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Continuous embeddings, completions and complementation in Krein spaces

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Abstract. Let the Krein space $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ be continuously embedded in the Krein space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$. A unique self-adjoint operator A in $\mathcal{K}$ can be associated with $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ via the adjoint of the inclusion mapping of $\mathcal{A}$ in $\mathcal{K}$. Then $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ is a Krein space completion of $\mathcal{R}(A)$ equipped with an A-inner product. In general this completion is not unique. If, additionally, the embedding of $\mathcal{A}$ in $\mathcal{K}$ is t-bounded then the operator A is definitizable in $\mathcal{K}$ and $\mathcal{R}(A)$ equipped with the A-inner product has unique Krein space completion. The spectral function of A yields some information about the embedding of $\mathcal{A}$ in $\mathcal{K}$. Applications to the complementation theory of de Branges are given.

1. Introduction

1. According to de Branges [dB, p. 284] (see also [ADRS, Section 1.1.5]), if t is a positive number, a Krein space $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ is *t–boundedly embedded* in another Krein space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$ if $\mathcal{A} \subset \mathcal{K}$ and

$$[f,f]_{\mathcal{K}} \leq t\,[f,f]_{\mathcal{A}}, \quad f \in \mathcal{A}, \tag{1.1}$$

and $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ is *continuously embedded* in $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$ if $\mathcal{A} \subset \mathcal{K}$ and the identity mapping

$$\imath : \mathcal{A} \to \mathcal{K}, \quad \imath f := f, \quad f \in \mathcal{A},$$

is continuous. If $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ is continuously and t–boundedly embedded in the Krein space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$, then the bounded self–adjoint operator $A := \imath\imath^*$

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in $\mathcal{K}$, which we call the *$\mathcal{K}$-adjoint of the inclusion* $\imath$ of $\mathcal{A}$ in $\mathcal{K}$, has the characteristic property that the operator $tA - A^2$ is non-negative in the Krein space $\mathcal{K}$:

$$t\,[Ax, x]_{\mathcal{K}} - [Ax, Ax]_{\mathcal{K}} \geq 0, \quad x \in \mathcal{K}, \quad \text{or} \quad t\,A - A^2 \geq_{\mathcal{K}} 0. \tag{1.2}$$

Fix a fundamental symmetry J in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ and denote by $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{K}}$ the Hilbert inner product in $\mathcal{K}$ generated by J. As a main result of this note we show in Section 3 that if $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ is continuously and t–boundedly embedded in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ then

(i) $\mathcal{A}$ is the range of the positive self–adjoint operator $|AJ|^{1/2}$ in the Hilbert space $(\mathcal{K}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{K}})$,

(ii) $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ is the unique Krein space completion of the inner product space $(\mathcal{R}(A), [\,\cdot\,,\,\cdot\,]_{A})$, where the inner product on the range $\mathcal{R}(A)$ of the operator A is defined as follows:

$$[Ax, Ay]_{A} := [Ax, y]_{\mathcal{K}}, \quad x, y \in \mathcal{K}. \tag{1.3}$$

The relation (1.2) means that the operator A is definitizable with the definitizing polynomial $p(\lambda) = t\lambda - \lambda^2$, see [L] and the next subsection. Therefore A has a spectral function with the possible critical points 0 and t. This spectral function gives some more information about the embedding of $\mathcal{A}$ in $\mathcal{K}$, see Theorems 3.6 and 3.7.

The embedding of $\mathcal{A}$ in $\mathcal{K}$ is called *contractive* if in (1.1) $t = 1$, that is if

$$[f, f]_{\mathcal{K}} \leq [f, f]_{\mathcal{A}}, \quad f \in \mathcal{A},$$

and *isometric*, if (1.1) specifies to

$$[f, f]_{\mathcal{K}} = [f, f]_{\mathcal{A}}, \quad f \in \mathcal{A}.$$

If the embedding of $\mathcal{A}$ in $\mathcal{K}$ is continuous and isometric, then $\mathcal{A}$ is a Krein subspace of $\mathcal{K}$ and the operator A is the orthogonal projection onto $\mathcal{A}$ in $\mathcal{K}$, and the last relation in (1.2) reduces to $A^2 = A$.

In Section 4, using the representation of the embedded subspace $\mathcal{A}$ as an operator range as in (i), alternative proofs for the existence, uniqueness and the properties of the complementary subspace of a continuously and contractively embedded subspace of a Krein space are given, comp. [dB, Theorem 2], [DR, Theorem 11] and [H, Theorem 5]. The essential new feature of complementation in a Krein space compared with complementation in a Hilbert space is that the embedded space $\mathcal{A}$ can be a degenerated subspace of the Krein space $\mathcal{K}$. We conclude Section 4 with a detailed analysis of the corresponding extremal case when $\mathcal{A}$ is even a neutral subspace of $\mathcal{K}$.

As a preparation for Section 3, in Section 2 we consider a Krein space $(\mathcal{S}, [\cdot,\cdot]_{\mathcal{S}})$, which is only continuously but not necessarily t–boundedly embedded in a Hilbert or Krein space. We prove in Theorems 2.3 and 2.15 that a bounded self–adjoint operator S is the adjoint of the corresponding inclusion operator if and only if $(\mathcal{S}, [\cdot,\cdot]_{\mathcal{S}})$ is a Krein space completion of the range $\mathcal{R}(S)$, equipped with the inner product $[\cdot,\cdot]_S$ as in (1.3). It turns out that $(\mathcal{R}(S), [\cdot,\cdot]_S)$ has in general infinitely many Krein space completions. In order to formulate in Theorem 2.7 a criterion for the uniqueness of this Krein space completion we use a result of T. Hara [H, Theorem 6]. For the convenience of the reader in the Appendix we give a proof of this uniqueness result, which is partly close to Hara's proof, but our proof yields infinitely many Krein space completions. In Theorem 2.8 we give necessary and sufficient conditions under which $(\mathcal{S}, [\cdot,\cdot]_{\mathcal{S}})$ can be represented as an operator range.

2. Our terminology follows mainly that of the book [B] and of [L]. In particular, a *fundamental decomposition* of the Krein space $(\mathcal{K}, [\cdot,\cdot])$ is a representation of $\mathcal{K}$ as the direct and $[\cdot,\cdot]$–orthogonal sum of two of its subspaces $\mathcal{K}_+$ and $\mathcal{K}_-$:

$$\mathcal{K} = \mathcal{K}_+ \dotplus \mathcal{K}_-, \tag{1.4}$$

such that $(\mathcal{K}_+, [\cdot,\cdot])$ is a Hilbert and $(\mathcal{K}_-, [\cdot,\cdot])$ is an anti-Hilbert space (which means that $(\mathcal{K}_-, -[\cdot,\cdot])$ is a Hilbert space). If the decomposition of an element $x \in \mathcal{K}$ according to (1.4) is denoted by $x = x_+ + x_-$, then the inner product

$$\langle x, y\rangle := [x_+, y_+] - [x_-, y_-], \quad x, y \in \mathcal{K},$$

is a Hilbert inner product on $\mathcal{K}$, and the operator J:

$$Jx := x_+ - x_-, \quad x, y \in \mathcal{K},$$

is called the *fundamental symmetry* corresponding to the decomposition (1.4). If in some (and hence in all) fundamental decompositions one of the components $\mathcal{K}_\pm$ is finite–dimensional the Krein space $\mathcal{K}$ is called a *Pontryagin space*; if e.g. $\dim \mathcal{K}_- = \kappa < +\infty$, the Pontryagin space $\mathcal{K}$ or its inner product are said to have *negative index* κ.

The bounded self-adjoint operator A in the Krein space $(\mathcal{K}, [\cdot,\cdot])$ is called *definitizable* (positizable in [B]) if there exists a polynomial p such that $p(A)$ is a non-negative operator in $\mathcal{K}$: $[p(A)x, x] \geq 0,\ x \in \mathcal{K}$; in this case p is called a *definitizing polynomial* for A. An interval $\Delta \subset \mathbb{R}$ is called *admissible* for the definitizable operator A if there exists a definitizing polynomial p such that the endpoints of Δ are not zeros of p. The *spectral function* of the definitizable operator A associates with each admissible interval Δ for

A a self–adjoint projection $E(\Delta)$ in $\mathcal{K}$, such that the range $E(\Delta)\mathcal{K}$, called the *spectral subspace of* A at Δ, is invariant under A and it is the maximal subspace $\mathcal{L}_\Delta$ with respect to the property

$$\sigma(A|\mathcal{L}_\Delta) \subset \overline{\Delta}.$$

The subspace $\big(E(\Delta)\mathcal{K}, [\,\cdot\,,\cdot\,]\big)$ is a Hilbert space if $p > 0$ on Δ and an anti–Hilbert space if $p < 0$ on Δ. Later we use the fact that the elements $x \in E(\Delta)\mathcal{K}$ are characterized by the property that the function $(A - z)^{-1}x$, which is holomorphic on $\rho(A)$, has an analytic continuation at least to $\mathbb{C}\backslash\overline{\Delta}$. A point $\lambda \in \sigma(A)$ is said to have *finite negative index* κ if for each sufficiently small admissible interval Δ containing λ the inner product $[\,\cdot\,,\cdot\,]$ has finite negative index κ on $E(\Delta)\mathcal{K}$.

Two Krein spaces $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ and $\big(\mathcal{L}, [\,\cdot\,,\cdot\,]_\mathcal{L}\big)$ are called *isomorphic* if there exists a continuous and continuously invertible linear bijection $T : \mathcal{K} \to \mathcal{L}$ such that

$$[Tx, Ty]_\mathcal{L} = [x, y]_\mathcal{K}, \quad x, y \in \mathcal{K};$$

in this case the mapping T is called an *isomorphism* between $\mathcal{K}$ and $\mathcal{L}$.

2. Krein space completions and continuous embeddings

1. The Krein space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ is said to be a *Krein space completion* of the inner product space $\big(\mathcal{L}, [\,\cdot\,,\cdot\,]_\mathcal{L}\big)$ if $\mathcal{L}$ is a dense subspace of $\mathcal{K}$ and $[x, y]_\mathcal{L} = [x, y]_\mathcal{K}$ for all x, $y \in \mathcal{L}$. An inner product space $\big(\mathcal{L}, [\,\cdot\,,\cdot\,]_\mathcal{L}\big)$ can have more than one Krein space completion. That is, there may exist two Krein spaces $\big(\mathcal{K}_1, [\,\cdot\,,\cdot\,]_{\mathcal{K}_1}\big)$ and $\big(\mathcal{K}_2, [\,\cdot\,,\cdot\,]_{\mathcal{K}_2}\big)$ which are Krein space completions of $\big(\mathcal{L}, [\,\cdot\,,\cdot\,]_\mathcal{L}\big)$ and such that there is no isomorphism $U : \big(\mathcal{K}_1, [\,\cdot\,,\cdot\,]_{\mathcal{K}_1}\big) \to \big(\mathcal{K}_2, [\,\cdot\,,\cdot\,]_{\mathcal{K}_2}\big)$ with $Ux = x$ for all $x \in \mathcal{L}$. Examples will come up later and in the Appendix. We say that the inner product space $\big(\mathcal{L}, [\,\cdot\,,\cdot\,]_\mathcal{L}\big)$ has a *unique Krein space completion* if any two Krein space completions of $\mathcal{L}$ are isomorphic with an isomorphism which acts as the identity on $\mathcal{L}$.

Special Krein space completions often arise as follows. The non-degenerated inner product space $\big(\mathcal{L}, [\,\cdot\,,\cdot\,]_\mathcal{L}\big)$ is said to have a *Hilbert majorant* $(\,\cdot\,,\cdot\,)_\mathcal{L}$ if the latter is a positive definite inner product on $\mathcal{L}$, such that $\big(\mathcal{L}, (\,\cdot\,,\cdot\,)_\mathcal{L}\big)$ is a Hilbert space, and with a positive number γ the relation

$$|[x, y]_\mathcal{L}| \leq \gamma (x, x)_\mathcal{L}^{1/2} \, (y, y)_\mathcal{L}^{1/2}, \quad x, y \in \mathcal{L},$$

holds. The latter inequality implies that there exists a bounded self-adjoint and injective operator G in $\big(\mathcal{L}, (\,\cdot\,,\cdot\,)_\mathcal{L}\big)$ such that

$$[x, y]_\mathcal{L} = (Gx, y)_\mathcal{L}, \quad x, y \in \mathcal{L}. \tag{2.1}$$

The operator G is called the *Gram operator of* $[\,\cdot\,,\cdot\,]_{\mathcal{L}}$ in $\big(\mathcal{L},(\,\cdot\,,\cdot\,)_{\mathcal{L}}\big)$. Using the spectral function E_G of G we introduce the spaces

$$\mathcal{L}_+ := E_G\big((0,+\infty)\big)\mathcal{L}, \quad \mathcal{L}_- := E_G\big((-\infty,0)\big)\mathcal{L}.$$

and consider the pre–Hilbert spaces $\big(\mathcal{L}_\pm,(|G|\,\cdot\,,\,\cdot\,)_{\mathcal{L}}\big)$. They are orthogonal to each other with respect to all the inner products $[\,\cdot\,,\cdot\,]_{\mathcal{L}}$, $(\,\cdot\,,\cdot\,)_{\mathcal{L}}$, and $(|G|\,\cdot\,,\,\cdot\,)_{\mathcal{L}}$. Denoting the completions of these pre–Hilbert spaces by $\big(\mathcal{G}_\pm,\langle\,\cdot\,,\cdot\,\rangle_\pm\big)$ the space

$$\mathcal{G} := \mathcal{G}_+ \dot{+} \mathcal{G}_-$$

equipped with the inner product

$$[x,y]_{\mathcal{G}} := \langle x_+,y_+\rangle_+ - \langle x_-,y_-\rangle_-, \quad x = x_+ + x_-,\ y = y_+ + y_-, \quad x_\pm, y_\pm \in \mathcal{G}_\pm,$$

is a Krein space completion of $\big(\mathcal{L},[\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$. The definition of this inner product is correct since on $\mathcal{L}$ it coincides with the inner product (2.1). We call the space $(\mathcal{G},[\,\cdot\,,\cdot\,]_{\mathcal{G}})$ the *canonical Krein space completion* of $\big(\mathcal{L},[\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$ with respect to the Hilbert majorant $(\,\cdot\,,\cdot\,)_{\mathcal{H}}$.

It follows from [B, Theorem V.2.1] that the canonical Krein space completion of a non-degenerated inner product space $\big(\mathcal{L},[\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$ with a Hilbert majorant is uniquely determined, that is any two canonical Krein space completions of $\mathcal{L}$ are isomorphic with an isomorphism which acts as the identity on $\mathcal{L}$.

Let the Krein space $\big(\mathcal{S},[\,\cdot\,,\cdot\,]_{\mathcal{S}}\big)$ be continuously embedded in the Hilbert space $\big(\mathcal{H},\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$. Then the adjoint $\imath^* : \mathcal{H} \to \mathcal{S}$, of the inclusion $\imath$ of $\mathcal{S}$ in $\mathcal{H}$ defined by

$$\langle \imath f, x\rangle_{\mathcal{H}} = [f, \imath^* x]_{\mathcal{S}}, \qquad f \in \mathcal{S},\ x \in \mathcal{H},$$

is also continuous. The operator $\imath\imath^*$ is a bounded self–adjoint operator in the Hilbert space $\mathcal{H}$, called the *$\mathcal{H}$–adjoint of the inclusion* $\imath$ of $\big(\mathcal{S},[\,\cdot\,,\cdot\,]_{\mathcal{S}}\big)$ in $\big(\mathcal{H},\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$.

Remark 2.1 Let the Krein space $\big(\mathcal{S},[\,\cdot\,,\cdot\,]_{\mathcal{S}}\big)$ be continuously embedded in the Hilbert space $\big(\mathcal{H},\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$ and assume $\mathcal{S} = \mathcal{H}$. Then it is well known that the norm topologies on $\big(\mathcal{S},[\,\cdot\,,\cdot\,]_{\mathcal{S}}\big)$ and $\big(\mathcal{H},\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$ are equivalent, see e.g. [B, Theorem IV.6.4]. If S is the $\mathcal{H}$–adjoint of the inclusion of $\mathcal{S}$ in $\mathcal{H}$ and Q is the Gram operator of $[\,\cdot\,,\cdot\,]_{\mathcal{S}}$ in $\big(\mathcal{H},\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$, then for $x,y \in \mathcal{H} = \mathcal{S}$ we have

$$\langle x,y\rangle_{\mathcal{H}} = \langle \imath x,y\rangle_{\mathcal{H}} = [x,\imath^* y]_{\mathcal{S}} = [x,Sy]_{\mathcal{S}} = \langle x,QSy\rangle_{\mathcal{H}},$$

therefore $S = Q^{-1}$.

Remark 2.2 With the above notation for the canonical Krein space completion $(\mathcal{G}, [\,\cdot\,,\cdot\,]_{\mathcal{G}})$ of $(\mathcal{L}, [\,\cdot\,,\cdot\,]_{\mathcal{L}})$we have

$$\langle f, f\rangle_{\mathcal{G}} = (|G|f, f)_{\mathcal{L}} \leq \gamma(f, f)_{\mathcal{L}}, \quad f \in \mathcal{L}.$$

Therefore the Hilbert space $(\mathcal{L}, (\,\cdot\,,\cdot\,)_{\mathcal{L}})$ is continuously embedded in the Hilbert space $(\mathcal{G}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{G}})$. Let $H := \imath\imath^*$ be the $\mathcal{G}$–adjoint of the inclusion $\imath$ of $(\mathcal{L}, (\,\cdot\,,\cdot\,)_{\mathcal{L}})$ in $(\mathcal{G}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{G}})$. It follows from the relation

$$(f, Hg)_{\mathcal{L}} = (f, \imath^* g)_{\mathcal{L}} = \langle \imath f, g\rangle_{\mathcal{G}} = \langle f, g\rangle_{\mathcal{G}} = (|G|f, g)_{\mathcal{L}} = (f, |G|g)_{\mathcal{L}}, \quad f, g \in \mathcal{L}\,,$$

that H is the continuous extension of $|G|$ to $\mathcal{G}$.

For a bounded self–adjoint operator S in the Hilbert space $(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}})$, on its range $\mathcal{R}(S)$ an inner product $(\,\cdot\,,\cdot\,)_S$ is defined by the relation

$$(u, v)_S := \langle Sx, y\rangle_{\mathcal{H}}, \quad \text{where} \quad u = Sx,\ v = Sy, \quad x, y \in \mathcal{H}. \tag{2.2}$$

The inner product $(\,\cdot\,,\cdot\,)_S$ on $\mathcal{R}(S)$ is well defined by (2.2) since $\langle Sx, y\rangle_{\mathcal{H}} = 0$ whenever x or y belongs to $\ker S$, and it is non–degenerate since, by (2.2), $(u, Sy)_S = 0$ for all $y \in \mathcal{H}$ implies that $u = 0$.

