), and use either a Banach type contractive condition of the form

$$d(Ax, By) \leq hm(x, y), \quad 0 \leq h < 1, \tag{1}$$

where

$$m(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Sx, By) + d(Ax, Ty)]/2\},$$

or, a Meir-Keeler type (ε, δ)–contractive condition of the form given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\varepsilon \leq m(x, y) < \varepsilon + \delta \Rightarrow d(Ax, By) < \varepsilon, \tag{2}$$

or, a ϕ–contractive condition of the form

$$d(Ax, By) \leq \phi(m(x, y)), \tag{3}$$

involving a contractive gauge function $\phi : R_+ \to R_+$ is such that $\phi(t) < t$ for each $t > 0$.

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Clearly, condition (1) is a special case of both conditions (2) and (3). A ϕ–contractive condition (3) does not guarantee the existence of a fixed point unless some additional condition is assumed. Therefore, to ensure the existence of common fixed point under the contractive condition (3), the following conditions on the function ϕ have been introduced and used by various authors.

(I) $\phi(t)$ is non decreasing and $t/(t-\phi(t))$ is non increasing ([2]),
(II) $\phi(t)$ is non decreasing and $\lim_n \phi^n(t) = 0$ for each $t > 0$ ([3], [8]),
(III) ϕ is upper semi continuous ([1], [3], [7], [11]) or equivalently,
(IV) ϕ is non decreasing and continuous from right ([17]).

It is now known (e.g., [3], [12]) that if any of the conditions (I), (II), (III) or (IV) is assumed on ϕ, then a ϕ–contractive condition (3) implies an analogous (ε, δ)–contractive condition (2) and both the contractive conditions hold simultaneously. Similarly, a Meir– Keeler type (ε, δ)–contractive condition does not ensure the existence of a fixed point. The following example illustrates that an (ε, δ)–contractive condition of type (2) neither ensures the existence of a fixed point nor implies an analogous ϕ–contractive condition (3).

Example 1. ([12]) Let $X = [0, 2]$ and d be the Euclidean metric on X. Define $f : X \to X$ by $f(x) = (1 + x)/2$ if $x < 1$; $f(x) = 0$ if $x \geq 1$. Then, it satisfies the contractive condition

$$\varepsilon \leq \max\{d(x,y), d(x,fx), d(y,fy), [d(x,fy)+d(y,fx)]/2\} < \varepsilon+\delta \Rightarrow d(fx,fy) < \varepsilon,$$

with $\delta(\varepsilon) = 1$ for $\varepsilon \geq 1$ and $\delta(\varepsilon) = 1-\varepsilon$ for $\varepsilon < 1$ but f does not have a fixed point. Also f does not satisfy the contractive condition

$$d(fx,fy) \leq \phi \max\{d(x,y), d(x,fx), d(y,fy), [d(x,fy) + d(y,fx)]/2\}),$$

since the desired function $\phi(t)$ cannot be defined at $t = 1$.

Hence, the two type of contractive conditions (2) and (3) are independent of each other. Thus, to ensure the existence of common fixed point under the contractive condition (2), the following conditions on the function δ have been introduced and used by various authors.

(V) δ is non decreasing ([10], [11])
(VI) δ is lower semi continuous ([5], [6]).

Jachymski [3] has shown that the (ε, δ)–contractive condition (2) with a non decreasing δ implies a ϕ–contractive condition (3). Also, Pant et al. [12] have shown that the (ε, δ)–contractive condition (2) with a lower semi continuous δ, implies a ϕ–contractive condition (3). Thus, we see that if additional conditions are assumed on δ then the (ε, δ)–contractive condition (2) implies an analogous ϕ–contractive condition (3) and both the contractive conditions hold simultaneously.

It is thus clear that contractive conditions (2) and (3) hold simultaneously whenever (2) or (3) is assumed with an additional condition on δ or ϕ respectively. It follows, therefore, that the known common fixed point theorems can be extended and generalized if instead of assuming one of the contractive condition (2) or (3) with additional conditions on δ and ϕ, we assume contractive condition (2) together with the following condition of the form

$$d(Ax, By) < k[d(Sx, Ty) + d(Ax, Sx) + d(By, Ty) + d(Sx, By) + d(Ax, Ty)],$$

for $0 \le k \le 1/3$.

