RADOVI MATEMATIČKI
Vol. 12 (2003), 91–98

On Ricci curvature of a quaternion CR–submanifold in a quaternion space form

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Abstract. B. Y. Chen [4] established a sharp relationship between the Ricci curvature and the squared mean curvature for a submanifold in a Riemannian space form with arbitrary codimension. Recently Liu Ximin [8] obtained results on Ricci curvature of a totally real submanifold in a quaternion projective space extending the results of Chen. In this article, we wish to estimate the Ricci curvature of a quaternion CR–submanifold in a quaternion space form.

1. Preliminaries

Let $\overline{M}^m$ be a $4m$–dimensional Riemannian manifold with metric g. $\overline{M}^m$ is called a quaternion Kaehlerian manifold if there exists a 3–dimensional vector space E of tensors of type $(1,1)$ with local basis of almost Hermitian structures I, J, K such that

(a) $IJ = -JI = K, JK = -KJ = I, KI = -IK = J$,
(b) for any local cross-section ϕ of E and any vector X tangent to $\overline{M}$, $\overline{\nabla}_X \phi$ is also a cross–section in E, where $\overline{\nabla}$ denotes the Riemannian connection in $\overline{M}$.

Condition (b) is equivalent to the following condition:

(b') there exist local 1–forms p, q, r such that

$$\overline{\nabla}_X I = r(X)J - q(X)K,$$
$$\overline{\nabla}_X J = -r(X)I + p(X)K,$$
$$\overline{\nabla}_X K = q(X)I - p(X)J.$$

Let X be a unit vector in $\overline{M}$, then X, IX, JX and KX form an orthonormal set on $\overline{M}$. We denote by $Q(X)$ the 4-plane spanned by them.

2000 Mathematics Subject Classification: 53C40, 53C25.

For any orthonormal vectors X, Y on $\overline{M}$, if $Q(X)$ and $Q(Y)$ are orthogonal, the plane $\pi(X, Y)$ spanned by X, Y is called a totally real plane. Any 2-plane in $Q(X)$ is called a quaternionic plane. The sectional curvature of a quaternionic plane π is called a quaternionic sectional curvature. A quaternion Kaehler manifold $\overline{M}$ is a quaternion space form if its quaternionic sectional curvatures are constant.

It is well known that a quaternion Kaehlerian manifold $\overline{M}$ is a quaternion space form $\overline{M}(c)$ if and only if its curvature tensor $\overline{R}$ is of the following form (see [1],[6])

$$\begin{aligned}\overline{R}(X,Y)Z = \frac{c}{4}\Big\{ & g(Y,Z)X - g(X,Z)Y \\ & + g(IY,Z)IX - g(IX,Z)IY + 2g(X,IY)IZ \\ & + g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ \\ & + g(KY,Z)KX - g(KX,Z)KY + 2g(X,KY)KZ\Big\},\end{aligned} \tag{1.1}$$

for vectors X, Y, Z tangent to $\overline{M}$.

A submanifold M of a quaternion Kaehler manifold $\overline{M}$ is called *quaternion* (resp. *totally real)* submanifold if each tangent space of M is carried into itself (resp. the normal space) by each section in E.

The curvature tensor R of M is related to the curvature tensor $\overline{R}$ of $\overline{M}$ by the following Gauss equation

$$R(X,Y;Z,W) = \overline{R}(X,Y;Z,W) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)). \tag{1.2}$$

In the following, for simplicity, we denote $\phi_1 = I, \phi_2 = J$ and $\phi_3 = K$.

Definition [2]. A submanifold M of a quaternion Kaehler manifold $\overline{M}$ is called a *quaternion CR–submanifold* if there exists two orthogonal complementry distributions D_x and $D_x^{\perp}$ such that D_x is invariant under quaternion structure, that is, $\phi_i(D_x) \subseteq D_x$, $i = 1, 2, 3$, $\forall x \in M$ and $D_x^{\perp}$ is totally real, that is, $\phi_i(D_x^{\perp}) \subseteq T_x^{\perp}M$, $i = 1, 2, 3$.

A submanifold M of a quaternion Kaehler manifold $\overline{M}$ is a quaternion submanifold (resp. totally real submanifold) if dim $D^{\perp} = 0$ (resp. dim $D = 0$).

For any X tangent to M, we put

$$\phi_i X = T_i X + F_i X, \qquad i = 1, 2, 3. \tag{1.3}$$

where $T_i X$ (resp. $F_i X$) denotes tangential (resp. normal) component of $\phi_i X$.

Let dim $D = 4p$, dim $D^{\perp} = q$. Now, we choose an orthogonal local frame $\{e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}, e_{p+q+1}, \ldots, e_m, Ie_1, \ldots, Ie_m, Je_1, \ldots, Je_m, Ke_1, \ldots, Ke_m\}$ on $\overline{M}$ such that restricted to M, $\{e_1, \ldots, e_p, Ie_1, \ldots, Ie_p, Je_1, \ldots, Je_p, Ke_1, \ldots, Ke_p\}$ are in D and $\{e_{p+1}, \ldots, e_{p+q}\}$ are in $D^{\perp}$.