Theorem 2.3. *Let the Krein space $(\mathcal{S}, [\,\cdot\,,\cdot\,]_{\mathcal{S}})$ be continuously embedded in the Hilbert space $(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}})$ and let S be a bounded self–adjoint operator in $\mathcal{H}$. The operator S is the $\mathcal{H}$–adjoint of the inclusion $\imath$ of $(\mathcal{S}, [\,\cdot\,,\cdot\,]_{\mathcal{S}})$ in $(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}})$ if and only if $(\mathcal{S}, [\,\cdot\,,\cdot\,]_{\mathcal{S}})$ is a Krein space completion of the inner product space $(\mathcal{R}(S), (\,\cdot\,,\cdot\,)_S)$.*

Proof. Suppose that S is the $\mathcal{H}$-adjoint of $\imath$: $S = \imath\imath^*$. Then $\mathcal{R}(S) = \mathcal{R}(\imath^*) \subset \mathcal{S}$ and

$$\langle f, x\rangle_{\mathcal{H}} = \langle \imath\, f, x\rangle_{\mathcal{H}} = [f, \imath^* x]_{\mathcal{S}} = [f, Sx]_{\mathcal{S}}, \quad f \in \mathcal{S},\ x \in \mathcal{H}. \tag{2.3}$$

It follows that $\mathcal{R}(S)$ is dense in $\mathcal{S}$. Indeed, if for some $f_0 \in \mathcal{S}$ it holds $[f_0, Sx]_{\mathcal{S}} = 0$ for all $x \in \mathcal{H}$, then, by (2.3), $f_0 = 0$. From the definition of $(\,\cdot\,,\cdot\,)_S$ and (2.3) for $u = Sx, v = Sy$, we have

$$(u, v)_S = (Sx, Sy)_S = \langle Sx, y\rangle_{\mathcal{H}} = [Sx, Sy]_{\mathcal{S}} = [u, v]_{\mathcal{S}}\,,$$

therefore the inner products $(\,\cdot\,,\cdot\,)_S$ and $[\,\cdot\,,\cdot\,]_{\mathcal{S}}$ coincide on $\mathcal{R}(S)$. Thus $(\mathcal{S}, [\,\cdot\,,\cdot\,]_{\mathcal{S}})$ is a Krein space completion of $(\mathcal{R}(S), (\,\cdot\,,\cdot\,)_S)$.

Conversely, assume that $(\mathcal{S}, [\,\cdot\,,\cdot\,]_{\mathcal{S}})$ is a Krein space completion of the inner product space $(\mathcal{R}(S), (\,\cdot\,,\cdot\,)_S)$. Then for all $x \in \mathcal{H}$ and $f \in \mathcal{R}(S)$ we have

$$[f, Sx]_{\mathcal{S}} = (f, Sx)_S = \langle f, x\rangle_{\mathcal{H}}. \tag{2.4}$$

Since for fixed $x \in \mathcal{H}$ the functionals $[\,\cdot\,, Sx]_{\mathcal{S}}$ and $\langle\,\cdot\,, x\rangle_{\mathcal{H}}$ are continuous on $\mathcal{S}$, we conclude that (2.4) holds for all $f \in \mathcal{S}$. Therefore,

$$\langle Sx, y\rangle_{\mathcal{H}} = \langle x, Sy\rangle_{\mathcal{H}} = \langle x, \imath\, Sy\rangle_{\mathcal{H}} = [\imath^* x, Sy]_{\mathcal{S}} = \langle \imath\imath^* x, y\rangle_{\mathcal{H}}, \quad x, y \in \mathcal{H}\,,$$

and, consequently, $\imath\imath^* = S$.

2. Let T be a bounded non-negative operator in the Hilbert space $(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}})$. We equip the range $\mathcal{R}(T)$ with the inner product $(\,\cdot\,,\cdot\,)_T$ as in (2.2)

$$(u, v)_T = \langle Tx, y\rangle_{\mathcal{H}}, \qquad \text{where} \qquad u = Tx,\ v = Ty, \quad x, y \in \mathcal{H};$$

it turns $\mathcal{R}(T)$ into a pre–Hilbert space $\big(\mathcal{R}(T), (\,\cdot\,,\cdot\,)_T\big)$. The relation

$$\begin{aligned} |(u,v)_T|^2 = |\langle Tx, y\rangle_{\mathcal{H}}|^2 &\le \langle Tx, x\rangle_{\mathcal{H}} \langle Ty, y\rangle_{\mathcal{H}} \\ &= (u,u)_T (v,v)_T = \|T^{1/2}x\|_{\mathcal{H}}^2 \|T^{1/2}y\|_{\mathcal{H}}^2 \end{aligned}$$

implies that the inner product $(\,\cdot\,,\cdot\,)_T$ can be extended to $\mathcal{R}(T^{1/2})$ by continuity with respect to the norm $\|T^{1/2}\cdot\|_{\mathcal{H}}$. We denote this extension also by $(\,\cdot\,,\cdot\,)_T$.

Lemma 2.4. *Let T be a bounded non–negative operator in the Hilbert space $(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}})$. Then the Hilbert space completion of $\big(\mathcal{R}(T), (\,\cdot\,,\cdot\,)_T\big)$ is the Hilbert space $\big(\mathcal{R}(T^{1/2}), (\,\cdot\,,\cdot\,)_T\big)$, and the latter is continuously embedded in $(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}})$.*

Proof. For $u = Tx,\ x \in \mathcal{H}$, we have

$$\langle u, u\rangle_{\mathcal{H}} = \langle Tx, Tx\rangle_{\mathcal{H}} \le \|T\| \langle Tx, x\rangle_{\mathcal{H}} = \|T\| (u,u)_T. \tag{2.5}$$

Therefore, if (u_n), $u_n = Tx_n,\ n \in \mathbb{N}$, is a Cauchy sequence in $\big(\mathcal{R}(T), (\,\cdot\,,\cdot\,)_T\big)$, then both (u_n) and $(T^{1/2}x_n)$ are Cauchy sequences in $(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}})$, and if $T^{1/2}x_n \to y$ and $u_n = Tx_n \to v$ in $(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}})$, $y, v \in \mathcal{H}$, then $v = T^{1/2}y$ and $u_n \to v$ in $\big(\mathcal{R}(T^{1/2}), (\,\cdot\,,\cdot\,)_T\big)$. The inequality (2.5) shows that $\big(\mathcal{R}(T^{1/2}), (\,\cdot\,,\cdot\,)_T\big)$ is continuously embedded in $(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}})$.

Corollary 2.5. *If the Hilbert space $(\mathcal{T}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{T}})$ is continuously embedded in the Hilbert space $(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}})$, then $(\mathcal{T}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{T}}) = \big(\mathcal{R}(T^{1/2}), (\,\cdot\,,\cdot\,)_T\big)$, where T denotes the $\mathcal{H}$-adjoint of the inclusion of $\mathcal{T}$ in $\mathcal{H}$.*

Proof. By Theorem 2.3, the Hilbert space $(\mathcal{T}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{T}})$ is a completion of the pre–Hilbert space $\big(\mathcal{R}(T), (\,\cdot\,,\cdot\,)_T\big)$. Therefore T is a positive operator in $(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}})$. It follows from Lemma 2.4 that the Hilbert space $\big(\mathcal{R}(T^{1/2}),(\,\cdot\,,\cdot\,)_T\big)$ is also a completion of $\big(\mathcal{R}(T),(\,\cdot\,,\cdot\,)_T\big)$. Since both $\big(\mathcal{R}(T),(\,\cdot\,,\cdot\,)_T\big)$ and $\big(\mathcal{R}(T^{1/2}),(\,\cdot\,,\cdot\,)_T\big)$ are continuously embedded in $\mathcal{H}$ and since the completion of $\big(\mathcal{R}(T),(\,\cdot\,,\cdot\,)_T\big)$ contained in $\big(\mathcal{H},\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$ is unique, the equality in the corollary follows.

Remark 2.6. In the notation of Lemma 2.4 we have

$$\big(T^{1/2}x,T^{1/2}y\big)_T=\langle Px,y\rangle_{\mathcal{H}},\qquad x,y\in\mathcal{H},$$

where $P:\mathcal{H}\to\mathcal{H}$ is the orthogonal projection onto the closure of $\mathcal{R}(T^{1/2})$ in $\big(\mathcal{H},\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$.

Now let S be a bounded self–adjoint operator in a Hilbert space $\big(\mathcal{H},\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$, $S=S_+-S_-$, where $S_+\geq 0$ and $S_->0$ are the non–negative and negative parts of S, respectively. The inner product space $\big(\mathcal{R}(S),(\,\cdot\,,\cdot\,)_S\big)$ is decomposable with one fundamental decomposition being

$$\mathcal{R}(S)=\mathcal{R}(S_+)+\mathcal{R}(S_-)\,, \tag{2.6}$$

that is, this sum is direct and orthogonal with respect to $(\,\cdot\,,\cdot\,)_S$ and $\mathcal{R}(S_+)$ is a positive, $\mathcal{R}(S_-)$ is a negative subspace of $\big(\mathcal{R}(S),(\,\cdot\,,\cdot\,)_S\big)$. The positive definite inner product corresponding to the decomposition (2.6) is $(\,\cdot\,,\cdot\,)_{|S|}$, where $|S|:=S_++S_-$; note that $\mathcal{R}(S)=\mathcal{R}(|S|)$. The inner product space $\big(\mathcal{R}(S),(\,\cdot\,,\cdot\,)_{|S|}\big)$ is a pre-Hilbert space, its completion in $\big(\mathcal{H},\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$ is the Hilbert space $\big(\mathcal{R}(|S|^{1/2}),(\,\cdot\,,\cdot\,)_{|S|}\big)$. Since for $u=Sx, v=Sy,\ x,y\in\mathcal{H}$, we have

$$|(u,v)_S|^2=|\langle Sx,y\rangle_{\mathcal{H}}|^2\leq\langle|S|x,x\rangle_{\mathcal{H}}\langle|S|y,y\rangle_{\mathcal{H}}=(u,u)_{|S|}(v,v)_{|S|}\,,$$

the inner product $(\,\cdot\,,\cdot\,)_S$ can also be extended by continuity to $\mathcal{R}(|S|^{1/2})$, and this extension is also denoted by $(\,\cdot\,,\cdot\,)_S$. The inner product space $\big(\mathcal{R}(|S|^{1/2}),(\,\cdot\,,\cdot\,)_S\big)$ is a Krein space which is a Krein space completion of $\big(\mathcal{R}(S),(\,\cdot\,,\cdot\,)_S\big)$. The topology of this Krein space is the topology of the Hilbert space $\big(\mathcal{R}(|S|^{1/2}),(\,\cdot\,,\cdot\,)_{|S|}\big)$. It follows from Lemma 2.4 that the Krein space $\big(\mathcal{R}(|S|^{1/2}),(\,\cdot\,,\cdot\,)_S\big)$ is continuously embedded in $\big(\mathcal{H},\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$.

Theorem 2.7. *Let S be a bounded self-adjoint operator in the Hilbert space $\big(\mathcal{H},\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$. The following statements are equivalent.*

(a) *For some $\varepsilon>0$ at least one of the intervals $(-\varepsilon,0)$, $(0,\varepsilon)$ belongs to $\rho(S)$.*

(b) *The Krein space $\big(\mathcal{R}(|S|^{1/2}),(\,\cdot\,,\cdot\,)_S\big)$ is the unique Krein space which is continuously embedded in $\mathcal{H}$ and such that the $\mathcal{H}$–adjoint of its inclusion in $\big(\mathcal{H},\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$ is the operator S.*

(c) *The Krein space* $\big(\mathcal{R}(|S|^{1/2}), (\,\cdot\,,\cdot\,)_S\big)$ *is the unique continuously in* $\mathcal{H}$ *embedded Krein space completion of the inner product space* $\big(\mathcal{R}(S), (\,\cdot\,,\cdot\,)_S\big)$.

(d) *The inner product space* $\big(\mathcal{R}(S), (\,\cdot\,,\cdot\,)_S\big)$ *has a unique Krein space completion.*

Proof. Theorem 2.3 implies that (b) and (c) are equivalent. Clearly (d) implies (c). To prove the equivalence of (a) and (d) note that the inner product space $\big(\mathcal{R}(S), (\,\cdot\,,\cdot\,)_S\big)$ has a Hilbert majorant. Indeed, $(\mathcal{R}(S), (\,\cdot\,,\cdot\,)_{S^2})$ is a Hilbert space since $\mathcal{R}(S) = \mathcal{R}((S^2)^{1/2})$, and the inequality

$$|(x,x)_S| \le \|S\|(x,x)_{S^2}, \quad x \in \mathcal{R}(|S|) = \mathcal{R}(S),$$

shows that $(\,\cdot\,,\cdot\,)_S$ is continuous on this Hilbert space. Therefore Theorem 5.2 in the Appendix implies that (a) is equivalent to the completeness of at least one of the inner product spaces $\big(\mathcal{R}(S_+), (\,\cdot\,,\cdot\,)_{|S|}\big)$, $\big(\mathcal{R}(S_-), (\,\cdot\,,\cdot\,)_{|S|}\big)$. Since $\big(\mathcal{R}(S_+), (\,\cdot\,,\cdot\,)_{|S|}\big)$ $\big(\big(\mathcal{R}(S_-), (\,\cdot\,,\cdot\,)_{|S|}\big)$, respectively $\big)$ is complete if and only if for some $\varepsilon > 0$ the interval $(0,\varepsilon)$ $((-\varepsilon,0)$, respectively$)$ belongs to $\rho(S)$, the equivalence of (a) and (d) follows.

It is of interest to characterize those Krein spaces $\big(\mathcal{S}, [\,\cdot\,,\cdot\,]_{\mathcal{S}}\big)$ which are continuously embedded in the Hilbert space $\big(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$ and for which $\big(\mathcal{S}, [\,\cdot\,,\cdot\,]_{\mathcal{S}}\big) = \big(\mathcal{R}(|S|^{1/2}), (\,\cdot\,,\cdot\,)_S\big)$ holds, where S is the $\mathcal{H}$-adjoint of the continuous inclusion of $\big(\mathcal{S}, [\,\cdot\,,\cdot\,]_{\mathcal{S}}\big)$ in $\big(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$. We give two characterizations; (c) is a modification of [F, Theorem 3]. Recall that, given a positive operator $\widehat{S}$ in a Krein space $\big(\mathcal{S}, [\,\cdot\,,\cdot\,]_{\mathcal{S}}\big)$ (that means $\big[\widehat{S}x, x\big]_{\mathcal{S}} > 0$ for $x \in \mathcal{S}$, $x \ne 0$), zero is a *regular critical point* of $\widehat{S}$ if for the spectral function $E_{\widehat{S}}$ of $\widehat{S}$ the projections $E_{\widehat{S}}(\Delta)$, Δ an arbitrary admissible interval for $\widehat{S}$, are uniformly bounded; this is equivalent to the fact that for the spectral function $E_{\widehat{S}}$ also the projections $E_{\widehat{S}}\big((-\infty,0)\big)$ and $E_{\widehat{S}}\big((0,+\infty)\big)$ exist and satisfy $E_{\widehat{S}}\big((-\infty,0)\big) + E_{\widehat{S}}\big((0,+\infty)\big) = I$.

Theorem 2.8. *Let the Krein space* $\big(\mathcal{S}, [\,\cdot\,,\cdot\,]_{\mathcal{S}}\big)$ *be continuously embedded in the Hilbert space* $\big(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$, *and let* S *be the* $\mathcal{H}$*–adjoint of the inclusion of* $\big(\mathcal{S}, [\,\cdot\,,\cdot\,]_{\mathcal{S}}\big)$ *in* $\big(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$. *The following statements are equivalent:*

(a) $\big(\mathcal{S}, [\,\cdot\,,\cdot\,]_{\mathcal{S}}\big) = \big(\mathcal{R}(|S|^{1/2}), (\,\cdot\,,\cdot\,)_S\big)$.

(b) *There exists a fundamental decomposition* $\mathcal{S} = \mathcal{S}_+[+]_{\mathcal{S}}\mathcal{S}_-$ *of* $\mathcal{S}$ *such that* $\mathcal{S}_+$ *and* $\mathcal{S}_-$ *are mutually orthogonal in* $\big(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$.

(c) *The operator* $\widehat{S} := S|_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$ *is a positive bounded operator in* $\big(\mathcal{S}, [\,\cdot\,,\cdot\,]_{\mathcal{S}}\big)$ *and* 0 *is not a singular critical point of* $\widehat{S}$.

Proof. Assume that (a) holds. Denote by E_S the spectral function of S and put $\mathcal{H}_- = E_S\big((-\infty,0)\big)$, $\mathcal{H}_+ = E_S\big([0,+\infty)\big)$, and $S_\pm = S|_{\mathcal{H}_\pm}$. Then

$\mathcal{H} = \mathcal{H}_-\langle\dot{+}\rangle_{\mathcal{H}}\mathcal{H}_+$, $|S| = -S_- \dot{+} S_+$ and $|S|^{1/2} = (-S_-)^{1/2} \dot{+} S_+^{1/2}$. Now define $\mathcal{S}_- = \mathcal{R}\big((-S_-)^{1/2}\big)$ and $\mathcal{S}_+ = \mathcal{R}\big(S_+^{1/2}\big)$. It is clear from the considerations preceding Theorem 2.7 that $\mathcal{S} = \mathcal{S}_-(\dot{+})_{\mathcal{S}}\mathcal{S}_+$ is a fundamental decomposition of $\big(\mathcal{S}, [\,\cdot\,,\cdot\,]_{\mathcal{S}}\big)$ which satisfies (b).

Assume (b) and let $\mathcal{S}_\pm^c$ be the closure of $\mathcal{S}_\pm$ in $\big(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$ and let $\mathcal{S}^o$ be the orthogonal complement of $\mathcal{S}_+^c \dot{+} \mathcal{S}_-^c$ in $\big(\mathcal{H}, [\,\cdot\,,\cdot\,]_{\mathcal{H}}\big)$. The Hilbert spaces $\big(\mathcal{S}_\pm, \pm[\,\cdot\,,\cdot\,]_{\mathcal{S}}\big)$ are continuously embedded in the Hilbert spaces $\big(\mathcal{S}_\pm^c, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$. Therefore Corollary 2.5 implies that there exist bounded positive operators $T_\pm$ such that

$$\big(\mathcal{S}_\pm, \pm[\,\cdot\,,\cdot\,]_{\mathcal{S}}\big) = \big(\mathcal{R}(T_\pm^{1/2}), (\,\cdot\,,\cdot\,)_{T_\pm}\big).$$

Put $S = -T_- \dot{+} T_+ \dot{+} 0$ (the direct sum with respect to $\mathcal{H} = \mathcal{S}_+^c \dot{+} \mathcal{S}_-^c \dot{+} \mathcal{S}^o$). Then

$$\big(\mathcal{S}, [\,\cdot\,,\cdot\,]_{\mathcal{S}}\big) = \big(\mathcal{S}_- + \mathcal{S}_+, [\,\cdot\,,\cdot\,]_{\mathcal{S}}\big) = \big(\mathcal{R}(|S|^{1/2}), (\,\cdot\,,\cdot\,)_S\big).$$

Thus (a) holds and S is the $\mathcal{H}$–adjoint of the inclusion of $\mathcal{S}$ in $\mathcal{H}$. Clearly $\widehat{S} = -T_-\big|_{\mathcal{S}_-} + T_+\big|_{\mathcal{S}_+}$, and $\widehat{S}$ is a positive operator in $\big(\mathcal{S}, [\,\cdot\,,\cdot\,]_{\mathcal{S}}\big)$. From the construction of S it follows that the fundamental decomposition in (b) reduces $\widehat{S}$. Therefore (c) holds. Note that along the way we have also proved that (b) implies (a).

It remains to prove that (c) implies (b). Assume that $\widehat{S}$ is a positive bounded operator in $\big(\mathcal{S}, [\,\cdot\,,\cdot\,]_{\mathcal{S}}\big)$ and that 0 is not a singular critical point of $\widehat{S}$. Then there exists a fundamental decomposition $\mathcal{S} = \mathcal{S}_-[\dot{+}]_{\mathcal{S}}\mathcal{S}_+$ of $\big(\mathcal{S}, [\,\cdot\,,\cdot\,]_{\mathcal{S}}\big)$ which reduces $\widehat{S}$, that is $\widehat{S}\mathcal{S}_\pm \subset \mathcal{S}_\pm$. For arbitrary $x_\pm \in \mathcal{S}_\pm$ we have

$$\langle x_+, x_-\rangle_{\mathcal{H}} = [x_+, Sx_-]_{\mathcal{S}} = 0\,,$$

and therefore $\mathcal{S}_+$ and $\mathcal{S}_-$ are mutually orthogonal in $\big(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$.