2. Main results

In this paper we prove a common fixed point theorem for four mappings adopting this approach. This gives a new approach of ensuring the existence of fixed points under an (ε, δ)–contractive condition that consists of assuming additional conditions which are independent of the ϕ–contractive condition implied by (V) and (VI).

Two self–mappings A and S of a metric space (X, d) are called **compatible** (see Jungck [5]) if, $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = t$ for some t in X. It is easy to see that compatible maps commute at their coincidence points.

To prove our theorem, we shall use the following Lemma of Jachymski [3]:

Lemma (2.2 of [3]): *Let A, B, S and T be self mappings of a metric space (X, d) such that $AX \subset TX, BX \subset SX$. Assume further that given $\varepsilon > 0$ there exists $\delta > 0$ such that for all x, y in X*

$$\varepsilon < M(x, y) < \varepsilon + \delta \Rightarrow d(Ax, By) \le \varepsilon, \tag{4}$$

and

$$d(Ax, By) < M(x, y), \text{ whenever } M(x, y) > 0 \tag{5}$$

where $M(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Sx, By) + d(Ax, Ty)]/2\}$. Then for each x_0 in X, the sequence $\{y_n\}$ in X defined by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1};\ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$$

is a Cauchy sequence.

Jachymski [3] has shown that contractive condition (2) implies (4) but contractive condition (4) does not imply the contractive condition (2).

Theorem 1. *Let (A, S) and (B, T) be compatible pairs of self mappings of a complete metric space (X, d) such that*

(i) $AX \subset TX, BX \subset SX$,

(ii) *given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x, y in X,*

$$\varepsilon \leq M(x, y) < \varepsilon + \delta \Rightarrow d(Ax, By) < \varepsilon, \text{ and}$$

(iii) $d(Ax, By) < k[d(Sx, Ty) + d(Ax, Sx) + d(By, Ty) + d(Sx, By) + d(Ax, Ty)]$, *for $0 \leq k \leq 1/3$. If one of the mappings A, B, S and T is continuous then A, B, S and T have unique common fixed point.*

Proof. Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1};\ y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}. \tag{6}$$

This can be done by virtue of (i). Since the contractive condition (ii) of this theorem implies the contractive conditions (4) and (5) of Lemma 2.2 of Jachymski, so using this Lemma, we conclude that $\{y_n\}$ is a Cauchy sequence in X. But X is complete so there exists a point z in X such that $y_n \to z$. Also, using (6), we have

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z. \tag{7}$$

Suppose that S is continuous. Then $SSx_{2n} \to Sz, SAx_{2n} \to Sz$ and compatibility of A and S implies that $ASx_{2n} \to Sz$. Using (iii), we have

$$\begin{aligned} d(ASx_{2n}, Bx_{2n+1}) < k[&d(SSx_{2n}, Tx_{2n+1}) + d(SSx_{2n}, ASx_{2n}) + d(Tx_{2n+1}, Bx_{2n+1}) \\ &+ d(SSx_{2n}, Bx_{2n+1}) + d(Tx_{2n+1}, ASx_{2n})]. \end{aligned}$$

Letting $n \to \infty$ we have $d(Sz, z) \leq 3kd(Sz, z)$, which implies that $Sz = z$. Further, by (iii), we have

$$\begin{aligned} d(Az, Bx_{2n+1}) < k[&d(Sz, Tx_{2n+1}) + d(Sz, Az) + d(Tx_{2n+1}, Bx_{2n+1}) \\ &+ d(Sz, Bx_{2n+1}) + d(Tx_{2n+1}, Az)] \end{aligned}$$

and letting n tend to infinity, we get $d(Az, z) \leq 2kd(z, Az)$ which implies that $z = Az$. This means that z is in the range of A and since $AX \subset TX$, there exists a point u in X such that $Tu = z$. Using (iii), we have successively

$$d(Az, Bu) < k[d(Sz, Tu) + d(Sz, Az) + d(Tu, Bu) + d(Sz, Bu) + d(Tu, Az)];$$

that is, $d(z, Bu) < 2kd(z, Bu)$ which implies that $z = Bu$. Also, since $Tu = Bu = z$, by the compatibility of (B, T), it follows that $BTu = TBu$ and so $Bz = BTu = TBu = Tz$. Thus, from (iii), we have successively

$$d(Az, Bz) < k[d(Sz, Tz) + d(Sz, Az) + d(Tz, Bz) + d(Sz, Bz) + d(Tz, Az)];$$

that is, $d(z, Bz) < 3kd(z, Bz)$ which implies that $z = Bz$. Thus $z = Bz = Tz$. Therefore we have proved that z is a common fixed point of A, B, S, T. The same result holds if we assume that T is continuous instead of S.