We denote by $\vec{H}$ the mean curvature vector, that is

$$\vec{H}\,(x) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{1.4}$$

and by H the mean curvature, i.e. the length of $\vec{H}$. Also, we set

$$h_{ij}^{r} = g\big(h(e_i, e_j), e_r\big),$$

and

$$||h||^2 = \sum_{i,j=1}^{n} g\big(h(e_i, e_j), h(e_i, e_j)\big). \tag{1.5}$$

Recall that for a submanifold M in a Riemannian manifold, the relative null space of M at a point $x \in M$ is defined by

$$N_x = \{X \in T_xM \,|\, h(X, Y) = 0 \text{ for all } Y \in T_xM\}.$$

We will need the following algebraic lemma due to Chen [3].

Lemma 1.1. *Let $a_1, ..., a_n$, c be $n+1$ $(n \geq 2)$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + c\right). \tag{1.6}$$

Then $2a_1a_2 \geq c$ with equality holding if and only if $a_1 + a_2 = a_3 = \ldots = a_n$.

In [4] B. Y. Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for submanifolds in real space form. Recently, Liu Ximin [8] obtained some results on Ricci curvature of a totally real submanifold in a quaternion projective space extending the result of Chen. In this article, we estimate the Ricci curvature of a quaternion CR–submanifold of a quaternion space form. In fact, we prove the following.

Theorem 1.2. *Let M be an n–dimensional quaternion CR–submanifold of a quaternion space form $\overline{M}(c)$. Then:*

(i_1) *For each unit vector* $X \in D_x^{\perp}$*, we have*

$$Ric\ (X) \leq (n-1)\frac{c}{4} + \frac{n^2}{4}H^2. \tag{1.7}$$

(i_2) *For each unit vector* $X \in D_x$*, we have*

$$Ric\ (X) \leq (n+8)\frac{c}{4} + \frac{n^2}{4}H^2. \tag{1.8}$$

(ii) *If* $H(x) = 0$*, then a unit tangent vector* X *at* x *satisfies the equality case of* (1.7) *(respectively* (1.8)*) if and only if* $X \in D_x^{\perp} \cap N_x$ *(respectively* $X \in D_x \cap N_x$*)*

2. Proof of Theorem 1.2

Let M be a quaternion CR–submanifold of a quaternion space form $\overline{M}(c)$. Then using equation (1.3) in Gauss equation we have

$$R(X,Y;Z,W) = \frac{c}{4}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)$$
$$+\sum_{i=1}^{3}[g(T_iY,Z)g(T_iX,W) - g(T_iX,Z)g(T_iY,W) + 2g(X,T_iY)g(T_iZ,W)]\}$$
$$+g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)).$$

for any vector fields X, Y, Z, W tangent to M.

Let $x \in M$ and an orthonormal basis $\{e_1, \ldots, e_n = X\}$ in T_xM.

The Ricci tensor $S(X,Y)$ is given by

$$S(X,Y) = \sum_{j=1}^{n} R(e_j, X; Y, e_j)$$
$$= \frac{c}{4}\sum_{j=1}^{n}\{g(X,Y)g(e_j,e_j) - g(e_j,Y)g(X,e_j)$$
$$+\sum_{i=1}^{3}\left[g(T_iX,Y)g(T_ie_j,e_j) - g(T_ie_j,Y)g(T_iX,e_j) + 2g(e_j,T_iX)g(T_iY,e_j)\right]\}$$
$$+g\big(h(e_j,e_j),h(X,Y)\big) - g\big(h(e_j,Y),h(X,e_j)\big)$$
$$= \frac{c}{4}\{(n-1)g(X,Y) + 3\sum_{i=1}^{3} g(T_iX,T_iY)\}$$
$$+\sum_{j=1}^{n} g\big(h(e_j,e_j),h(X,Y)\big) - g\big(h(e_j,Y),h(X,e_j)\big).$$

The scalar curvature ρ is given by

$$\rho = \sum_{l=1}^{n} S(e_l, e_l) = \frac{c}{4}\big[(n-1)n + 12p\big] + n^2H^2 - ||h||^2. \tag{2.1}$$

Now we put

$$\delta = \rho - n(n-1)\frac{c}{4} - 3pc - \frac{n^2}{2}H^2.$$

Then from equation (2.1), we get

$$n^2H^2 = 2(\delta + ||h||^2). \tag{2.2}$$

With respect to the above orthogonal basis, equation (2.2) takes the form

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = 2\left\{\delta + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{j=1}^{n}(h_{ij}^{r})^2\right\}. \tag{2.3}$$

Now we put $a_1 = h_{11}^{n+1}$, $a_2 = \sum_{i=2}^{n-1} h_{ii}^{n+1}$, $a_3 = h_{nn}^{n+1}$ in the above equation and we get

$$\left(\sum_{i=1}^{3} a_i\right)^2 = 2\left\{\delta + \sum_{i=1}^{3} a_i^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2 - \sum_{2\leq\alpha\neq\beta\leq n-1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1}\right\}. \tag{2.4}$$

By Lemma 1.1, we know that if $(\sum_{i=1}^{3} a_i)^2 = 2(b + \sum_{i=1}^{3} a_i^2)$, then $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3$.