The statements of Theorem 2.8 do not depend on the choice of the inner product on $\mathcal{H}$. In order to prove this, a lemma is needed which is a consequence of the Closed Graph Theorem and the Heinz inequality, see [Kr, Theorem 7.1].

Lemma 2.9 *Let G and H be bounded self-adjoint operators in the Hilbert spaces $\big(\mathcal{G}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{G}}\big)$ and $\big(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$, respectively. If $F\colon \big(\mathcal{G}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{G}}\big) \longrightarrow \big(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$ is a bounded operator such that $F\mathcal{R}(G) \subseteq \mathcal{R}(H)$ and $0 \le \alpha \le 1$, then $F\mathcal{R}(|G|^\alpha) \subseteq \mathcal{R}(|H|^\alpha)$ and also*

$$F : \big(\mathcal{R}(|G|^\alpha), (\,\cdot\,,\cdot\,)_{|G|^{2\alpha}}\big) \longrightarrow \big(\mathcal{R}(|H|^\alpha), (\,\cdot\,,\cdot\,)_{|H|^{2\alpha}}\big)$$

is a bounded operator.

We shall use the following consequence of this lemma. Let $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $\langle\cdot,\cdot\rangle'_{\mathcal{H}}$ be two Hilbert inner products on $\mathcal{H}$ and denote by G the corresponding Gram operator: $\langle x,y\rangle'_{\mathcal{H}} = \langle Gx,y\rangle_{\mathcal{H}}$, $x,y\in\mathcal{H}$. If S is a bounded self-adjoint operator in $\big(\mathcal{H},\langle\cdot,\cdot\rangle_{\mathcal{H}}\big)$ and $S' = SG$, then the operator S' is self–adjoint in $\big(\mathcal{H},\langle\cdot,\cdot\rangle'_{\mathcal{H}}\big)$ and

$$\big(\mathcal{R}(|S|^{1/2}),(\cdot,\cdot)_S\big) = \big(\mathcal{R}(|S'|^{1/2}),(\cdot,\cdot)_{S'}\big). \tag{2.7}$$

This follows immediately from the equality $\mathcal{R}(S)=\mathcal{R}(SG)$ and Lemma 2.9 with F being chosen to be the identity operator from $\big(\mathcal{H},\langle\cdot,\cdot\rangle_{\mathcal{H}}\big)$ to $\big(\mathcal{H},\langle\cdot,\cdot\rangle'_{\mathcal{H}}\big)$.

The following corollary is an immediate consequence of Theorem 2.8 and (2.7).

Corollary 2.10. *Let the Krein space $\big(\mathcal{S},[\cdot,\cdot]_{\mathcal{S}}\big)$ be continuously embedded in the Hilbert spaces $\big(\mathcal{H},\langle\cdot,\cdot\rangle_{\mathcal{H}}\big)$ and $\big(\mathcal{H},\langle\cdot,\cdot\rangle'_{\mathcal{H}}\big)$ and denote the $\mathcal{H}$–adjoints of these inclusions by S and S'. Then the following statements are equivalent:*

(a) $\big(\mathcal{S},[\cdot,\cdot]_{\mathcal{S}}\big) = \big(\mathcal{R}(|S|^{1/2}),(\cdot,\cdot)_S\big)$.

(b) $\big(\mathcal{S},[\cdot,\cdot]_{\mathcal{S}}\big) = \big(\mathcal{R}(|S'|^{1/2}),(\cdot,\cdot)_{S'}\big)$.

(c) *There exists a fundamental decomposition $\mathcal{S} = \mathcal{S}'_+[\dotplus]_{\mathcal{S}}\mathcal{S}'_-$ of $\mathcal{S}$ such that $\mathcal{S}'_+$ and $\mathcal{S}'_-$ are mutually orthogonal in $\big(\mathcal{H},\langle\cdot,\cdot\rangle'_{\mathcal{H}}\big)$.*

(d) *The operator $\widehat{S}' := S'|_{\mathcal{S}} : \mathcal{S}\to\mathcal{S}$ is a positive bounded operator in $\big(\mathcal{S},[\cdot,\cdot]_{\mathcal{S}}\big)$ and 0 is not a singular critical point of $\widehat{S}'$.*

Remark 2.11. Obviously, each statement in Corollary 2.10 is equivalent to each statement in Theorem 2.8. In particular, the statements (c) in Theorem 2.8 and (c) in Corollary 2.10 are equivalent.

3. In this subsection we consider a Krein space $\big(\mathcal{A},[\cdot,\cdot]_{\mathcal{A}}\big)$ which is continuously embedded in another Krein space $\big(\mathcal{K},[\cdot,\cdot]_{\mathcal{K}}\big)$. If a fundamental symmetry J is chosen in $\mathcal{K}$ and

$$\langle x,y\rangle_{\mathcal{K}} := [Jx,y]_{\mathcal{K}}, \quad x,y\in\mathcal{K},$$

is the corresponding Hilbert inner product on $\mathcal{K}$, then we are in the situation of the foregoing subsection. The Hilbert space $\big(\mathcal{K},\langle\cdot,\cdot\rangle_{\mathcal{K}}\big)$ will also be denoted by $\big(\mathcal{H},\langle\cdot,\cdot\rangle_{\mathcal{H}}\big)$. The adjoint $\imath^+$ of the inclusion

$$\imath : \big(\mathcal{A},[\cdot,\cdot]_{\mathcal{A}}\big) \to \big(\mathcal{K},[\cdot,\cdot]_{\mathcal{K}}\big)$$

defined by

$$[\imath f,x]_{\mathcal{K}} = [f,\imath^+x]_{\mathcal{A}}, \qquad f\in\mathcal{A},\ x\in\mathcal{K},$$

is a continuous mapping from $\mathcal{K}$ to $\mathcal{A}$. The operator $\imath^+$ is a bounded self-adjoint operator in the Krein space $\mathcal{K}$, called the $\mathcal{K}$-*adjoint of the inclusion* $\imath$ of $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$. As before let $\imath^*$ be the adjoint of the inclusion

$$\imath : (\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}) \to (\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}).$$

Then, for $f \in \mathcal{A}$, $x \in \mathcal{K}$ we have

$$[f, \imath^+ x]_{\mathcal{A}} = [\imath f, x]_{\mathcal{K}} = \langle \imath f, Jx\rangle_{\mathcal{K}} = [f, \imath^* Jx]_{\mathcal{A}},$$

which implies that $\imath^+ = \imath^* J$. Consequently, the operator A is the $\mathcal{K}$-adjoint of the inclusion $\imath$ of $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ if and only if the operator AJ is the $\mathcal{H}$–adjoint of the inclusion $\imath$ of $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ in $(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}})$.

Remark 2.12. In Remark 2.2 it was shown that the Hilbert space $(\mathcal{L}, (\,\cdot\,,\cdot\,)_{\mathcal{L}})$ is continuously embedded in the Hilbert space $(\mathcal{G}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{G}})$. Therefore the Hilbert space $(\mathcal{L}, (\,\cdot\,,\cdot\,)_{\mathcal{L}})$ is continuously embedded in the Krein space $(\mathcal{G}, [\,\cdot\,,\cdot\,]_{\mathcal{G}})$. Let $L := \imath^+$ be the $\mathcal{K}$–adjoint of the corresponding inclusion. It follows from

$$(f, Lg)_{\mathcal{L}} = (f, \imath^+ g)_{\mathcal{L}} = [if, g]_{\mathcal{G}} = (Gf, g)_{\mathcal{L}} = (f, Gg)_{\mathcal{L}}, \quad f, g \in \mathcal{L},$$

that L is the continuous extension of G to $\mathcal{G}$.

Let A be a bounded self–adjoint operator in the Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$, and set $S := AJ$. Then S is a bounded self–adjoint operator in $(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}})$ with $\mathcal{R}(A) = \mathcal{R}(S)$. On the range $\mathcal{R}(A)$ we define the inner product $[\,\cdot\,,\cdot\,]_A$ by the relation

$$[u, v]_A := [Ax, y]_{\mathcal{K}}, \qquad \text{where} \qquad u = Ax,\ v = Ay, \quad x, y \in \mathcal{K}.$$

Then $[\,\cdot\,,\cdot\,]_A$ coincides with $(\,\cdot\,,\cdot\,)_S$ on $\mathcal{R}(A) = \mathcal{R}(S)$. Indeed, for $u = Ax$, $v = Ay$, we have

$$[u, v]_A = [Ax, y]_{\mathcal{K}} = [SJx, y]_{\mathcal{K}} = \langle SJx, Jy\rangle_{\mathcal{K}} = (SJx, SJy)_S = (u, v)_S.$$

Consequently the inner product $[\,\cdot\,,\cdot\,]_A$ can be extended by continuity to $\mathcal{R}(|AJ|^{1/2})$. This extension is also denoted by $[\,\cdot\,,\cdot\,]_A$, and it coincides with $(\,\cdot\,,\cdot\,)_S$. Since

$$\big(\mathcal{R}(|AJ|^{1/2}), [\,\cdot\,,\cdot\,]_A\big) = \big(\mathcal{R}(|S|^{1/2}), (\,\cdot\,,\cdot\,)_S\big), \tag{2.8}$$

the inner product space $(\mathcal{R}(|AJ|^{1/2}), [\,\cdot\,,\cdot\,]_A)$ is a Krein space and it is a Krein space completion of $(\mathcal{R}(A), [\,\cdot\,,\cdot\,]_A)$. Note that by Lemma 2.4 the Krein space

$\big(\mathcal{R}(|AJ|^{1/2}), [\,\cdot\,,\cdot\,]_A\big)$ is continuously embedded in $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$. We summarize these facts in

Theorem 2.13. *Let A be a bounded self–adjoint operator in the Krein space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$. Then $\big(\mathcal{R}(|AJ|^{1/2}), [\,\cdot\,,\cdot\,]_A\big)$ is continuously embedded in $\big(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}}\big)$ and a Krein space completion of $\big(\mathcal{R}(A), [\,\cdot\,,\cdot\,]_A\big)$.*

Lemma 2.9 implies that $\mathcal{R}(|AJ|^{1/2})$ does not depend on the choice of J.

Corollary 2.14. *Let J_1 and J_2 be fundamental symmetries in the Krein space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$ and let A be a bounded self–adjoint operator in $\mathcal{K}$. Then*

$$\mathcal{R}\big(|AJ_1|^{1/2}\big) = \mathcal{R}\big(|AJ_2|^{1/2}\big).$$

Proof. The corollary follows from the relation $\mathcal{R}(AJ_1) = \mathcal{R}(AJ_2)$ and Lemma 2.9, applied to the identity operator on $\mathcal{R}(AJ_1) = \mathcal{R}(AJ_2)$.

Theorem 2.15. *Let the Krein space $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ be continuously embedded in the Krein space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$ and let A be a bounded self–adjoint operator in $\mathcal{K}$. Then $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ is a Krein space completion of the inner product space $\big(\mathcal{R}(A), [\,\cdot\,,\cdot\,]_A\big)$ if and only if the operator A is the $\mathcal{K}$–adjoint of the inclusion of $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ in $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$; in this case*

$$[f, Ay]_{\mathcal{A}} = [f, y]_{\mathcal{K}}, \qquad f \in \mathcal{A},\ y \in \mathcal{K}. \tag{2.9}$$

Proof. Let J be a fundamental symmetry on $\mathcal{K}$ and $S = AJ$. At the beginning of this subsection we remarked that the operator A is the $\mathcal{K}$–adjoint of the inclusion $\imath$ of $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ in $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$ if and only if the operator S is $\mathcal{H}$–adjoint of the inclusion $\imath$ of $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ in $\big(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big) = \big(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}}\big)$. By Theorem 2.3, S is $\mathcal{H}$–adjoint of the inclusion $\imath$ of $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ in $\big(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\big)$ if and only if $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ is a Krein space completion of the inner product space $\big(\mathcal{R}(S), (\,\cdot\,,\cdot\,)_S\big)$. Since $\big(\mathcal{R}(S), (\,\cdot\,,\cdot\,)_S\big) = \big(\mathcal{R}(A), [\,\cdot\,,\cdot\,]_A\big)$, the equivalence in the theorem is proved.

Since $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ is a Krein space completion of $\big(\mathcal{R}(A), [\,\cdot\,,\cdot\,]_A\big)$ we have

$$[Av, Ay]_{\mathcal{A}} = [Av, Ay]_A = [Av, y]_{\mathcal{K}}, \qquad v,\ y \in \mathcal{K}.$$

The extension of the last equality in the topology of $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ yields (2.9).

Corollary 2.16. *Let the Krein space $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ be continuously embedded in the Krein space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$ and let A be the $\mathcal{K}$–adjoint of the inclusion of $\mathcal{A}$ in $\mathcal{K}$. Then $\mathcal{A}$ is dense in $\mathcal{K}$ if and only if $0 \notin \sigma_p(A)$; $\mathcal{A} = \mathcal{K}$ if and only if $0 \in \rho(A)$; in the latter case the norm topologies on $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ and $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$ coincide.*

Proof. If $0 \notin \sigma_p(A)$, then $\mathcal{R}(A)$ is dense in $\mathcal{K}$. By Theorem 2.15 $\mathcal{R}(A) \subset \mathcal{A}$, and therefore $\mathcal{A}$ is dense in $\mathcal{K}$. Assume that $\mathcal{A}$ is dense in $\mathcal{K}$. By Theorem 2.15 $\mathcal{R}(A)$ is dense in $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$. Since by assumption $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ is continuously embedded in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$, $\mathcal{R}(A)$ is also dense in $\mathcal{A}$ in the topology of $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$. Therefore $\mathcal{R}(A)$ is dense in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$, and thus $0 \notin \sigma_p(A)$. This proves the first statement.

If $0 \in \rho(A)$, then $\mathcal{K} = \mathcal{R}(A) = \mathcal{A}$. Conversely, if $\mathcal{A} = \mathcal{K}$ then, since $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ is continuously embedded in the Hilbert space $(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}})$, Remark 2.1 implies that the operator $S = AJ$ has bounded inverse. Consequently $0 \in \rho(A)$. The last statement is also an easy consequence of Remark 2.1.

Corollary 2.17. *Let the Krein space $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ be continuously embedded in the Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$. Assume that the subspace $\mathcal{F}$ is dense in $\mathcal{K}$, $\mathcal{F} \subset \mathcal{A}$, and*

$$[f, g]_{\mathcal{A}} = [f, g]_{\mathcal{K}}, \qquad f,\, g \in \mathcal{F}. \tag{2.10}$$

Then $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}) = (\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$.

Proof. Let A be the $\mathcal{K}$–adjoint of the inclusion of $\mathcal{A}$ in $\mathcal{K}$. Combining (2.9) and (2.10) we get

$$[f, Ag]_{\mathcal{A}} = [f, g]_{\mathcal{K}} = [f, g]_{\mathcal{A}}, \qquad f, g \in \mathcal{F}.$$

Since $\mathcal{F}$ is dense in $\mathcal{K}$ the last relation yields $A = I$ and then Theorem 2.15 implies the claim.

Remark 2.18. Let A be a bounded self–adjoint operator in the Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$. It follows from Theorem 2.13 that $(\mathcal{R}(|AJ|^{1/2}), [\,\cdot\,,\cdot\,]_A)$ is continuously embedded in $\mathcal{K}$ and a Krein space completion of $(\mathcal{R}(A), [\,\cdot\,,\cdot\,]_A)$. Therefore Theorem 2.15 implies that the operator A is the $\mathcal{K}$–adjoint of the inclusion $\imath$ of $(\mathcal{R}(|AJ|^{1/2}), [\,\cdot\,,\cdot\,]_A)$ in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$.

Remark 2.19. In the notation of Theorem 2.15, let A be the $\mathcal{K}$–adjoint of the inclusion of $\mathcal{A}$ in $\mathcal{K}$. Then for $f \in \mathcal{A}$ the relation

$$0 = [f, Ax]_{\mathcal{A}} = [f, x]_{\mathcal{K}}, \qquad x \in \mathcal{A}\,,$$

implies that the range $\mathcal{R}(A|_{\mathcal{A}})$ of the restriction of A to $\mathcal{A}$ is dense in $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ if and only if $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ is non–degenerate.

Corollary 2.20. *Let the Krein space $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ be continuously embedded in the Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ and let A be the $\mathcal{K}$–adjoint of the inclusion of $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$. Then $\mathcal{A}$ is a non-negative (non-positive,* *respectively) subspace of* $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ *if and only if* A^2 *is a non–negative (non–positive, respectively) operator in* $\mathcal{K}$*; in particular,* $\mathcal{A}$ *is a neutral subspace of* $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ *if and only if* $A^2 = 0$.

Proof. By Theorem 2.15 $\mathcal{R}(A)$ is dense in $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ and since $\mathcal{A}$ is continuously embedded in $\mathcal{K}$, $\mathcal{R}(A)$ is also dense in $\mathcal{A}$ with respect to the norm topology of $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$. Therefore $\mathcal{A}$ is non–negative in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ if and only if $\mathcal{R}(A)$ is non–negative in $\mathcal{K}$, and this is equivalent to $A^2 \geq_{\mathcal{K}} 0$.

3. Continuous and t–bounded embeddings

1. Recall that, for a positive number t, the Krein space $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ is said to be *t–boundedly* (*contractively, isometrically*, respectively) *embedded* in the Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ if $\mathcal{A} \subset \mathcal{K}$ and for all $f \in \mathcal{A}$ we have

$$[f,f]_{\mathcal{K}} \leq t\,[f,f]_{\mathcal{A}} \quad \big([f,f]_{\mathcal{K}} \leq [f,f]_{\mathcal{A}}, \quad [f,f]_{\mathcal{K}} = [f,f]_{\mathcal{A}}, \text{ respectively}\big). \tag{3.1}$$

If $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ is continuously and t–boundedly embedded in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ we denote the $\mathcal{K}$–adjoint of the inclusion of $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ again by A. Applying the inequality (3.1) to the element $f = Ax$, $x \in \mathcal{K}$, and observing (2.9), we obtain

$$[A^2x, x]_{\mathcal{K}} = [Ax, Ax]_{\mathcal{K}} \leq t\,[Ax, x]_{\mathcal{K}}, \quad x \in \mathcal{K}\,,$$

that is

$$t\,A \; - \; A^2 \;\; \geq_{\mathcal{K}} \;\; 0. \tag{3.2}$$

If $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ is continuously and contractively (isometrically, respectively) embedded in the Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$, the relation (3.2) becomes

$$A \; - \; A^2 \; \geq_{\mathcal{K}} \; 0 \qquad (A \; - \; A^2 = 0, \;\; \text{respectively}).$$

In the latter case this means that A is the orthogonal projection onto $\mathcal{A}$ in $\mathcal{K}$.