Now suppose that A is continuous. Then by the continuity of A and the compatibility of (A, S), we have that ASx_{2n} and SAx_{2n} converge to Az. Using inequality (iii), we have

$$\begin{aligned} d(AAx_{2n}, Bx_{2n+1}) < k[&d(SAx_{2n}, Tx_{2n+1}) + d(SAx_{2n}, AAx_{2n}) + d(Tx_{2n+1}, Bx_{2n+1}) \\ &+ d(SAx_{2n}, Bx_{2n+1}) + d(Tx_{2n+1}, AAx_{2n})]. \end{aligned}$$

Letting $n \to \infty$, we have $d(Az, z) \leq 3kd(Az, z)$, which implies that $z = Az$. This means that z is in the range of A and since $AX \subset TX$, there exists a point v in X such that $Tv = z$. Thus, by using (iii), we have

$$\begin{aligned} d(AAx_{2n}, Bv) < k[&d(SAx_{2n}, Tv) + d(SAx_{2n}, AAx_{2n}) + d(Tv, Bv) \\ &+ d(SAx_{2n}, Bv) + d(Tv, AAx_{2n})]. \end{aligned}$$

Letting $n \to \infty$, we get $d(z, Bv) \leq 2kd(z, Bv)$, which implies that $z = Bv$. Again, since $Tv = Bv = z$, by compatibility of (B, T), it follows that $Bz = BTv = TBv = Tz$. Thus from (iii), we have

$$\begin{aligned} d(Ax_{2n}, Bz) < k[&d(Sx_{2n}, Tz) + d(Sx_{2n}, Ax_{2n}) + d(Tz, Bz) + d(Sx_{2n}, Bz) \\ &+ d(Tz, Ax_{2n})]. \end{aligned}$$

Letting $n \to \infty$ we obtain $d(z, Bz) \leq 3kd(z, Bz)$, which implies that $z = Bz = Tz$. This means that z is in the range of B and since $BX \subset SX$, there exists a point w in X such that $Sw = z$. Then, by using (iii), we have successively

$$d(Aw, Bz) < k[d(Sw, Tz) + d(Sw, Aw) + d(Tz, Bz) + d(Sw, Bz) + d(Tz, Aw)];$$

that is, $d(Aw, z) \leq 2kd(Aw, z)$ which implies that $z = Aw = Sw$. By the compatibility of (A, S), it follows that $z = Az = ASw = SAw = Sz$. Thus $z = Sz$. Therefore we have proved that z is a common fixed point of A, B, S, T. The same result holds if we assume that B is continuous instead of A. This establishes the theorem. We now give an example to illustrate the above theorem.

Example 2. Let $X = [2, 20]$ and d be the Euclidean metric on X. Define A, B, S and $T : X \to X$ as follows:

$$\begin{array}{ll} Ax = 2 \text{ for each } x & \\ Bx = 2 \text{ if } x < 4 \text{ or } \geq 5, & Bx = 3 + x \text{ if } 4 \leq x < 5 \\ Sx = x \text{ if } x \leq 8, & Sx = 8 \text{ if } x > 8; \\ Tx = 2 \text{ if } x < 4 \text{ or } \geq 5, & Tx = 9 + x \text{ if } 4 \leq x < 5. \end{array}$$

Then A, B, S and T satisfy all the conditions of the above theorem and have a unique common fixed point $x = 2$. It can be seen in this example that A, B, S and T satisfy the condition (ii) when $\delta(\varepsilon) = 1$ if $\varepsilon \geq 6$ and $\delta(\varepsilon) = 6 - \varepsilon$ if $\varepsilon < 6$. Thus, $\delta(\varepsilon)$ is neither non decreasing nor lower semi continuous. However, A, B, S, and T do not satisfy the ϕ–contractive condition (3) since the required function $\phi(t)$ can not be defined at $t = 6$. Hence we see that the present example does not satisfy the conditions of any previously known common fixed point theorem for contractive type mappings, since neither the mappings satisfy a ϕ–contractive condition nor δ is lower semi continuous or non decreasing.