Thus, from equation (2.4), we get

$$\sum_{1\leq\alpha\neq\beta\leq n-1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1} \geq \delta + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2, \tag{2.5}$$

or equivalently,

$$\begin{aligned} n(n-1)\frac{c}{4} + 3pc + \frac{n^2}{2}H^2 \geq \rho - &\sum_{1\leq\alpha\neq\beta\leq n-1} h_{\alpha\alpha}^{n+1}h_{\beta\beta}^{n+1} \\ &+ 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^{r})^2. \end{aligned} \tag{2.6}$$

We distinguish two cases:

(a) $e_n \in D_x^{\perp}$;

(b) $e_n \in D_x$.

Using again Gauss equation in case (a), we get

$$\rho - \sum_{1\le\alpha\neq\beta\le n-1} h_{\alpha\alpha}^{n+1} h_{\beta\beta}^{n+1} + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^r)^2$$
$$= 2S(e_n,e_n) + 3pc + (n-1)(n-2)\frac{c}{4}$$
$$+ 2\sum_{i<n}(h_{in}^{n+1})^2 + \sum_{r=n+2}^{4m}\Big\{(h_{nn}^r)^2 + 2\sum_{i=1}^{n-1}(h_{in}^r)^2 + \Big(\sum_{j=1}^{n-1} h_{jj}^r\Big)^2\Big\}. \qquad (2.7)$$

From equations (2.6) and (2.7), we have

$$\frac{n^2}{4}H^2 + (n-1)\frac{c}{4} \ge S(e_n,e_n) + 2\sum_{i<n}(h_{in}^{n+1})^2 + \sum_{r=n+2}^{4m}\Big\{\sum_{i=1}^{n}(h_{in}^r)^2 + \Big(\sum_{j=1}^{n-1} h_{jj}^r\Big)^2\Big\}. \qquad (2.8)$$

Thus we have

$$S(e_n,e_n) \le (n-1)\frac{c}{4} + \frac{n^2}{4}H^2.$$

Analogously, using Gauss equation in case (b), we find

$$\rho - \sum_{1\le\alpha\neq\beta\le n-1} h_{\alpha\alpha}^{n+1} h_{\beta\beta}^{n+1} + 2\sum_{i<j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h_{ij}^r)^2$$
$$= 2S(e_n,e_n) + 3pc - \frac{3c}{2}\sum_{i=1}^{3}||T_i e_n||^2 + (n-1)(n-2)\frac{c}{4} + 2\sum_{i<n}(h_{in}^{n+1})^2$$
$$+ \sum_{r=n+2}^{4m}\Big\{(h_{nn}^r)^2 + 2\sum_{i=1}^{n-1}(h_{in}^r)^2 + \Big(\sum_{j=1}^{n-1} h_{jj}^r\Big)^2\Big\}. \qquad (2.9)$$

Similar arguments as in case (a) lead to the desired result.

(ii) Assume $H(x) = 0$. Equality holds in (1.7) (respectively (1.8)) if and only if

$$\begin{cases} h_{1n}^r = \ldots = h_{n-1,n}^r = 0 \\ h_{nn}^r = \sum_{i=1}^{n-1} h_{ii}^r \end{cases}, \quad r \in \{n+1,\ldots,4m\}. \qquad (2.10)$$

Then $h_{in}^r = 0, \forall i \in \{1,\ldots,n\}$, $r \in \{n+1,\ldots,4m\}$, i.e. $X \in D_x^{\perp} \cap N_x$ (respectively $X \in D_x \cap N_x$).

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(Received: June 13, 2003)
(Revised: August 21, 2003)

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O Ricci krivinama CR–podmnogostrukosti kvaterniona u prostoru kvaterniona

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Sadržaj

B.Y. Chen [4] izveo je relacije izmedju Ricci krivine i srednje kvadratne krivine za mnogostrukosti u Riemannovim prostorima proizvoljne kodimenzije. Nedavno je Liu Ximin [8] dobio rezultate o Ricci krivinama na total-

no realnim podmnogostrukostima na projektivnom prostoru kvaterniona proširujući rezultate Chena. U ovom članku, želimo da procijenimo CR–podmnogostrukosti u prostoru kvaterniona.