Theorem 3.1. *Let* $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ *be a Krein space and let* A *be a bounded definitizable operator in* $\mathcal{K}$ *with the definitizing polynomial*

$$p(\lambda) = t\,\lambda - \lambda^2, \tag{3.3}$$

where $t > 0$*. Then the Krein space* $(\mathcal{R}(|AJ|^{1/2}), [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ *is continuously and* t*–boundedly embedded in the Krein space* $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$*. Moreover, the mapping*

$$A \quad \longmapsto \quad \big(\mathcal{R}(|AJ|^{1/2}), [\,\cdot\,,\cdot\,]_{A}\big) \tag{3.4}$$

establishes a bijective correspondence between all bounded definitizable operators A in $\mathcal{K}$ with a definitizing polynomial (3.3) *and all Krein spaces* $(\mathcal{A},[\,\cdot\,,\cdot\,]_{\mathcal{A}})$ *which are continuously and t–boundedly embedded in the Krein space* $(\mathcal{K},[\,\cdot\,,\cdot\,]_{\mathcal{K}})$*. The inverse of the mapping* (3.4) *maps each Krein space* $(\mathcal{A},[\,\cdot\,,\cdot\,]_{\mathcal{A}})$*, which is continuously and t–boundedly embedded in* $(\mathcal{K},[\,\cdot\,,\cdot\,]_{\mathcal{K}})$*, to the $\mathcal{K}$–adjoint A of the inclusion of* $(\mathcal{A},[\,\cdot\,,\cdot\,]_{\mathcal{A}})$ *in* $(\mathcal{K},[\,\cdot\,,\cdot\,]_{\mathcal{K}})$*; in particular*

$$\big(\mathcal{A},[\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)=\big(\mathcal{R}(|AJ|^{1/2}),[\,\cdot\,,\cdot\,]_{A}\big). \tag{3.5}$$

Proof. By Theorem 2.13 the Krein space $\big(\mathcal{R}(|AJ|^{1/2}),[\,\cdot\,,\cdot\,]_A\big)$ is continuously embedded in $(\mathcal{K},[\,\cdot\,,\cdot\,]_{\mathcal{K}})$. For $f=Ax$, $x\in\mathcal{K}$, the inequality $tA-A^2\geq_{\mathcal{K}}0$ yields

$$[f,f]_{\mathcal{K}}=[Ax,Ax]_{\mathcal{K}}\leq t\,[Ax,x]_{\mathcal{K}}=t\,[f,f]_{\mathcal{A}}, \tag{3.6}$$

which extends by continuity to $\mathcal{R}(|AJ|^{1/2})$. Thus $\big(\mathcal{R}(|AJ|^{1/2}),[\,\cdot\,,\cdot\,]_A\big)$ is continuously and t–boundedly embedded in $\mathcal{K}$, and the first statement of the theorem is proved.

Now let $(\mathcal{A},[\,\cdot\,,\cdot\,]_{\mathcal{A}})$ be a Krein space which is continuously and t–boundedly embedded in $\mathcal{K}$; denote by A the $\mathcal{K}$–adjoint of the inclusion of $(\mathcal{A},[\,\cdot\,,\cdot\,]_{\mathcal{A}})$ in $(\mathcal{K},[\,\cdot\,,\cdot\,]_{\mathcal{K}})$. From Theorem 2.15 we conclude that $(\mathcal{A},[\,\cdot\,,\cdot\,]_{\mathcal{A}})$ is a Krein space completion of the inner product space $\big(\mathcal{R}(A),[\,\cdot\,,\cdot\,]_A\big)$. Since $(\mathcal{A},[\,\cdot\,,\cdot\,]_{\mathcal{A}})$ is t–boundedly embedded in $\mathcal{K}$ the considerations preceding Theorem 3.1 show that A is a bounded definitizable operator with the definitizing polynomial (3.3). To show that the image of A under the mapping (3.4) is $(\mathcal{A},[\,\cdot\,,\cdot\,]_{\mathcal{A}})$ we shall show that for some fundamental symmetry J in $\mathcal{K}$ we have $[-\varepsilon,0)\subset\rho(AJ)$ with an $\varepsilon>0$. Fix d, $0<d<t$, and consider the intervals $\Delta_0:=(-\infty,d]$ and $\Delta_1:=(d,+\infty)$. The corresponding spectral subspaces of A are denoted by $\mathcal{K}_{\Delta_0}$ and $\mathcal{K}_{\Delta_1}$, respectively. They are Krein spaces and their orthogonal sum is the Krein space $\mathcal{K}$. We choose fundamental symmetries J_0 and J_1 in $\mathcal{K}_{\Delta_0}$ and $\mathcal{K}_{\Delta_1}$, respectively, and in $\mathcal{K}$ the fundamental symmetry $J:=J_0\dotplus J_1$. If the restrictions of A to $\mathcal{K}_{\Delta_0}$ and $\mathcal{K}_{\Delta_1}$ are denoted by A_0 and A_1, respectively, then A is the direct and orthogonal sum of A_0 and A_1. The operator A_0 in the Krein space $\mathcal{K}_0$ is non–negative, and hence also $\sigma(A_0J_0)$ is non–negative. Since 0 belongs to $\rho(A_1)$ and hence also to $\rho(A_1J_1)$, an interval of the form $[-\varepsilon,0)$ belongs to $\rho(AJ)$.

Now Theorem 2.7 implies that the Krein space $\big(\mathcal{R}(|AJ|^{1/2}),[\,\cdot\,,\cdot\,]_A\big)$ is the unique continuously in $\mathcal{H}$ embedded Krein space completion of the inner product space $\big(\mathcal{R}(A),[\,\cdot\,,\cdot\,]_A\big)$. Since $(\mathcal{A},[\,\cdot\,,\cdot\,]_{\mathcal{A}})$ is also continuously embedded in $\mathcal{K}$ and a Krein space completion of the inner product space $\big(\mathcal{R}(A),[\,\cdot\,,\cdot\,]_A\big)$ we conclude that (3.5) holds.

Corollary 3.2. *If the Hilbert space* $(\mathcal{A},(\,\cdot\,,\cdot\,)_{\mathcal{A}})$ *is continuously embedded in the Krein space* $(\mathcal{K},[\,\cdot\,,\cdot\,]_{\mathcal{K}})$*, then there exist a $t>0$ and a unique* *non–negative bounded operator A in $(\mathcal{K},[\,\cdot\,,\cdot\,]_{\mathcal{K}})$ such that $(\mathcal{A},(\,\cdot\,,\cdot\,)_{\mathcal{A}})$ is t–boundedly embedded in the Krein space $(\mathcal{K},[\,\cdot\,,\cdot\,]_{\mathcal{K}})$ and*

$$\big(\mathcal{A},(\,\cdot\,,\cdot\,)_{\mathcal{A}}\big)=\big(\mathcal{R}(AJ^{1/2}),[\,\cdot\,,\cdot\,]_{A}\big).$$

The mapping (3.4) *establishes a bijective correspondence between all bounded non–negative operators A in $(\mathcal{K},[\,\cdot\,,\cdot\,]_{\mathcal{K}})$ and all Hilbert spaces $(\mathcal{A},[\,\cdot\,,\cdot\,]_{\mathcal{A}})$ which are continuously embedded in $\mathcal{K}$.*

Proof. Let J be a fundamental symmetry on $(\mathcal{K},[\,\cdot\,,\cdot\,]_{\mathcal{K}})$, and denote by $\langle\cdot,\cdot\rangle_{\mathcal{K}}$ the corresponding Hilbert space inner product. Since the inclusion $\imath:\mathcal{A}\to\mathcal{K}$ is continuous, there exists a $t>0$ such that $\langle x,x\rangle_{\mathcal{K}}\le t(x,x)_{\mathcal{A}},\ x\in\mathcal{A}$. The relation $[x,x]_{\mathcal{K}}\le|[x,x]_{\mathcal{K}}|\le\langle x,x\rangle_{\mathcal{K}},\ x\in\mathcal{K}$, implies that $[x,x]_{\mathcal{K}}\le t(x,x)_{\mathcal{A}},\ x\in\mathcal{A}$, that is, $(\mathcal{A},(\,\cdot\,,\cdot\,)_{\mathcal{A}})$ is t–boundedly embedded in the Krein space $(\mathcal{K},[\,\cdot\,,\cdot\,]_{\mathcal{K}})$. Now the Corollary 3.2 follows from Theorem 3.1.

2. Let again the Krein space $(\mathcal{A},[\,\cdot\,,\cdot\,]_{\mathcal{A}})$ be continuously and t–boundedly embedded in the Krein space $(\mathcal{K},[\,\cdot\,,\cdot\,]_{\mathcal{K}})$ and let A be the $\mathcal{K}$–adjoint of the corresponding inclusion. Then A is a definitizable operator with the definitizing polynomial p from (3.3). Hence A is determined by its spectral function E and two non-negative nilpotent operators N_0, N_1 in the Krein space $\mathcal{K}$ with the properties

$$N_0^2=N_1^2=0,\quad N_0\,N_1=0,\quad N_0\,E(\Delta)=0\ \text{ if }\ 0\notin\Delta,\ N_1\,E(\Delta)=0\ \text{ if }\ t\notin\Delta,$$

for all intervals Δ with endpoints different from 0 and t. In fact with the intervals Δ_0, Δ_1 in the proof of Theorem 3.1 we have

$$A=\int_{\Delta_0}'\lambda\,dE_\lambda\,+\,N_0\,+\,N_1\,+\int_{\Delta_1}'(\lambda-t)\,dE_\lambda\,+\,t\,E(\Delta_1);$$

the prime at the integrals indicates that they are improper at 0 and t in the strong operator topology.

Since $\mathcal{R}(A)$ is contained in $\mathcal{A}$, the operator A maps also the space $(\mathcal{A},[\,\cdot\,,\cdot\,]_{\mathcal{A}})$ into itself. Denote the restriction of A to $\mathcal{A}$ by $\widehat{A}$. Then the operator $\widehat{A}$ is continuous in $\mathcal{A}$ since it is closed in $\mathcal{A}$. Indeed, if $\{x_n,\widehat{A}x_n\}\to\{u,v\},\ n\to+\infty$, in $\mathcal{A}\oplus\mathcal{A}$, then, since $\mathcal{A}$ is continuously embedded in $\mathcal{K}$, $x_n\to u$ and $\widehat{A}x_n=Ax_n\to v$ in $\mathcal{K}$, and hence $v=Au$.

Lemma 3.3. *The operator $\widehat{A}$ in the Krein space $(\mathcal{A},[\,\cdot\,,\cdot\,]_{\mathcal{A}})$ is definitizable with the definitizing polynomial $\widehat{p}(\lambda)=t-\lambda$.*

Proof. If $f=Ax$, then

$$[(t-A)f,f]_{\mathcal{A}}=[A(t-A)x,Ax]_{\mathcal{A}}=[A(t-A)x,x]_{\mathcal{K}}\ge 0,$$

and by continuity this relation extends to all elements $f \in \mathcal{A}$.

For simplicity we formulate the following statements only for an admissible intervals Δ. They can be extended in an obvious way to finite unions of admissible intervals and also to more general (measurable) sets. Having in mind only the operators A and $\widehat{A}$ with the definitizing polynomials $p(\lambda) = t\lambda - \lambda^2$ and $\widehat{p}(\lambda) = t - \lambda$, an admissible interval for A ($\widehat{A}$, respectively) denotes here an interval Δ for which 0 and t (t, respectively) are not boundary points of Δ. Further, the spectral subspace of A in $\mathcal{K}$ ($\widehat{A}$ in $\mathcal{A}$, respectively) corresponding to Δ is denoted by $\mathcal{K}_\Delta$ ($\mathcal{A}_\Delta$, respectively).

Lemma 3.4. *If Δ is an admissible interval for A then*

$$\mathcal{A}_\Delta = \mathcal{K}_\Delta \cap \mathcal{A}, \tag{3.7}$$

Proof. For $g \in \mathcal{A}$ we have

$$f := (\widehat{A} - z)g = (A - z)g \in \mathcal{A}$$

and hence

$$(\widehat{A} - z)^{-1} f = (\widehat{A} - z)^{-1} f, \quad z \in \rho(\widehat{A}) \cap \rho(A),\ f \in \mathcal{A}. \tag{3.8}$$

If $f \in \mathcal{A}_\Delta$ then $f \in \mathcal{A}$ and, as a function of z, $(\widehat{A} - z)^{-1} z$ has a holomorphic continuation outside $\overline{\Delta}$. According to (3.8) and because of the continuity of the inclusion of $\mathcal{A}$ in $\mathcal{K}$ also the function $(A - z)^{-1} z$ has a holomorphic continuation outside $\overline{\Delta}$, and hence $f \in \mathcal{K}_\Delta$. Conversely, if $f \in \mathcal{K}_\Delta \cap \mathcal{A}$ then (3.8) implies that the function

$$\big[(\widehat{A} - z)^{-1} f, Ay\big]_{\mathcal{A}} = \big[(A - z)^{-1} f, y\big]_{\mathcal{K}}$$

has a holomorphic continuation outside $\overline{\Delta}$. Since $\|(\widehat{A} - z)^{-1} f\|_{\mathcal{A}} \le C_z \|f\|_{\mathcal{A}}$, with constants C_z which can be chosen locally uniformly bounded with respect to z outside $\overline{\Delta}$, also $(\widehat{A} - z)^{-1} f$ has a holomorphic continuation in $\mathcal{A}$ outside $\overline{\Delta}$, and hence $f \in \mathcal{A}_\Delta$.

According to [L, Theorem 3.1], as an immediate consequence of the definitizability of the operators A and $\widehat{A}$ we have the following

Corollary 3.5. (a) *Let Δ be an admissible interval for A. If $\Delta \subset (0,t)$ then $\big(\mathcal{K}_\Delta, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ is a Hilbert space; if $\Delta \subset \mathbb{R} \setminus (0,t)$, then $\big(\mathcal{K}_\Delta, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ is an anti–Hilbert space.*

(b) *Let Δ be an admissible interval for $\widehat{A}$. If $\Delta \subset (-\infty,t)$ then $\big(\mathcal{A}_\Delta, [\,\cdot\,,\cdot\,]_\mathcal{A}\big)$ is a Hilbert space; if $\Delta \subset (t,+\infty)$, then $\big(\mathcal{A}_\Delta, [\,\cdot\,,\cdot\,]_\mathcal{A}\big)$ is an anti-Hilbert space.*

Theorem 3.6. *Suppose that the Krein space $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_\mathcal{A}\big)$ is continuously and t–boundedly embedded in the Krein space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$, and denote by A the $\mathcal{K}$–adjoint of the corresponding inclusion. If Δ is an admissible interval for A, then the following statements hold:*

(a) *If $0 \notin \Delta$ then $\mathcal{A}_\Delta = \mathcal{K}_\Delta \subset \mathcal{R}(A)$.*

(b) *If $\Delta \subset (0,+\infty)$, then the Krein spaces $\big(\mathcal{K}_\Delta, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ and $\big(\mathcal{A}_\Delta, [\,\cdot\,,\cdot\,]_\mathcal{A}\big)$ are isomorphic, if $\Delta \subset (-\infty,0)$, then the Hilbert spaces $\big(\mathcal{K}_\Delta, -[\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ and $\big(\mathcal{A}_\Delta, [\,\cdot\,,\cdot\,]_\mathcal{A}\big)$ are isomorphic.*

(c) *The space $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_\mathcal{A}\big)$ is a Pontryagin space with negative index κ if and only if the total multiplicity of the spectrum of A in $(t,+\infty)$ and the negative index of t as a spectral point of A in $\mathcal{K}$ are both finite and their sum equals κ; it is a Hilbert space if and only if $\sigma(A) \subset (-\infty,t]$ and $\big(\ker(t-A), [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ is a Hilbert space.*

Proof. If $0 \notin \Delta$ then $\mathcal{K}_\Delta \subset \mathcal{R}(A)$ and the first claim of (a) follows from (3.7). To prove (b) let $\Delta \subset (0,+\infty)$. Then $\mathcal{K}_\Delta = \mathcal{A}_\Delta$, and if the restriction of A to $\mathcal{K}_\Delta$ is denoted by A_1, we have $\sigma(A_1) \subset (0,+\infty)$ and

$$[x,y]_\mathcal{A} = [A_1^{-1}x, y]_\mathcal{K}, \quad x,y \in \mathcal{K}_\Delta.$$

By means of the Riesz-Dunford functional calculus a bounded, boundedly invertible and self–adjoint operator $A_1^{-1/2} : \mathcal{K}_\Delta = \mathcal{A}_\Delta \to \mathcal{K}_\Delta$ can be defined in $\big(\mathcal{K}_\Delta, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ such that $(A_1^{-1/2})^2 = A_1$, and the relation (3.9) becomes

$$[x,y]_\mathcal{A} = [A_1^{-1/2}x, A_1^{-1/2}y]_\mathcal{K}, \qquad x,y \in \mathcal{K}_\Delta = \mathcal{A}_\Delta.$$

Therefore $A_1^{-1/2}$ is an isomorphism between $\big(\mathcal{A}_\Delta, [\,\cdot\,,\cdot\,]_\mathcal{A}\big)$ and $\big(\mathcal{K}_\Delta, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$. A similar reasoning, applied to a negative interval Δ, yields the second statement of (b).

In the next theorem, under some additional assumptions on the operator A we give a characterization of $\mathcal{A}$ by means of the spectral function of A.

Theorem 3.7. *Suppose that the Krein space $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_\mathcal{A}\big)$ is continuously and t–boundedly embedded in the Krein space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$, and denote by A the $\mathcal{K}$–adjoint of the corresponding inclusion. If Δ_0 is an admissible* *interval for A such that $0 \in \Delta_0$, $1 \notin \Delta_0$, and $\ker A$ is projectionally complete in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$, then*

$$\mathcal{A} = \left\{ x \in \mathcal{K} \ : \ x[\perp]_{\mathcal{K}} \ker A, \ \int_{\Delta_0}' \frac{1}{\lambda} \, d[E_\lambda x, x] < +\infty \right\}.$$

Proof. Denote by A_0 the restriction of the operator A to the subspace $\mathcal{K}_{\Delta_0}$, and let J_0 be a fundamental symmetry in $\mathcal{K}_{\Delta_0}$. Since $\ker A$ is projectionally complete in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$, it follows from [Ć, Proposition 3.1] that

$$\mathcal{R}\big(\big|A_0 J_0\big|^{1/2}\big) = \left\{ x \in \mathcal{K}_{\Delta_0} \ : \ x[\perp]_{\mathcal{K}} \ker A, \ \int_{\Delta_0}' \frac{1}{\lambda} \, d[E_\lambda x, x] < +\infty \right\}.$$

The fact that the restriction of the operator A to $\mathcal{K}_{\mathbb{R}\setminus\Delta_0}$ has a bounded inverse implies

$$\mathcal{R}\big(\big|AJ\big|^{1/2}\big) = \mathcal{K}_{\mathbb{R}\setminus\Delta_0}[\dot{+}]_{\mathcal{K}} \mathcal{R}\big(\big|A_0 J_0\big|^{1/2}\big).$$

Denote by y the $[\,\cdot\,,\cdot\,]_{\mathcal{K}}$–orthogonal projection of x onto $\mathcal{K}_{\Delta_0}$. Then $x \in \mathcal{R}\big(\big|AJ\big|^{1/2}\big)$ if and only if $y \in \mathcal{R}\big(\big|A_0 J_0\big|^{1/2}\big)$. Since the integrals

$$\int_{\Delta_0}' \frac{1}{\lambda} \, d[E_\lambda x, x] \qquad \text{and} \qquad \int_{\Delta_0}' \frac{1}{\lambda} \, d[E_\lambda y, y]$$

converge simultaneously the theorem is proved.

4. Continuous contractive embeddings and complementation

1. Let the Krein space $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ be continuously and contractively embedded in the Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$. Then the $\mathcal{K}$–adjoint A of the inclusion of $\mathcal{A}$ in $\mathcal{K}$ is definitizable:

$$A \ - \ A^2 \ \geq_{\mathcal{K}} \ 0, \tag{4.1}$$

and, moreover, by Theorem 3.1 the mapping

$$A \quad \longmapsto \quad \big(\mathcal{R}(|AJ|^{1/2}), [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$$

establishes a bijective correspondence between all bounded definitizable operators in $\mathcal{K}$ satisfying (4.1) and all Krein spaces $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$, which are continuously and contractively embedded in the Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$. We call the operator A the *generalized projection* corresponding to the Krein space $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$.