Remarks. Pant [15] has shown that condition (iii) of the above Theorem 1 is independent of ϕ–contractive conditions. Our result extends the results of Pant and Jha [14] and Pant [15] and gives a generalization of Meir–Keeler type common fixed point theorem. Also, as various assumptions either on ϕ or on δ have been considered to ensure the existence of common fixed points under contractive conditions, so our Theorem 1 improves and generalizes all other similar results of fixed points.

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REFERENCES

[1] **D.W. Boyd and J.S. Wong,** *On nonlinear contractions,* Proc. Amer. Math. Soc, 20 (1969), 458–464.

[2] **A. Carbone, B.E. Rhoades and S.P. Singh,** *A fixed point theorem for generalized contraction map,* Indian J. Pure Appl. Math, 20 (1989), 543–548.

[3] **J. Jachymski,** *Common fixed point theorems for some families of mappings,* Indian J. Pure Appl. Math, 25 (1994), 925–937.

[4] **J. Jachymski,** *Equivalent conditions and Meir–Keeler type theorems,* J. Math. Anal. Appl, 194 (1995), 293–303.

[5] **G. Jungck,** *Compatible mappings and common fixed points,* Internat. J. Math. Math. Sci, 9 (1986), 771–779.

[6] **G. Jungck, K.B. Moon, S. Park and B.E. Rhoades,** *On generalizations of the Meir–Keeler type contractive maps: Corrections,* J. Math. Anal. Appl, 180 (1993), 221–222.

[7] **M. Maiti and T.K. Pal,** *Generalization of two fixed point theorems,* Bull. Cal. Math. Soc, 70 (1978), 57–61.

[8] **J. Matkowski,** *Integrable solutions of functional equations,* Diss. Math, 127 (1975).

[9] **A. Meir and E. Keeler,** *A theorem on contraction mappings,* J. Math. Anal. Appl, 28 (1969), 326–329.

[10] **R.P. Pant,** *Common fixed points of weakly commuting mappings,* Math. Student, 62 (1993), 97–102.

[11] **R.P. Pant,** *Common fixed points of contractive maps,* J. Math. Anal. Appl, 226 (1998), 251–258.

[12] **R.P. Pant, P.C. Joshi and V. Gupta,** *A Meir–Keeler type fixed point theorem,* Indian J. Pure Appl. Math, 32 (6) (2001), 779–787.

[13] **R.P. Pant, A.B. Lohani and K. Jha,** *Meir–Keeler type fixed point theorem and reciprocal continuity,* Bull. Cal. Math. Soc, 94 (6) (2002), 459–466.

[14] **R.P. Pant and K. Jha,** *A generalization of Meir–Keeler type common fixed point theorem for four mappings,* J. Nat. Phys. Sci, 16 (1–2) (2002), 77–84.

[15] **R.P. Pant,** *Fixed point theorems and dynamics of functions,* J. Indian Math. Soc, 69 (1–4) (2002), (to appear).

[16] **S. Park and J.S. Bae,** *Extension of a fixed point theorem of Meir–Keeler,* Ark. Math, 19 (1991), 223–228.

[17] **S. Park and B.E. Rhoades,** *Extension of some fixed point theorems of Hegedus and Kasahara,* Math. Seminar Notes, 9 (1981), 113–118.

[18] **B.E. Rhoades,** *Contractive definitions and continuity,* Contemporary Math, 72 (1988), 233–247.

[19] **S.L. Singh and S. Kasahara,** *On some recent results on common fixed points,* Indian J. Pure Appl. Math, 13 (1982), 757–761.

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Zajedničke fiksne tačke za kompatibilno preslikavanje u metričkim prostorima

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Sadržaj

U radu je dokazan teorem zajedničke fiksne tačke za dva para kompatibilnog preslikavanja pod novim uvjetom kontrakcije, koji je nezavisan od poznatih definicija kontrakcije.