If $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ is continuously and *isometrically* embedded in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ then $A^2 = A$ and A is the orthogonal projection onto $\mathcal{A}$ in $\mathcal{K}$. There is another "extremal" case, namely $A^2 = 0$, $A \geq_{\mathcal{K}} 0$. This case will be considered in more detail at the end of this section.

If the operator A satisfies the relation (4.1) also the operator $B := I - A$ does:

$$B - B^2 = (I - A) - (I - A)^2 = A - A^2 \geq_{\mathcal{K}} 0.$$

Therefore also B is the $\mathcal{K}$–adjoint of the inclusion of the Krein space

$$\big(\mathcal{R}(|BJ|^{1/2}), [\,\cdot\,,\cdot\,]_B\big) = \big(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{B}}\big)$$

in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$, which is continuously and contractively embedded in $\mathcal{K}$. As we show in Theorem 4.2 below, the two Krein spaces $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ and $(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{B}})$ are complementary in the sense of the following definition of L. de Branges.

Definition 4.1. If the Krein spaces $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ and $(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{B}})$ are continuously embedded in the Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$, they are said to be *complementary* in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$, or $(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{B}})$ is said to be *complementary to* $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$, if

(i) $c = a + b$ with $a \in \mathcal{A}$, $b \in \mathcal{B}$ implies

$$[c, c]_{\mathcal{K}} \leq [a, a]_{\mathcal{A}} + [b, b]_{\mathcal{B}}. \tag{4.2}$$

(ii) Each element $c \in \mathcal{K}$ admits some decomposition $c = a + b$, $a \in \mathcal{A}$, $b \in \mathcal{B}$, for which the equality sign in (4.2) holds: $[c, c]_{\mathcal{K}} = [a, a]_{\mathcal{A}} + [b, b]_{\mathcal{B}}$.

The decomposition in (ii) is called a *minimal decomposition* of $c \in \mathcal{K}$, and the space $(\mathcal{A} \cap \mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{A}} + [\,\cdot\,,\cdot\,]_{\mathcal{B}})$ is called the *overlapping space* of the complementary spaces $\mathcal{A}$ and $\mathcal{B}$.

The relation (4.2) with $b = 0$ or $a = 0$ implies that complementary Krein spaces $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ and $(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{B}})$ are contractively embedded in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$. Below we show that to each Krein space $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$, continuously and contractively embedded in the Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$, there corresponds a unique Krein space $(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{B}})$ which is also continuously and contractively embedded in the Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ and complementary to $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$, and the complementary space of $(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{B}})$ is $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$.

Theorem 4.2. *Let the Krein space* $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ *be continuously and contractively embedded in the Krein space* $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$. *Let* A *be the corresponding generalized projection; hence* $\mathcal{A} = \big(\mathcal{R}(|AJ|^{1/2}), [\,\cdot\,,\cdot\,]_A\big)$. *If* $B = I - A$, *then the Krein space* $\big(\mathcal{R}(|BJ|^{1/2}), [\,\cdot\,,\cdot\,]_B\big) = \big(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{B}}\big)$ *is the unique complementary space to* $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ *in* $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$*, and each element* $c \in \mathcal{K}$ *has the unique minimal decomposition* $c = a + b$ *with* $a = Ac$ *and* $b = Bc$*.*

Proof. Let A be the generalized projection for $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$, that is A is a self–adjoint operator in the Krein space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$ such that

$$\big(\mathcal{R}(|AJ|^{1/2}), [\,\cdot\,,\cdot\,]_A\big) = \big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$$

and $A - A^2 \geq_{\mathcal{K}} 0$. Set $B = I - A$. Then $AB = A(I - A) = BA$ is also a non–negative operator in $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$, therefore $\big(\mathcal{R}(AB), [\,\cdot\,,\cdot\,]_{AB}\big)$ is a pre–Hilbert space. Its completion is the Hilbert space $\big(\mathcal{R}((ABJ)^{1/2}), [\,\cdot\,,\cdot\,]_{AB}\big)$. It follows from Lemma 2.9 that the mappings

$$A : \big(\mathcal{R}(|BJ|^{1/2}), [\,\cdot\,,\cdot\,]_B\big) \to \big(\mathcal{R}((ABJ)^{1/2}), [\,\cdot\,,\cdot\,]_{AB}\big) \tag{4.3}$$

and

$$B : \big(\mathcal{R}(|AJ|^{1/2}), [\,\cdot\,,\cdot\,]_A\big) \to \big(\mathcal{R}((BAJ)^{1/2}), [\,\cdot\,,\cdot\,]_{BA}\big) \tag{4.4}$$

are bounded.

To prove that the Krein space $\big(\mathcal{R}(|BJ|^{1/2}), [\,\cdot\,,\cdot\,]_B\big)$ is a complementary space to $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ note that for $x, y \in \mathcal{K}$ we have

$$\begin{aligned}
[Ax + By, &Ax + By]_{\mathcal{K}} - [Ax, Ax]_A - [By, By]_B \\
&= [Ax, Ax + By]_{\mathcal{K}} + [By, Ax + By]_{\mathcal{K}} - [Ax, x]_{\mathcal{K}} - [By, y]_{\mathcal{K}} \\
&= [y - x, AB(x - y)]_{\mathcal{K}} \\
&= -[AB(x - y), AB(x - y)]_{AB} \;\leq\; 0.
\end{aligned} \tag{4.5}$$

Since $\mathcal{R}(A)$ is dense in $\mathcal{R}(|AJ|^{1/2})$ and $\mathcal{R}(B)$ is dense in $\mathcal{R}(|BJ|^{1/2})$, for given $a \in \mathcal{R}(|AJ|^{1/2})$ and $b \in \mathcal{R}(|BJ|^{1/2})$ there exist sequences (x_n) and (y_n) in $\mathcal{K}$ such that $Ax_n \to a$ $(n \to +\infty)$ in $\mathcal{R}(|AJ|^{1/2})$ and $By_n \to b$ $(n \to +\infty)$ in $\mathcal{R}(|BJ|^{1/2})$. It follows from the boundedness of A and B in (4.3) and (4.4) that $BAx_n \to Ba$ and $ABy_n \to Ab$ $(n \to +\infty)$ in $\big(\mathcal{R}((ABJ)^{1/2}), [\,\cdot\,,\cdot\,]_{AB}\big)$. This, together with (4.5), implies

$$[a + b, a + b]_{\mathcal{K}} - [a, a]_A - [b, b]_B = -[Ba - Ab, Ba - Ab]_{AB} \leq 0 \tag{4.6}$$

for all $a \in \mathcal{R}(|AJ|^{1/2})$ and $b \in \mathcal{R}(|BJ|^{1/2})$, and the inequality (4.2) is proved. It is clear that with $a = Ac$ and $b = Bc$ in (4.2) the equality sign holds for arbitrary $c \in \mathcal{K}$. Therefore the Krein space $\big(\mathcal{R}(|BJ|^{1/2}), [\,\cdot\,,\cdot\,]_B\big)$ is a complementary space to $\big(\mathcal{R}(|AJ|^{1/2}), [\,\cdot\,,\cdot\,]_A\big)$.

It remains to show that the minimal decomposition $c = a + b$ of an element $c \in \mathcal{K}$ is unique. The relation (4.6) implies that

$$[c,c]_{\mathcal{K}} = [a,a]_A + [b,b]_B,\ c = a+b,\ a \in \mathcal{R}(|AJ|^{1/2}),\ b \in \mathcal{R}(|BJ|^{1/2}) \tag{4.7}$$

is equivalent to $[Ab - Ba, Ab - Ba]_{AB} = 0$. Since the inner product $[\,\cdot\,,\cdot\,]_{AB}$ is positive definite, we have $Ab = Ba$. Therefore,

$$Ac = Aa + Ab = Aa + Ba = a, \quad Bc = Ba + Bb = Ab + Bb = b,$$

and, consequently, a and b are uniquely determined by (4.7).

The uniqueness of the complementary subspace will be proved in Theorem 4.4 below.

2. In Theorem 4.4 we give another characterization of the complementary space which implies its uniqueness. Our proof is different from the proofs in [DB] and [DR]. It is based on the following variation of a characterization of operator ranges due to Šmul′jan [Š] (see also [FW]).

Lemma 4.3. *Let $(\mathcal{H}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}})$ be a Hilbert space and let S be a bounded operator in $\mathcal{H}$. Then $y \in \mathcal{R}(S)$ if and only if*

$$\sup\bigl\{2|\langle x,y\rangle_{\mathcal{H}}| - \langle S^*x, S^*x\rangle_{\mathcal{H}} : x \in \mathcal{H}\bigr\} < +\infty.$$

Proof. A simple calculation shows that

$$\begin{aligned}
&\sup\bigl\{2|\langle x,y\rangle_{\mathcal{H}}| - \langle S^*x, S^*x\rangle_{\mathcal{H}} : x \in \mathcal{H}\bigr\} \\
&\quad = \sup\bigl\{2|\langle x,y\rangle_{\mathcal{H}}| - \langle S^*x, S^*x\rangle_{\mathcal{H}} : x \notin \ker(S^*)\bigr\} \\
&\quad = \sup\bigl\{2t|\langle x,y\rangle_{\mathcal{H}}| - t^2\langle S^*x, S^*x\rangle_{\mathcal{H}} : t > 0, x \notin \ker(S^*)\bigr\} \\
&\quad = \sup\left\{\frac{|\langle x,y\rangle_{\mathcal{H}}|^2}{\langle S^*x, S^*x\rangle_{\mathcal{H}}} : x \notin \ker(S^*)\right\}.
\end{aligned}$$

Now the lemma follows from [Š, Lemma 3].

Theorem 4.4. *Let $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ be a Krein space which is continuously and contractively embedded in the Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$, and let A be the corresponding generalized projection. Then the complementary space $(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{B}})$ to $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ in $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ is uniquely determined: It is the set of all $b \in \mathcal{K}$ such that*

$$\tau(b) := \sup\Bigl\{[b+a, b+a]_{\mathcal{K}} - [a,a]_{\mathcal{A}} : a \in \mathcal{A}\Bigr\} < +\infty$$

with inner product given by the relation

$$\big[b_1, b_2\big]_{\mathcal{B}} := \frac{1}{4}\big(\tau(b_1+b_2) - \tau(b_1-b_2) + i\,\tau(b_1+i\,b_2) - i\,\tau(b_1-i\,b_2)\big),$$

where $i = \sqrt{-1}$.

Proof. Denote by $\mathcal{B}$ the set of all elements $b \in \mathcal{K}$ such that $\tau(b) < +\infty$. First we prove that $\mathcal{B} = \mathcal{R}(|BJ|^{1/2})$, where $B = I - A$. Since $\mathcal{R}(A)$ is dense in $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ we have

$$\tau(b) = \sup\big\{[b+Ax, b+Ax]_{\mathcal{K}} - [Ax, Ax]_A : x \in \mathcal{K}\big\}.$$

For all $x \in \mathcal{K}$,

$$[b+Ax, b+Ax]_{\mathcal{K}} - [Ax, Ax]_A = [b, b]_{\mathcal{K}} + 2\,[Ax, b]_{\mathcal{K}} - [A(I-A)x, x]_{\mathcal{K}},$$

therefore

$$\begin{aligned} \tau(b) &= [b,b]_{\mathcal{K}} + \sup\big\{2\,[Ax,b]_{\mathcal{K}} - [A(I-A)x,x]_{\mathcal{K}} : x \in \mathcal{K}\big\} \\ &= [b,b]_{\mathcal{K}} + \sup\big\{2|[Ax,b]_{\mathcal{K}}| - [ABx,x]_{\mathcal{K}} : x \in \mathcal{K}\big\} \qquad (4.8) \\ &= [b,b]_{\mathcal{K}} + \sup\big\{2|\langle x, Ab\rangle_{\mathcal{K}}| - \langle ABJx, x\rangle_{\mathcal{K}} : x \in \mathcal{K}\big\}. \end{aligned}$$

By Lemma 4.3 the last supremum is finite if and only if $Ab \in \mathcal{R}\big((ABJ)^{1/2}\big)$. (Note that the operator ABJ is positive in the Hilbert space $\big(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}}\big)$.) We have $Ab \in \mathcal{R}\big((ABJ)^{1/2}\big)$ if and only if $b \in \mathcal{R}(|BJ|^{1/2})$. Indeed, (4.3) implies that if $b \in \mathcal{R}(|BJ|^{1/2})$, then $Ab \in \mathcal{R}\big((ABJ)^{1/2}\big)$. Conversely, if $Ab \in \mathcal{R}\big((ABJ)^{1/2}\big)$, then, by Theorem 2.9, $Ab \in \mathcal{R}(|BJ|^{1/2})$. Therefore, $b = Ab + Bb \in \mathcal{R}(|BJ|^{1/2})$. Hence, the supremum in (4.8) is finite if and only if $b \in \mathcal{R}(|BJ|^{1/2})$. This yields $\mathcal{B} = \mathcal{R}(|BJ|^{1/2}) = \mathcal{R}\big(|(I-A)J|^{1/2}\big)$.

It follows from (4.8) and Lemma 4.3 that $\tau(b) = [b,b]_{\mathcal{K}} + [Ab, Ab]_{AB},\ b \in \mathcal{R}(|BJ|^{1/2}) = \mathcal{B}$. This and (4.3) imply that $\tau : \mathcal{R}(|BJ|^{1/2}) \to \mathbb{R}$ is a continuous function. It is not difficult to see that $\tau(b) = [b,b]_B$ for all $b \in \mathcal{R}(B)$. Since $\mathcal{R}(B)$ is dense in $\mathcal{R}(|BJ|^{1/2})$, we have $\tau(b) = [b,b]_B$ for all $b \in \mathcal{R}(|BJ|^{1/2})$.

Let the Krein space $(\mathcal{C}, [\,\cdot\,,\cdot\,]_{\mathcal{C}})$ be complementary to

$$\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big) = \big(\mathcal{R}(|AJ|^{1/2}), [\,\cdot\,,\cdot\,]_A\big).$$

Then $\mathcal{C} \subseteq \mathcal{B} = \mathcal{R}(|BJ|^{1/2})$ and

$$[b,b]_{\mathcal{C}} \geq \tau(b) = [b,b]_B \quad \text{for all} \quad b \in \mathcal{B}.$$

If $c = a + b,\ a \in \mathcal{A},\ b \in \mathcal{C}$ and if $[c,c]_{\mathcal{K}} = [a,a]_{\mathcal{A}} + [b,b]_{\mathcal{C}}$, then

$$[a,a]_A + [b,b]_B \geq [c,c]_{\mathcal{K}} = [a,a]_{\mathcal{A}} + [b,b]_{\mathcal{C}} \geq [a,a]_A + [b,b]_B.$$

Hence $[b,b]_B = [b,b]_{\mathcal{C}} = [c,c]_{\mathcal{K}} - [a,a]_{\mathcal{A}}$. Now the relation (4.6) implies $b = Bc$ and $a = Ac$. Since $c \in \mathcal{K}$ was arbitrary, it follows that $\mathcal{R}(B) \subset \mathcal{C}$ and that $[\,\cdot\,,\cdot\,]_{\mathcal{C}}$ coincides with $[\,\cdot\,,\cdot\,]_B$ on $\mathcal{R}(B)$. Since $\big(\mathcal{C}, [\,\cdot\,,\cdot\,]_{\mathcal{C}}\big)$ is continuously embedded in $\big(\mathcal{R}(|BJ|^{1/2}), [\,\cdot\,,\cdot\,]_B\big)$ we can apply Corollary 2.17 with $\mathcal{F} = \mathcal{R}(B)$, $\mathcal{A} = \mathcal{C}$ and $\mathcal{K} = \mathcal{R}(|BJ|^{1/2})$ to conclude $\mathcal{C} = \mathcal{R}(|BJ|^{1/2})$.

Theorem 4.5. *Let Krein spaces $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ and $\big(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{B}}\big)$ be continuously and contractively embedded in the Krein space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are complementary to each other in the Krein space $\mathcal{K}$ and let A and $B = I - A$ be the corresponding generalized projections. Then for the overlapping space we have*

$$\big(\mathcal{A} \cap \mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{A}} + [\,\cdot\,,\cdot\,]_{\mathcal{B}}\big) = \big(\mathcal{R}\big((ABJ)^{1/2}\big), [\,\cdot\,,\cdot\,]_{AB}\big).$$

In particular, the space $\big(\mathcal{A} \cap \mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{A}} + [\,\cdot\,,\cdot\,]_{\mathcal{B}}\big)$ is a Hilbert space and the intersection of the sets of non–positive vectors of $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ and $\big(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{B}}\big)$ is $\{0\}$.

Proof. Let $u \in \mathcal{R}\big(|AJ|^{1/2}\big) \cap \mathcal{R}\big(|BJ|^{1/2}\big)$. Then there exist sequences (x_n), (y_n) in $\mathcal{K}$, such that for $n \to +\infty$

$$By_n \to u \quad \text{in} \quad \big(\mathcal{R}(|BJ|^{1/2}), [\,\cdot\,,\cdot\,]_B\big), \quad Ax_n \to u \quad \text{in} \quad \big(\mathcal{R}(|AJ|^{1/2}), [\,\cdot\,,\cdot\,]_A\big).$$

By (4.3) and (4.4), it follows that for $n \to +\infty$

$$ABy_n \to Au \quad \text{and} \quad BAx_n \to Bu \quad \text{in} \quad \big(\mathcal{R}((ABJ)^{1/2}), [\,\cdot\,,\cdot\,]_{AB}\big).$$

Thus, $Au, Bu \in \mathcal{R}\big((ABJ)^{1/2}\big)$ and, consequently, $u = Au + Bu \in \mathcal{R}\big((ABJ)^{1/2}\big)$. It follows that

$$\mathcal{R}\big(|AJ|^{1/2}\big) \cap \mathcal{R}\big(|BJ|^{1/2}\big) \subset \mathcal{R}\big((ABJ)^{1/2}\big).$$

To prove the converse inclusion note that, since the identity mapping on $\mathcal{K}$ maps $\mathcal{R}(ABJ)$ into $\mathcal{R}(AJ)$, Lemma 2.9 implies

$$\mathcal{R}\big((ABJ)^{1/2}\big) \subset \mathcal{R}\big(|AJ|^{1/2}\big), \quad \mathcal{R}\big((ABJ)^{1/2}\big) \subset \mathcal{R}\big(|BJ|^{1/2}\big).$$

Hence $\mathcal{R}\big((ABJ)^{1/2}\big)$ is the overlapping subspace of the complementary spaces $(\mathcal{R}(|AJ|^{1/2}), [\,\cdot\,,\cdot\,]_A)$ and $(\mathcal{R}(|BJ|^{1/2}), [\,\cdot\,,\cdot\,]_B)$. For $u = ABx, v = ABy$ we have

$$[u,v]_A + [u,v]_B = [ABx, By]_{\mathcal{K}} + [ABx, Ay]_{\mathcal{K}} = [ABx, y]_{\mathcal{K}} = [u,v]_{AB}. \tag{4.9}$$

By Lemma 2.9, the topology on $\mathcal{R}\big((ABJ)^{1/2}\big)$ is stronger than the topologies on $\mathcal{R}\big(|AJ|^{1/2}\big)$ and $\mathcal{R}\big(|BJ|^{1/2}\big)$, therefore the equality (4.9) extends by continuity to $\mathcal{R}\big((ABJ)^{1/2}\big)$. Further, since $\big(\mathcal{R}\big((ABJ)^{1/2}\big), [\,\cdot\,,\cdot\,]_{AB}\big)$ is a Hilbert space, also $\big(\mathcal{A}\cap\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{A}} + [\,\cdot\,,\cdot\,]_{\mathcal{B}}\big)$ is a Hilbert space. If $x \in \mathcal{A}\cap\mathcal{B}$ is a non–positive vector in both $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ and $\big(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{B}}\big)$, then we have

$$0 \le [x,x]_{\mathcal{A}} + [x,x]_{\mathcal{B}} \le 0,$$

and, since $\big(\mathcal{A}\cap\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{A}} + [\,\cdot\,,\cdot\,]_{\mathcal{B}}\big)$ is a Hilbert space, $x = 0$.

Corollary 4.6. *Let Krein spaces $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ and $\big(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{B}}\big)$ be complementary in the Krein space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$ and let A and $B = I - A$ be the corresponding generalized projections. Let $\mathcal{B} = \mathcal{B}_-[\dot{+}]_{\mathcal{B}}\mathcal{B}_+$ be a fundamental decomposition of $\mathcal{B}$.*

(a) *The sum $\mathcal{K} = \mathcal{A} + \mathcal{B}$ is direct if and only if A is a projection, that is, if and only if $A = A^2$.*

(b) *If $0 \in \rho(A)$, then the norm topologies of $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ and $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$ coincide, and on $\mathcal{B}_-$ they are equivalent to the Hilbert space topology of $\big(\mathcal{B}_-, -[\,\cdot\,,\cdot\,]_{\mathcal{B}}\big)$.*

(c) *If Δ is an admissible interval for A and B and if $0, 1 \notin \Delta$ then $\mathcal{A}_\Delta = \mathcal{B}_\Delta = \mathcal{K}_\Delta$; in particular, $\mathcal{A}\cap\mathcal{B}$ contains all such subspaces $\mathcal{K}_\Delta$.*

Proof. To prove (a) assume that the sum $\mathcal{K} = \mathcal{A} + \mathcal{B}$ is direct. Theorem 4.5 implies that $AB = 0$ and consequently $A = A^2$. The converse is clear.

Let $\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{A}}$ and $\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}}$ be Hilbert space inner products on $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ and $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$, respectively. Assume that $0 \in \rho(A)$. Since by Corollary 2.16 the norm topologies on $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ and $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$ are equivalent, there exist $\gamma_1, \gamma_2 > 0$ such that

$$\gamma_1 \langle x,x\rangle_{\mathcal{K}} \le \langle x,x\rangle_{\mathcal{A}} \le \gamma_2 \langle x,x\rangle_{\mathcal{K}}, \qquad x \in \mathcal{K}. \tag{4.10}$$

The assumption $0 \in \rho(A)$ implies that $\mathcal{A} = \mathcal{K}$. Therefore $\mathcal{B}_- \subset \mathcal{B} = \mathcal{A}\cap\mathcal{B}$ and by Theorem 4.5

$$0 \le -[x,x]_{\mathcal{B}} \le [x,x]_{\mathcal{A}}, \qquad x \in \mathcal{B}_-. \tag{4.11}$$

Clearly

$$[x,x]_{\mathcal{A}} \le \langle x,x\rangle_{\mathcal{A}}, \qquad x \in \mathcal{A}. \tag{4.12}$$

Since the Krein space $\big(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{B}}\big)$ is continuously embedded in $\big(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}}\big)$, the Hilbert space $\big(\mathcal{B}_-, -[\,\cdot\,,\cdot\,]_{\mathcal{B}}\big)$ is also continuously embedded in $\big(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}}\big)$. Therefore there exist $\gamma_3 > 0$ such that

$$\langle x,x\rangle_{\mathcal{K}} \le -\gamma_3\, [x,x]_{\mathcal{B}}, \qquad x \in \mathcal{B}_-. \tag{4.13}$$

Combining the inequalities (4.13), (4.11), (4.12) and (4.10) we get

$$\langle x,x\rangle_{\mathcal{K}} \le -\gamma_3\,[x,x]_{\mathcal{B}} \le \gamma_3\,[x,x]_{\mathcal{A}} \le \gamma_3\,\langle x,x\rangle_{\mathcal{A}} \le \gamma_3\gamma_2\,\langle x,x\rangle_{\mathcal{K}}, \quad x\in\mathcal{B}_-.$$

This proves the statement about $(\mathcal{B}_-, -[\,\cdot\,,\cdot\,]_{\mathcal{B}})$.

The claim (c) follows immediately from Theorem 3.6.

3. In the next theorem we suppose that the embedded subspace $\mathcal{A}$ is a neutral subspace of $\mathcal{K}$. The representations of the elements of $\mathcal{K}$ and the operators in $\mathcal{K}$ that we use bellow refer to the chosen canonical decomposition.

Theorem 4.7. *Let the Krein space $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ be continuously and contractively embedded in the Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ with the corresponding generalized projection A and denote by $(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{B}})$ the corresponding complementary space. Let $\mathcal{K} = \mathcal{K}_+[\dot+]_{\mathcal{K}}\mathcal{K}_-$ be a fundamental decomposition of $\mathcal{K}$. The following statements are equivalent:*

(a) *$\mathcal{A}$ is a neutral subspace of $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$.*

(b) *$A^2 = 0$.*

(c) *There exists a bounded operator $Q : (\mathcal{K}_+, [\,\cdot\,,\cdot\,]_{\mathcal{K}}) \to (\mathcal{K}_-, -[\,\cdot\,,\cdot\,]_{\mathcal{K}})$ such that*

$$A = \begin{pmatrix} |Q| & -Q^* \\ Q & -|Q^*| \end{pmatrix}.$$

(d) *There exist a Hilbert space $(\mathcal{L}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{L}})$, which is continuously embedded in $(\mathcal{K}_+, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$, and an isometry $U : (\mathcal{L}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}) \to (\mathcal{K}_-, -[\,\cdot\,,\cdot\,]_{\mathcal{K}})$ such that*

$$\mathcal{A} = \left\{ \begin{pmatrix} x_+ \\ Ux_+ \end{pmatrix} : x_+ \in \mathcal{L} \right\},$$

and

$$\left[\begin{pmatrix} x_+ \\ Ux_+ \end{pmatrix}, \begin{pmatrix} y_+ \\ Uy_+ \end{pmatrix} \right]_{\mathcal{A}} = [x_+, y_+]_{\mathcal{L}}, \quad x_+, y_+ \in \mathcal{L}.$$

(e) *$\mathcal{B} = \mathcal{K}$, and $\mathcal{A}$ is a neutral subspace of $(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{B}})$.*

Proof. (a) $\Leftrightarrow$ (b) follows from Corollary 2.20.

(b) $\Rightarrow$ (c). Since A is a generalized projection we have $A - A^2 \ge_{\mathcal{K}} 0$, hence, by (b), $A \ge_{\mathcal{K}} 0$. With the chosen fundamental decomposition $\mathcal{K} = \mathcal{K}_+[\dot+]_{\mathcal{K}}\mathcal{K}_-$ the corresponding fundamental symmetry is $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and we write the self–adjoint operator A as

$$A = \begin{pmatrix} P & -Q^* \\ Q & R \end{pmatrix}$$

with P, R, Q, being bounded operators in or between the corresponding Hilbert spaces $(\mathcal{K}_+, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ and $(\mathcal{K}_-, -[\,\cdot\,,\cdot\,]_{\mathcal{K}})$, P and R being self–adjoint. The relation $A^2 = 0$ is equivalent to

$$P^2 = Q^* Q, \quad QP = -RQ, \quad R^2 = QQ^*,$$

and the non–negativity of A in the Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ implies $P \geq 0$, $R \leq 0$. This yields the following representation for A:

$$A = \begin{pmatrix} |Q| & -Q^* \\ Q & -|Q^*| \end{pmatrix}$$

and (c) is proved.

(c) $\Rightarrow$ (d). Let $(\mathcal{L}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{L}}) = (\mathcal{R}(|Q|^{1/2}), (\,\cdot\,,\cdot\,)_{|Q|})$. There exists an isometry

$$U : (\mathcal{R}(|Q|), [\,\cdot\,,\cdot\,]_{\mathcal{K}}) \to (\mathcal{K}_-, -[\,\cdot\,,\cdot\,]_{\mathcal{K}})$$

such that $Q = U\,|Q|$, $Q^* = |Q|\,U^{-1}$ and $|Q^*| = U\,|Q|\,U^{-1}$. Then

$$A = \begin{pmatrix} |Q| & -|Q|\,U^{-1} \\ U\,|Q| & -U\,|Q|\,U^{-1} \end{pmatrix} = \begin{pmatrix} I \\ U \end{pmatrix} |Q| \begin{pmatrix} I & -U^{-1} \end{pmatrix},$$

$$AJ = \begin{pmatrix} |Q| & |Q|\,U^{-1} \\ U\,|Q| & U\,|Q|\,U^{-1} \end{pmatrix} = \begin{pmatrix} I \\ U \end{pmatrix} |Q| \begin{pmatrix} I & U^{-1} \end{pmatrix}.$$

Since U is a continuous mapping it can be extended to the closure of $\mathcal{R}(|Q|)$ in $(\mathcal{K}_+, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$, and therefore also to the smaller subspace $\mathcal{L} = \mathcal{R}(|Q|^{1/2})$. This extension is still an isometry, and we denote it by U as well:

$$U : (\mathcal{L}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}) \to (\mathcal{K}_-, -[\,\cdot\,,\cdot\,]_{\mathcal{K}}).$$

It is easily calculated that

$$(AJ)^{1/2} = \begin{pmatrix} I \\ U \end{pmatrix} |Q|^{1/2} \begin{pmatrix} I & U^{-1} \end{pmatrix}. \tag{4.14}$$

The vector $y \in \mathcal{K}$ belongs to $\mathcal{R}((AJ)^{1/2})$ if and only if there exists an $x \in \mathcal{K}$ such that $(AJ)^{1/2} x = y$. By (4.14) this is equivalent to

$$y = \begin{pmatrix} |Q|^{1/2}(x_+ + U^{-1} x_-) \\ U\,|Q|^{1/2}(x_+ + U^{-1} x_-) \end{pmatrix}.$$

Since $\mathcal{A} = \mathcal{R}\big((AJ)^{1/2}\big)$ this proves the first equality in (d). Similarly

$$AJx = \begin{pmatrix} |Q|\left(x_+ + U^{-1}x_-\right) \\ \\ U\,|Q|\left(x_+ + U^{-1}x_-\right) \end{pmatrix}.$$

Let $Ax = \begin{pmatrix} |Q|u_+ \\ U\,|Q|u_+ \end{pmatrix}$ and $Ay = \begin{pmatrix} |Q|v_+ \\ U\,|Q|v_+ \end{pmatrix}$ with $u_+, v_+ \in \mathcal{K}_+$. Then

$$\begin{aligned}
[Ax, Ay]_{\mathcal{A}} &= (AJJx, AJJy)_{AJ} = \langle AJJx, Jy\rangle_{\mathcal{K}} = \left\langle \begin{pmatrix} |Q|u_+ \\ U\,|Q|u_+ \end{pmatrix}, \begin{pmatrix} y_+ \\ -\,y_- \end{pmatrix} \right\rangle_{\mathcal{K}} \\
&= \big[|Q|u_+, y_+\big]_{\mathcal{K}} + \big[U\,|Q|u_+, y_-\big]_{\mathcal{K}} = \big[|Q|u_+, y_+ + U^{-1}y_-\big]_{\mathcal{K}} \\
&= \big[|Q|u_+, v_+\big]_{\mathcal{K}} = \big(|Q|u_+, |Q|v_+\big)_{|Q|} = \big[|Q|u_+, |Q|v_+\big]_{\mathcal{L}}.
\end{aligned}$$

This shows that

$$\left[\begin{pmatrix} x_+ \\ Ux_+ \end{pmatrix}, \begin{pmatrix} y_+ \\ Uy_+ \end{pmatrix}\right]_{\mathcal{A}} = \big[x_+, y_+\big]_{\mathcal{L}}, \quad x_+, y_+ \in \mathcal{L}, \tag{4.15}$$

holds on a dense subspace of $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$. The equality (4.15) implies that the restriction $P_+|_{\mathcal{A}}$ of the orthogonal projection $P_+ : \mathcal{K} \to \mathcal{K}_+$ is an isometry between dense subspaces of the Hilbert spaces $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ and $\big(\mathcal{L}, [\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$. It follows that $P_+\big|_{\mathcal{A}}$ maps $\big(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}}\big)$ isometrically onto $\big(\mathcal{L}, [\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$, which proves (d).

(d) $\Rightarrow$ (a) is clear, hence we have proved that the statements from (a) to (d) are equivalent.

(b) $\Rightarrow$ (e). If (b) holds then the generalized projection $B = I - A$ has the property $0 \in \rho(B)$ and hence $\mathcal{R}(B) = \mathcal{B} = \mathcal{K}$. The inner product $[\,\cdot\,,\cdot\,]_{\mathcal{B}}$ is given by

$$[x, y]_{\mathcal{B}} = \big[(I - A)^{-1}x, y\big]_{\mathcal{K}} = [(I + A)x, y]_{\mathcal{K}}, \quad x, y \in \mathcal{K}.$$

Applying this to the elements in $\mathcal{R}(A)$ we get

$$[Ax, Ay]_{\mathcal{B}} = [(I + A)Ax, Ay]_{\mathcal{K}} = [Ax, Ay]_{\mathcal{K}} = [A^2x, y]_{\mathcal{K}} = 0, \quad x, y \in \mathcal{K},$$

which implies the second statement in (e).

(e) $\Rightarrow$ (b). Since $\mathcal{B} = \mathcal{K}$, by Corollary 4.6, B is invertible, and therefore

$$[x, y]_{\mathcal{B}} = \big[B^{-1}x, y\big]_{\mathcal{K}}, \quad x, y \in \mathcal{K}.$$

Since $\mathcal{R}(A) \subset \mathcal{A}$ is a neutral subspace of $(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{B}})$ we have

$$0 = [Ax, Ay]_{\mathcal{B}} = \left[B^{-1}Ax, Ay\right]_{\mathcal{K}} = \left[B^{-1}A^2x, y\right]_{\mathcal{K}} \quad x, y \in \mathcal{K}.$$

Consequently $B^{-1}A^2 = 0$ and therefore $A^2 = 0$.

Remark 4.8. In Theorem 4.7 the space $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ is a Hilbert space. Clearly $\mathcal{A} \cap \mathcal{B} = \mathcal{A}$ and the inner product on the overlapping space coincides with $[\,\cdot\,,\cdot\,]_{\mathcal{A}}$. Namely we have $AB = A(I - A) = A$ and

$$[x, y]_{\mathcal{A}} = [x, y]_{AB} = [x_+, y_+]_{\mathcal{L}}, \quad x, y \in \mathcal{A}.$$

5. Appendix: Uniqueness of Krein space completions

As mentioned in the introduction, in this section we prove a slightly extended version of a result of T. Hara [H, Theorems 5 and 6] about the uniqueness of the Krein space completion of an indefinite inner product space with a Hilbert majorant.

Lemma 5.1. *If $(\mathcal{L}, [\,\cdot\,,\cdot\,]_{\mathcal{L}})$ is a non–degenerate inner product space and $[\,\cdot\,,\cdot\,]_{\mathcal{L}}$ has a Hilbert majorant $(\,\cdot\,,\cdot\,)_{\mathcal{L}}$, then an equivalent Hilbert inner product $(\,\cdot\,,\cdot\,)'_{\mathcal{L}}$ can be chosen in $\mathcal{L}$ such that the spectrum of the corresponding Gram operator G' consists outside zero of isolated eigenvalues only.*

Proof. Let G be the Gram operator for the inner product $[\,\cdot\,,\cdot\,]_{\mathcal{L}}$, and denote by E its spectral function: $G = \int_{-\|G\|}^{\|G\|} \lambda \, dE_\lambda$. Note that G is injective and consider the function

$$\varphi(\lambda) := \begin{cases} -1 & \text{if} \quad \lambda \in (-\infty, -1], \\ -\frac{1}{n+1} & \text{if} \quad \lambda \in \left(-\frac{1}{n}, -\frac{1}{n+1}\right], \qquad n \in \mathbb{N}, \\ \frac{1}{n+1} & \text{if} \quad \lambda \in \left[\frac{1}{n+1}, \frac{1}{n}\right), \qquad n \in \mathbb{N}, \\ 1 & \text{if} \quad \lambda \in [1, +\infty). \end{cases}$$

Then

$$1 \leq \frac{\lambda}{\varphi(\lambda)} \leq \max\bigl\{\|G\|, 2\bigr\} \quad \text{if} \quad \lambda \in \bigl[-\|G\|, \|G\|\bigr] \setminus \{0\}.$$

Define on $\mathcal{L}$ the positive definite inner product

$$(x, y)'_{\mathcal{L}} := \int_{-\|G\|}^{\|G\|} \frac{\lambda}{\varphi(\lambda)} \, d(E_\lambda x, y)_{\mathcal{L}}.$$

With $\Delta_0^+ := \big[1, \|G\|\big]$, $\Delta_n^+ := \big[\frac{1}{n+1}, \frac{1}{n}\big)$, $\Delta_n^- := \big(-\frac{1}{n}, -\frac{1}{n+1}\big]$, $n \in \mathbb{N}$, and $\Delta_0^- := \big[-\|G\|, -1\big]$ we obtain:

$$\begin{aligned}
\langle x, y\rangle_{\mathcal{L}} = (Gx, y)_{\mathcal{L}} &= \int_{-\|G\|}^{\|G\|} \varphi(\lambda) \frac{\lambda}{\varphi(\lambda)}\, d(E_\lambda x, y)_{\mathcal{L}} \\
&= \int_{\Delta_0^+} \lambda\, d(E_\lambda x, y)_{\mathcal{L}} + \sum_{n=1}^{+\infty} \frac{1}{n+1} \int_{\Delta_n^+} \frac{\lambda}{\varphi(\lambda)}\, d(E_\lambda x, y)_{\mathcal{L}} \\
&\quad - \int_{\Delta_0^-} \lambda\, d(E_\lambda x, y)_{\mathcal{L}} - \sum_{n=1}^{+\infty} \frac{1}{n+1} \int_{\Delta_n^-} \frac{\lambda}{\varphi(\lambda)}\, d(E_\lambda x, y)_{\mathcal{L}} \\
&= (G'x, y)'_{\mathcal{L}},
\end{aligned}$$

where the operator G' is defined by the relation

$$G'x := \pm \frac{1}{n+1}\, x \quad \text{if} \quad x \in E(\Delta_n^{\pm})\mathcal{L}, \quad n \in \mathbb{N}.$$

Thus, the spectrum of G' is contained in $\big\{\pm(n+1)^{-1} : n \in \mathbb{N}\big\} \cup \{0\}$.

Theorem 5.2. *Let $\big(\mathcal{L}, [\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$ be a decomposable non–degenerate inner product space. If for one and hence for all fundamental decompositions $\mathcal{L} = \mathcal{L}_+ [\dot{+}]_{\mathcal{L}} \mathcal{L}_-$ at least one of the spaces $\big(\mathcal{L}_\pm, \pm[\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$ is a Hilbert space, then $\big(\mathcal{L}, [\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$ has a unique Krein space completion. Conversely, if $[\,\cdot\,,\cdot\,]_{\mathcal{L}}$ has a Hilbert majorant and $\big(\mathcal{L}, [\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$ has a unique Krein space completion, then at least one of the spaces $\big(\mathcal{L}_\pm, \pm[\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$ is a Hilbert space.*

Proof. It was proved in [B, Theorem IV.7.2] that if $\mathcal{L} = \mathcal{L}_+ [\dot{+}]_{\mathcal{L}} \mathcal{L}_-$ and $\mathcal{L}' = \mathcal{L}'_+ [\dot{+}]_{\mathcal{L}} \mathcal{L}'_-$ are two fundamental decompositions, then $\big(\mathcal{L}_\pm, \pm[\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$ is a Hilbert space if and only if $\big(\mathcal{L}'_\pm, \pm[\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$ is a Hilbert space. Assume that $\mathcal{L} = \mathcal{L}_+ [\dot{+}]_{\mathcal{L}} \mathcal{L}_-$ is a fundamental decomposition of $\big(\mathcal{L}, [\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$ and that for example $\big(\mathcal{L}_+, [\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$ is a Hilbert space. Let $\big(\mathcal{K}_1, [\,\cdot\,,\cdot\,]_{\mathcal{K}_1}\big)$ and $\big(\mathcal{K}_2, [\,\cdot\,,\cdot\,]_{\mathcal{K}_2}\big)$ be two Krein space completions of $\big(\mathcal{L}, [\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$. Then $\big(\mathcal{L}_+, [\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$ is a uniformly positive subspace of both $\big(\mathcal{K}_1, [\,\cdot\,,\cdot\,]_{\mathcal{K}_1}\big)$ and $\big(\mathcal{K}_2, [\,\cdot\,,\cdot\,]_{\mathcal{K}_2}\big)$. Denote by P_j the orthogonal projection onto $\mathcal{L}_+$ in $\big(\mathcal{K}_j, [\,\cdot\,,\cdot\,]_{\mathcal{K}_j}\big)$ and set $(1 - P_j)\mathcal{K}_j =: \mathcal{L}_j$, $j = 1, 2$. Then $\mathcal{L}_- \subset \mathcal{L}_j$, $j = 1, 2$, and for $x = x_+ + x_-$, $x_\pm \in \mathcal{L}_\pm$, we have $(1 - P_j)x = x_-$, $j = 1, 2$. Since for arbitrary $u \in \mathcal{L}_j$ and $(x_n) \subset \mathcal{L}$, such that $x_n \to u$ in $\mathcal{K}_j$, it follows that $x_{n,-} = (1 - P_j)x_n \to (1 - P_j)u = u$, $n \to +\infty$, the subspace $\mathcal{L}_-$ is dense in $\mathcal{L}_j$, $j = 1, 2$. Consequently, $\big(\mathcal{L}_1, [\,\cdot\,,\cdot\,]_{\mathcal{K}_1}\big)$ and $\big(\mathcal{L}_2, [\,\cdot\,,\cdot\,]_{\mathcal{K}_2}\big)$ are isometrically isomorphic anti–Hilbert spaces, containing both $\mathcal{L}_-$ as a dense subspace. Thus they can be identified; denote this space by $\big(\mathcal{L}_-^c, [\,\cdot\,,\cdot\,]_-\big)$. Finally, the space $\mathcal{L}^c := \mathcal{L}_+ \dot{+} \mathcal{L}_-^c$, equipped with the inner product

$$[x, y]_{\mathcal{L}^c} := [x_+, y_+]_{\mathcal{L}} - [x_-, y_-]_-,$$

where

$$x = x_+ + x_-, \quad y = y_+ + y_-, \quad x_+, y_+ \in \mathcal{L}_+, \quad x_-, y_- \in \mathcal{L}_-^c,$$

is the unique Krein space completion of $(\mathcal{L}, [\,\cdot\,,\cdot\,]_\mathcal{L})$.

It remains to show that if $(\mathcal{L}, [\,\cdot\,,\cdot\,]_\mathcal{L})$ has a Hilbert majorant $(\,\cdot\,,\cdot\,)_\mathcal{L}$ and for a fundamental decomposition $\mathcal{L} = \mathcal{L}_+[\dot{+}]_\mathcal{L}\mathcal{L}_-$ both components $(\mathcal{L}_+, [\,\cdot\,,\cdot\,]_\mathcal{L})$, $(\mathcal{L}_-, -[\,\cdot\,,\cdot\,]_\mathcal{L})$ are not complete then $(\mathcal{L}, [\,\cdot\,,\cdot\,]_\mathcal{L})$ has more than one Krein space completion. For this we give two proofs in items (I) and (II) below. In (I) we construct a pair of different Krein space completions in a straightforward way, the construction in (II) uses operator ranges and is therefore more related to the considerations in the first sections of this paper; besides, it supplies an infinite family of Krein space completions.

We begin with the preliminaries which are common to both constructions. Denote the Gram operator of the indefinite inner product $[\,\cdot\,,\cdot\,]_\mathcal{L}$ on $\mathcal{L}$ with respect to the Hilbert majorant $(\,\cdot\,,\cdot\,)_\mathcal{L}$ by G:

$$[x, y]_\mathcal{L} = (Gx, y)_\mathcal{L}, \quad x, y \in \mathcal{L}.$$

It is easy to see that the component $(\mathcal{L}_+, [\,\cdot\,,\cdot\,]_\mathcal{L})$ $\big((\mathcal{L}_-, -[\,\cdot\,,\cdot\,]_\mathcal{L})$, respectively$\big)$ is not complete if and only if zero is an accumulation point of $\sigma(G)$ from the right (left, respectively). Thus, since both components are supposed to be non-complete, zero is an accumulation point of $\sigma(G)$ from both sides. According to Lemma 5.1, without loss of generality we can also suppose that the spectrum of G consists outside zero of isolated eigenvalues only, hence it consists of two sequences of eigenvalues λ_n^+ and $-\lambda_n^-$, $n \in \mathbb{N}$, such that

$$-\lambda_1^- < -\lambda_2^- < \cdots < 0 < \cdots < \lambda_2^+ < \lambda_1^+, \quad \lim_{n\to+\infty} \lambda_n^- = \lim_{n\to+\infty} \lambda_n^+ = 0.$$

In each of the eigenspaces corresponding to $\pm\lambda_n^\pm$ we choose an eigenvector $e_n^\pm$, $n \in \mathbb{N}$, such that $(e_n^\pm, e_n^\pm)_\mathcal{L} = 1$. Instead of $\mathcal{L}$ consider the subspace

$$\mathcal{L}' := \overline{\operatorname{span}}\{e_n^+,\, e_n^- : n \in \mathbb{N}\}, \quad \text{the closure in the topology of } \big(\mathcal{L}, (\,\cdot\,,\cdot\,)_\mathcal{L}\big),$$

equipped with the Hilbert inner product $(\,\cdot\,,\cdot\,)_\mathcal{L}$ and with the indefinite inner product $[\,\cdot\,,\cdot\,]_\mathcal{L}$. If $(\mathcal{L}', [\,\cdot\,,\cdot\,]_\mathcal{L})$ has more than one Krein space completion then the same is true for $(\mathcal{L}, [\,\cdot\,,\cdot\,]_\mathcal{L})$. Therefore without loss of generality we can suppose that $\mathcal{L} = \mathcal{L}'$.

(I) Set $\mathcal{L}_n := \operatorname{span}\{e_n^+, e_n^-\}$ and $\mathcal{H}_n := \mathbb{C}^2$, $n \in \mathbb{N}$. All 2×2–matrices in this proof relate to the standard basis consisting of the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ of

$\mathcal{H}_n$. We introduce a positive definite inner product $(\,\cdot\,,\cdot\,)_{\mathcal{H}_n}$ on $\mathcal{H}_n$ by the Gram matrix

$$\widehat{A}_n := \begin{pmatrix} 1 & 0 \\ 0 & (\lambda_n^+\lambda_n^-)^{-1} \end{pmatrix}$$

and the operator $\widehat{G}_n$ on $\mathcal{H}_n$ by the matrix

$$\widehat{G}_n := \begin{pmatrix} \lambda_n^+ - \lambda_n^- & 1 \\ \lambda_n^+\lambda_n^- & 0 \end{pmatrix}.$$

The operator $\widehat{G}_n$ is self–adjoint with respect to the positive definite inner product $(\,\cdot\,,\cdot\,)_{\mathcal{H}_n}$ on $\mathcal{H}_n$. Indeed, an easy calculation shows that

$$\widehat{S}_n := \widehat{A}_n\,\widehat{G}_n = \begin{pmatrix} \lambda_n^+ - \lambda_n^- & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.1}$$

It is clear that the eigenvalues of the operator $\widehat{G}_n$ are $-\lambda_n^-$ and λ_n^+. A straightforward calculation gives that the corresponding eigenvectors, normalized with respect to $(\,\cdot\,,\cdot\,)_{\mathcal{H}_n}$, are

$$\phi_n^- = \sqrt{\frac{\lambda_n^-}{\lambda_n^+ + \lambda_n^-}}\begin{pmatrix} -1 \\ \lambda_n^+ \end{pmatrix}, \qquad \phi_n^+ = \sqrt{\frac{\lambda_n^+}{\lambda_n^+ + \lambda_n^-}}\begin{pmatrix} 1 \\ \lambda_n^- \end{pmatrix},$$

respectively. The linear mapping U_n defined, through

$$U_n \;:\; e_n^\pm \quad\longmapsto\quad \phi_n^\pm, \tag{5.2}$$

is an isomorphism between the (2–dimensional) Hilbert spaces $\big(\mathcal{L}_n, (\,\cdot\,,\cdot\,)_{\mathcal{L}}\big)$ and $\big(\mathcal{H}_n, (\,\cdot\,,\cdot\,)_{\mathcal{H}_n}\big)$. The definitions of $(\,\cdot\,,\cdot\,)_{\mathcal{H}_n}$, $\widehat{G}_n$, and U_n imply that

$$[u,v]_{\mathcal{L}} = \big(U_n v\big)^*\big(\widehat{A}_n\,\widehat{G}_n\,U_n u\big), \quad u,v\in\mathcal{L}_n. \tag{5.3}$$

Define the Hilbert space $\big(\mathcal{H}, (\,\cdot\,,\cdot\,)_{\mathcal{H}}\big)$ as follows: $\mathcal{H}\subset\ell^2$,

$$x = \big(\xi_n\big)_{n\in\mathbb{N}} \in \mathcal{H} \quad\Longleftrightarrow\quad \big(\xi_{2n-1}\big)_{n\in\mathbb{N}} \in \ell^2 \quad\text{and}\quad \left(\frac{\xi_{2n}}{\sqrt{\lambda_n^+\lambda_n^-}}\right)_{n\in\mathbb{N}} \in \ell^2,$$

and

$$(x,y)_{\mathcal{H}} := \sum_{n=1}^{+\infty}\left(\xi_{2n-1}\,\overline{\zeta}_{2n-1} + \frac{1}{\lambda_n^+\lambda_n^-}\,\xi_{2n}\,\overline{\zeta}_{2n}\right);$$

here and in the rest of the proof we use

$$x = \big(\xi_n\big)_{n\in\mathbb{N}},\; y = \big(\zeta_n\big)_{n\in\mathbb{N}} \in \mathcal{H}.$$

It follows from (5.1) that the formula

$$[x,y]_{\mathcal{H}} := \sum_{n=1}^{+\infty} \Big((\lambda_n^+ - \lambda_n^-)\xi_{2n-1}\overline{\zeta}_{2n-1} + \xi_{2n}\overline{\zeta}_{2n-1} + \xi_{2n-1}\overline{\zeta}_{2n} \Big) \tag{5.4}$$

defines an indefinite inner product on $\mathcal{H}$. Since

$$\mathcal{L} = \bigoplus_{n=1}^{+\infty} \mathcal{L}_n, \quad \text{infinite direct sum in } \big(\mathcal{L}, (\,\cdot\,,\cdot\,)_{\mathcal{L}}\big),$$
$$\mathcal{H} = \bigoplus_{n=1}^{+\infty} \mathcal{H}_n, \quad \text{infinite direct sum in } \big(\mathcal{H}, (\,\cdot\,,\cdot\,)_{\mathcal{H}}\big),$$

it follows from (5.2) that the Hilbert spaces $\big(\mathcal{L}, (\,\cdot\,,\cdot\,)_{\mathcal{L}}\big)$ and $\big(\mathcal{H}, (\,\cdot\,,\cdot\,)_{\mathcal{H}}\big)$ are isomorphic, the isomorphism being established by the mapping

$$U := \bigoplus_{n=1}^{+\infty} U_n.$$

Further, the relations (5.1), (5.3) and (5.4) yield

$$[u,v]_{\mathcal{L}} = [Uu, Uv]_{\mathcal{H}}, \qquad u,v \in \mathcal{L}.$$

Consequently, the indefinite inner product spaces $\big(\mathcal{L}, [\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$ and $\big(\mathcal{H}, [\,\cdot\,,\cdot\,]_{\mathcal{H}}\big)$, and also the Hilbert spaces $\big(\mathcal{L}, (\,\cdot\,,\cdot\,)_{\mathcal{L}}\big)$ and $\big(\mathcal{H}, (\,\cdot\,,\cdot\,)_{\mathcal{H}}\big)$, can be identified. Therefore the inner product space $\big(\mathcal{L}, [\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$ has a unique Krein space completion if and only if $\big(\mathcal{H}, [\,\cdot\,,\cdot\,]_{\mathcal{H}}\big)$ has a unique Krein space completion. To complete the proof we shall show that $\big(\mathcal{H}, [\,\cdot\,,\cdot\,]_{\mathcal{H}}\big)$ has at least two Krein space completions.

First consider the usual ℓ^2–inner product on $\mathcal{H}$:

$$(x,y)_2 := \sum_{n=1}^{+\infty} \xi_n \overline{\zeta}_n,$$

and denote by $\|\cdot\|_2$ the corresponding norm. The indefinite inner product $[\,\cdot\,,\cdot\,]_{\mathcal{H}}$ is continuous with respect to this norm since it clearly follows from (5.4) that

$$[x,y]_{\mathcal{H}} \le \big(2\|G\| + 1\big)\|x\|_2\,\|y\|_2, \quad x,y \in \mathcal{H}.$$

Since $\mathcal{H}$ contains all finite sequences, its completion with respect to the norm $\|\cdot\|_2$ is $\big(\ell^2, (\,\cdot\,,\cdot\,)_2\big)$. As the formula in (5.4) is valid for all elements in ℓ^2, the extension by continuity of $[\,\cdot\,,\cdot\,]_{\mathcal{H}}$ onto the entire space ℓ^2 is also given by (5.4). Thus $\big(\ell^2, [\,\cdot\,,\cdot\,]_{\mathcal{H}}\big)$ is a non–degenerate inner product space with the Hilbert majorant $(\cdot,\cdot)_2$. It follows from (5.1) that the Gram operator in $\big(\ell^2,(\cdot,\cdot)_2\big)$ of this inner product is

$$\widehat{S} := \bigoplus_{n=1}^{+\infty} \widehat{S}_n, \quad \text{infinite direct sum in } \big(\ell^2,(\cdot,\cdot)_2\big).$$

The eigenvalues of the operators $\widehat{S}_n$ are

$$\frac{1}{2}\left(\lambda_n^+ - \lambda_n^- \pm \sqrt{4+\big(\lambda_n^+ - \lambda_n^-\big)^2}\right), \quad n\in\mathbb{N},$$

consequently, they are $\neq 0$ and accumulate only at -1 and 1, yielding that $\widehat{S}$ is a bounded and boundedly invertible self–adjoint operator in $\big(\ell^2,(\cdot,\cdot)_2\big)$. Therefore $\big(\ell^2,[\cdot,\cdot]_{\mathcal{H}}\big)$ is a Krein space which is a Krein space completion of $\big(\mathcal{H},[\cdot,\cdot]_{\mathcal{H}}\big)$.

Note also that

$$\widehat{A} := \bigoplus_{n=1}^{+\infty} \widehat{A}_n, \quad \text{infinite direct sum in } \big(\ell^2,(\cdot,\cdot)_2\big),$$

is an unbounded uniformly positive operator in $\big(\ell^2,(\cdot,\cdot)_2\big)$, $\mathcal{H}=\mathcal{D}\big(\widehat{A}^{1/2}\big)$ and

$$\big(x,y\big)_{\mathcal{H}} = \big(\widehat{A}^{1/2}x,\widehat{A}^{1/2}y\big)_2, \qquad x,y\in\mathcal{H}.$$

To complete the proof we shall show that the canonical Krein space completion of $\big(\mathcal{H},[\cdot,\cdot]_{\mathcal{H}}\big)$ differs from ℓ^2. The operator

$$\widehat{G} = \bigoplus_{n=1}^{+\infty} \widehat{G}_n, \quad \text{infinite direct sum in } \big(\mathcal{H},(\cdot,\cdot)_{\mathcal{H}}\big),$$

is the Gram operator of the indefinite inner product $[\cdot,\cdot]_{\mathcal{H}}$ in the Hilbert space $\big(\mathcal{H},(\cdot,\cdot)_{\mathcal{H}}\big)$. Since the decomposition topology of $\big(\mathcal{H},[\cdot,\cdot]_{\mathcal{H}}\big)$ is given by the inner product $(|\widehat{G}|\ \cdot,\cdot)_{\mathcal{H}}$ to calculate the canonical Krein space completion of the inner product space $\big(\mathcal{H},[\cdot,\cdot]_{\mathcal{H}}\big)$ we need to study the operator $|\widehat{G}|$ in the Hilbert space $\big(\mathcal{H},(\cdot,\cdot)_{\mathcal{H}}\big)$. First observe that the subspace $\mathcal{F}(\subset\mathcal{H})$ consisting of all finite sequences is dense in $\big(\mathcal{H},(|\widehat{G}|\ \cdot,\cdot)_{\mathcal{H}}\big)$. Therefore the completion of $\big(\mathcal{H},(|\widehat{G}|\ \cdot,\cdot)_{\mathcal{H}}\big)$ coincides with the completion of $\big(\mathcal{F},(|\widehat{G}|\ \cdot,\cdot)_{\mathcal{H}}\big)$. A straightforward calculation shows that the matrix representation $|\widehat{G}|_n$ of the restriction of $|\widehat{G}|$ to $\mathcal{H}_n$ is

$$|\widehat{G}|_n = |\widehat{G}_n| = \frac{1}{\lambda_n^+ + \lambda_n^-}\begin{pmatrix} (\lambda_n^+)^2+(\lambda_n^-)^2 & \lambda_n^+ - \lambda_n^- \\ \lambda_n^+\lambda_n^-\big(\lambda_n^+ - \lambda_n^-\big) & 2\lambda_n^+\lambda_n^- \end{pmatrix}.$$

Since the operator $|\widehat{G}|$ has the same eigenvectors as $\widehat{G}$, the subspaces $\mathcal{H}_n$, $n \in \mathbb{N}$, are mutually orthogonal with respect to $(|\widehat{G}|\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$. The restriction of $(|\widehat{G}|\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$ to $\mathcal{H}_n$, $n \in \mathbb{N}$, is given by the positive operator $\widehat{B}_n := \widehat{A}_n\,|\widehat{G}|_n$ whose matrix representation is

$$\widehat{B}_n = \frac{1}{\lambda_n^+ + \lambda_n^-}\begin{pmatrix} (\lambda_n^+)^2 + (\lambda_n^-)^2 & \lambda_n^+ - \lambda_n^- \\ \lambda_n^+ - \lambda_n^- & 2 \end{pmatrix}, \quad n \in \mathbb{N}. \tag{5.5}$$

The operator $\widehat{B}$ defined by

$$\widehat{B} := \bigoplus_{n=1}^{+\infty} \widehat{B}_n, \quad \text{infinite direct sum in } \big(\ell^2, (\,\cdot\,,\cdot\,)_2\big),$$

is a positive self–adjoint operator in $\big(\ell^2, (\,\cdot\,,\cdot\,)_2\big)$. As each matrix in (5.5) has determinant 1, its eigenvalues are positive numbers which are reciprocal to each other. Since the traces of the matrices in (5.5) are unbounded as $n \to +\infty$, we conclude that the eigenvalues of the operator $\widehat{B}$ accumulate at 0 and $+\infty$. Therefore $\widehat{B}$ is not bounded and it does not have a bounded inverse. Consequently for the completion $\big(\mathcal{B}, (\,\cdot\,,\cdot\,)_{\mathcal{B}}\big)$ of the pre–Hilbert space $\big(\mathcal{D}(\widehat{B}), (\widehat{B}\,\cdot\,,\,\cdot\,)_2\big)$ we have that neither $\mathcal{B} \subseteq \ell^2$ nor $\ell^2 \subseteq \mathcal{B}$, see Remark 5.4. Clearly $\mathcal{F}$ is dense in $\big(\mathcal{D}(\widehat{B}), (\widehat{B}\,\cdot\,,\,\cdot\,)_2\big)$. Therefore the completion of $\big(\mathcal{F}, (\widehat{B}\,\cdot\,,\,\cdot\,)_2\big)$ is also $\mathcal{B}$. By the definitions above we have

$$\big(\mathcal{F}, (|\widehat{G}|\,\cdot\,,\,\cdot\,)_{\mathcal{H}}\big) = \big(\mathcal{F}, (\widehat{B}\,\cdot\,,\,\cdot\,)_2\big).$$

Thus the canonical Krein space completion of $\big(\mathcal{H}, [\,\cdot\,,\cdot\,]_{\mathcal{H}}\big)$ is $\big(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{H}}\big)$. Since neither $\mathcal{B} \subseteq \ell^2$ nor $\ell^2 \subseteq \mathcal{B}$, we have constructed two different Krein space completions.

(II) Let the Krein space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$ be the canonical Krein space completion of $\big(\mathcal{L}, [\,\cdot\,,\cdot\,]_{\mathcal{L}}\big)$. By Remark 2.12 the Hilbert space $\big(\mathcal{L}, (\,\cdot\,,\cdot\,)_{\mathcal{L}}\big)$ is continuously embedded in $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$ and the $\mathcal{K}$–adjoint $T : \mathcal{K} \to \mathcal{K}$ of the corresponding inclusion is the extension to $\mathcal{K}$ by continuity of the Gram operator $G : \mathcal{L} \to \mathcal{L}$. Therefore the eigenvalues and the eigenvectors of T coincide with the eigenvalues $\lambda_n^\pm$ and the corresponding eigenvectors $e_n^\pm$, $n \in \mathbb{N}$, of G. We normalize these eigenvectors:

$$\omega_n^\pm := \frac{1}{\sqrt{\lambda_n^\pm}}\, e_n^\pm, \quad n \in \mathbb{N};$$

then $\big[\omega_n^\pm, \omega_n^\pm\big]_{\mathcal{K}} = \pm 1$, $n \in \mathbb{N}$.

Let $(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}})$ be the Hilbert space completion of $(\mathcal{L}, (|G|\,\cdot\,,\,\cdot\,)_{\mathcal{L}})$. By Remark 2.2 the Hilbert space $(\mathcal{L}, (\,\cdot\,,\cdot\,)_{\mathcal{L}})$ is continuously embedded in $(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}})$ and the $\mathcal{H}$–adjoint $T_1 : \mathcal{K} \to \mathcal{K}$ of the corresponding inclusion is the extension to $\mathcal{K}$ by continuity of the operator $|G| : \mathcal{L} \to \mathcal{L}$. Then the extension to $\mathcal{K}$ of the signum of G is the fundamental symmetry connecting the inner products $[\,\cdot\,,\cdot\,]_{\mathcal{K}}$ and $\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}}$. We denote this fundamental symmetry by J. Clearly J commutes with T and consequently T is a self–adjoint operator in the Hilbert space $(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}})$. In this space, $|T| = JT = T_1$. It follows from Corollary 2.5 that

$$\big(\mathcal{L}, (\,\cdot\,,\cdot\,)_{\mathcal{L}}\big) = \big(\mathcal{R}(|T|^{1/2}), (\,\cdot\,,\cdot\,)_{|T|}\big).$$

The eigenvectors $\omega_n^{\pm}$, $n \in \mathbb{N}$, are also the eigenvectors of $|T|$ corresponding to the eigenvalues $\lambda_n^{\pm}$, $n \in \mathbb{N}$. Thus these vectors form an orthonormal system of vectors in the Hilbert space $(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}})$.

Earlier in this proof, without loss of generality, we assumed that

$$\mathcal{L} = \overline{\operatorname{span}}\big\{\omega_n^{\pm} : n \in \mathbb{N}\big\}, \quad \text{the closure in the topology of } \big(\mathcal{L}, (\,\cdot\,,\cdot\,)_{\mathcal{L}}\big).$$

This assumption implies that

$$\mathcal{K} = \overline{\operatorname{span}}\big\{\omega_n^{\pm} : n \in \mathbb{N}\big\}, \quad \text{the closure in the topology of } \big(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}}\big).$$

Let $\mathcal{K} = \mathcal{K}_+ \,[\dot{+}]_{\mathcal{K}}\, \mathcal{K}_-$ be the fundamental decomposition corresponding to the fundamental symmetry J. Then

$$\mathcal{K}_{\pm} = \overline{\operatorname{span}}\big\{\omega_n^{\pm} : n \in \mathbb{N}\big\}, \quad \text{the closure in the topology of } \big(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}}\big).$$

Let $\mathcal{K}_n = \operatorname{span}\{\omega_n^+, \omega_n^-\}$. Then

$$\mathcal{K} = \mathcal{K}_+ \,[\dot{+}]_{\mathcal{K}}\, \mathcal{K}_- = \bigoplus_{n=1}^{+\infty} \mathcal{K}_n.$$

Here and in the rest of the proof the infinite direct sums are considered in the topology of $(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}})$. The matrix $\begin{pmatrix} \lambda_n^+ & 0 \\ 0 & \lambda_n^- \end{pmatrix}$ is the representation of the restriction $|T|_n$ of $|T|$ to $\mathcal{K}_n$ with respect to the basis $\{\omega_n^+, \omega_n^-\}$. Then the equality

$$|T| = \bigoplus_{n=1}^{+\infty} |T|_n$$

holds.

Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence with the following properties:

$$\mu_n > 0, \quad n \in \mathbb{N}, \qquad \lim_{n\to+\infty} \mu_n = 0, \qquad \sup_{n\in\mathbb{N}} \frac{\lambda_n^{\pm}}{\mu_n} < +\infty. \tag{5.6}$$

Define the operators S_n and Q_n on $\mathcal{K}_n$ by their matrix representations with respect to the basis $\{\omega_n^+, \omega_n^-\}$:

$$S_n = \frac{1}{\sqrt{\mu_n}} \begin{pmatrix} \sqrt{1+\mu_n} & -1 \\ 1 & -\sqrt{1+\mu_n} \end{pmatrix}$$

and

$$Q_n = \frac{1}{\sqrt{\mu_n}} \begin{pmatrix} \sqrt{1+\mu_n} & -1 \\ -1 & \sqrt{1+\mu_n} \end{pmatrix}. \tag{5.7}$$

Setting $\nu_n := \sqrt{1+\mu_n} + \sqrt{\mu_n}$, the eigenvalues and the corresponding normalized in $(\mathcal{K}_n, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}})$ eigenvectors of the operator S_n are

$$\begin{aligned} -1 \qquad &\text{with} \qquad \phi_n^- = \frac{\sqrt{2}}{2\,\sqrt[4]{1+\mu_n}} \left(\frac{1}{\sqrt{\nu_n}}\,\omega_n^+ + \sqrt{\nu_n}\,\omega_n^- \right), \\ 1 \qquad &\text{with} \qquad \phi_n^+ = \frac{\sqrt{2}}{2\,\sqrt[4]{1+\mu_n}} \left(\sqrt{\nu_n}\,\omega_n^+ + \frac{1}{\sqrt{\nu_n}}\,\omega_n^- \right). \end{aligned}$$

The eigenvalues and the corresponding normalized in $(\mathcal{K}_n, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}})$ eigenvectors of the operator Q_n are

$$\begin{aligned} \frac{\sqrt{\mu_n}}{1+\sqrt{1+\mu_n}} \qquad &\text{with} \qquad \psi_n^- = \frac{1}{\sqrt{2}} \left(\omega_n^+ + \omega_n^- \right), \\ \frac{1+\sqrt{1+\mu_n}}{\sqrt{\mu_n}} \qquad &\text{with} \qquad \psi_n^+ = \frac{1}{\sqrt{2}} \left(\omega_n^+ - \omega_n^- \right). \end{aligned}$$

Put

$$S := \bigoplus_{n=1}^{+\infty} S_n \qquad \text{and} \qquad Q := \bigoplus_{n=1}^{+\infty} Q_n. \tag{5.8}$$

The operator Q is a positive self–adjoint operator in the Hilbert space $(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}})$. The assumptions (5.6) imply that the eigenvalues of Q form an unbounded sequence which also accumulates at 0. Therefore Q is unbounded and its inverse is also unbounded. Clearly, $S = JQ$. Consequently S is a positive self–adjoint operator in the Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$. It is also neither bounded nor it has a bounded inverse. It is important to note that S is idempotent, that is, $S = S^{-1}$. Put $\mathcal{M} = \mathcal{D}(S) = \mathcal{D}(Q)$. Here $\mathcal{D}(\cdot)$ denotes the domain of an operator.

Next we prove that $\mathcal{L} \subset \mathcal{M}$. Since $\mathcal{L} = \mathcal{R}\big(|T|^{1/2}\big)$ and $\mathcal{M} = \mathcal{D}(Q)$, this will be accomplished by proving that the operator $Q|T|^{1/2}$ is bounded in $(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}})$. The boundedness of $Q|T|^{1/2}$ is equivalent to the boundedness of $\big(Q|T|^{1/2}\big)^* Q|T|^{1/2}$, where * denotes the adjoint in $\big(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}}\big)$. Consider first the operators

$$\big(Q_n(|T|_n)^{1/2}\big)^* Q_n(|T|_n)^{1/2}, \quad n \in \mathbb{N},$$

whose matrix representation with respect to the basis $\{\omega_n^+, \omega_n^-\}$ is

$$\frac{2}{\mu_n}\begin{pmatrix} \lambda_n^+ & -\sqrt{\lambda_n^+\lambda_n^-(1+\mu_n)} \\ -\sqrt{\lambda_n^+\lambda_n^-(1+\mu_n)} & \lambda_n^- \end{pmatrix} + \begin{pmatrix} \lambda_n^+ & 0 \\ 0 & \lambda_n^- \end{pmatrix}. \tag{5.9}$$

The eigenvalues of the first matrix in (5.9) are

$$\frac{\lambda_n^+ + \lambda_n^-}{\mu_n} \pm \sqrt{\left(\frac{\lambda_n^+ + \lambda_n^-}{\mu_n}\right)^2 + \frac{4\lambda_n^+\lambda_n^-}{\mu_n}}, \qquad n \in \mathbb{N}. \tag{5.10}$$

The assumptions (5.6) about the sequence (μ_n) imply that the sequences in (5.10) are bounded. Since these sequences represent all the eigenvalues of the self–adjoint in $\big(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}}\big)$ operator $\big(Q|T|^{1/2}\big)^* Q|T|^{1/2} - |T|$ and since the corresponding normalized eigenvectors form an orthonormal basis for $\big(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}}\big)$, we conclude that $\big(Q|T|^{1/2}\big)^* Q|T|^{1/2} - |T|$ is bounded. Consequently $\big(Q|T|^{1/2}\big)^* Q|T|^{1/2}$, and therefore $Q|T|^{1/2}$, is bounded in $\big(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}}\big)$. It follows that $\mathcal{R}\big(|T|^{1/2}\big) = \mathcal{L} \subset \mathcal{M} = \mathcal{D}(Q)$.

Define a positive definite inner product on $\mathcal{M}$ by

$$(x, y)_{\mathcal{M}} = \langle Qx, y\rangle_{\mathcal{K}}, \qquad x, y \in \mathcal{M}.$$

With this inner product $\mathcal{M}$ is a pre-Hilbert space. Put

$$\mathcal{M}_+ = \ker(1 - S) \qquad \text{and} \qquad \mathcal{M}_- = \ker(1 + S).$$

Since S is a closed operator the subspaces $\mathcal{M}_\pm$ are closed in $\big(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}}\big)$ and

$$\mathcal{M} = \mathcal{M}_+[\dot{+}]_{\mathcal{K}}\mathcal{M}_-. \tag{5.11}$$

Let $x, y \in \mathcal{M}$ and let $x = x_+ + x_-$, $y = y_+ + y_-$ be the decompositions with respect to (5.11). Now we calculate

$$\begin{aligned}(x, y)_{\mathcal{M}} &= \langle Qx, y\rangle_{\mathcal{K}} = [Sx, y]_{\mathcal{K}} = [Sx_+, y_+]_{\mathcal{K}} + [Sx_-, y_-]_{\mathcal{K}} \\ &= [x_+, y_+]_{\mathcal{K}} - [x_-, y_-]_{\mathcal{K}}.\end{aligned}$$

Therefore the topology of $\big(\mathcal{M}, (\,\cdot\,,\cdot\,)_{\mathcal{M}}\big)$ is a decomposition topology on $\big(\mathcal{M}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\big)$. Since the operator Q is invertible and since the elements $\psi_n^\pm$, $n \in \mathbb{N}$, form a complete set in $\big(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{K}}\big)$, it follows that the (finite) linear combinations of the eigenvectors $\psi_n^{\pm}$, $n \in \mathbb{N}$, of Q, and therefore also the linear combinations of the elements $\omega_n^{\pm}$, $n \in \mathbb{N}$, form a dense subspace of $\bigl(\mathcal{M}, (\,\cdot\,,\cdot\,)_{\mathcal{M}}\bigr)$. Consequently, the subspace $\mathcal{L}$, which contains all the linear combinations of $\omega_n^{\pm}$, $n \in \mathbb{N}$, is dense in $\bigl(\mathcal{M}, (\,\cdot\,,\cdot\,)_{\mathcal{M}}\bigr)$. Since Q is neither bounded nor has it a bounded inverse for any completion $\bigl(\mathcal{Q}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{Q}}\bigr)$ of $\bigl(\mathcal{M}, (\,\cdot\,,\cdot\,)_{\mathcal{M}}\bigr)$ we have that neither $\mathcal{Q} \subseteq \mathcal{K}$ nor $\mathcal{K} \subseteq \mathcal{Q}$. Therefore the completion $\bigl(\mathcal{Q}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{Q}}\bigr)$ of $\bigl(\mathcal{M}, (\,\cdot\,,\cdot\,)_{\mathcal{M}}\bigr)$ gives rise to the Krein space completion $\bigl(\mathcal{Q}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\bigr)$ of $\bigl(\mathcal{M}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\bigr)$ which is different from $\bigl(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\bigr)$. Since $\mathcal{L}$ is dense in $\bigl(\mathcal{M}, (\,\cdot\,,\cdot\,)_{\mathcal{M}}\bigr)$ we have constructed two different Krein space completions of $\bigl(\mathcal{L}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\bigr)$. In fact, by choosing different sequences μ_n, $n \in \mathbb{N}$, we get infinitely many different Krein space completions for $\bigl(\mathcal{L}, [\,\cdot\,,\cdot\,]_{\mathcal{L}}\bigr)$.

Remark 5.3. Let (μ_n), (μ_n') be two sequences of real numbers such that

$$\mu_n > 0,\ n \in \mathbb{N}, \qquad \sup_{n\in\mathbb{N}} \mu_n < +\infty, \qquad \sup_{n\in\mathbb{N}} \frac{\lambda_n^{\pm}}{\mu_n} < +\infty, \tag{5.12}$$

and that the same relations hold for μ_n' instead of μ_n. Then the condition

$$\sup_{n\in\mathbb{N}} \frac{\mu_n}{\mu_n'} < +\infty, \qquad \sup_{n\in\mathbb{N}} \frac{\mu_n'}{\mu_n} < +\infty \tag{5.13}$$

is necessary and sufficient for the two sequences (μ_n) and (μ_n') to generate the same Krein space completion $\bigl(\mathcal{Q}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\bigr)$ of $\bigl(\mathcal{L}, [\,\cdot\,,\cdot\,]_{\mathcal{L}}\bigr)$ in the construction (II) of the above proof.

Indeed, let Q be the positive self–adjoint operator defined by (5.7) and (5.8). The eigenvalues of Q are

$$\left\{ f(\mu_n),\ \frac{1}{f(\mu_n)} : n \in \mathbb{N} \right\} \qquad \text{where} \qquad f(t) = \frac{\sqrt{t}}{1+\sqrt{1+t}},\ t > 0.$$

Clearly, $\lim\limits_{t\to+\infty} f(t)=1$, $\lim\limits_{t\to 0+} f(t)=0$. The Hilbert space completion $\bigl(\mathcal{Q}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{Q}}\bigr)$ of $\bigl(\mathcal{D}(Q), \langle Q\,\cdot\,,\cdot\,\rangle_{\mathcal{K}}\bigr)$ is isomorphic to the space $\widehat{\mathcal{Q}} \subset \mathbb{C}^{\mathbb{N}}$ defined by

$$(\xi_n)_{n\in\mathbb{N}} \in \widehat{\mathcal{Q}} \iff \bigl(\sqrt{f(\mu_n)}\,\xi_{2n-1}\bigr)_{n\in\mathbb{N}} \in \ell^2 \text{ and } \left(\frac{\xi_{2n}}{\sqrt{f(\mu_n)}}\right)_{n\in\mathbb{N}} \in \ell^2, \tag{5.14}$$

with the inner product

$$\langle x, y\rangle_{\widehat{\mathcal{Q}}} := \sum_{n=1}^{+\infty} \Bigl(f(\mu_n)\,\xi_{2n-1}\,\overline{\zeta}_{2n-1} + \frac{1}{f(\mu_n)}\,\xi_{2n}\,\overline{\zeta}_{2n} \Bigr).$$

Since $\lim_{t\to 0} f(t)/\sqrt{t} = 1$, from the definition (5.14) we have

$$(\xi_n)_{n\in\mathbb{N}} \in \widehat{\mathcal{Q}} \iff \left(\sqrt[4]{\mu_n}\ \xi_{2n-1}\right)_{n\in\mathbb{N}} \in \ell^2 \quad \text{and} \quad \left(\frac{\xi_{2n}}{\sqrt[4]{\mu_n}}\right)_{n\in\mathbb{N}} \in \ell^2, \tag{5.15}$$

and, because of (5.13) and (5.14), the inclusions on the right hand side are equivalent to

$$\left(\sqrt[4]{\mu_n'}\ \xi_{2n-1}\right)_{n\in\mathbb{N}} \in \ell^2 \quad \text{and} \quad \left(\frac{\xi_{2n}}{\sqrt[4]{\mu_n'}}\right)_{n\in\mathbb{N}} \in \ell^2.$$

By (5.12) and (5.13), the completion $\left(\mathcal{Q}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\right)$ coincides with $\left(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}}\right)$ if and only if $\inf\{\mu_n : n\in\mathbb{N}\} > 0$, in particular, if $\mu_n = 1$.

Remark 5.4. In part (I) of the proof of Theorem 5.2 we obtained two different Krein space completions, among them being the canonical one. In part (II) of the proof we obtained the canonical completion and a whole class of non–canonical completions. However, the non–canonical completion of part (I) is not among the non–canonical completions of part (II).

To see this we describe the canonical Krein space completion $\left(\mathcal{B}, [\,\cdot\,,\cdot\,]_{\mathcal{H}}\right)$ of $\left(\mathcal{H}, [\,\cdot\,,\cdot\,]_{\mathcal{H}}\right)$ as in (I). The eigenvalues of the positive self–adjoint operator $\widehat{B}$ in $\left(\ell^2, (\,\cdot\,,\cdot\,)_2\right)$ are β_n, $\frac{1}{\beta_n}$, $n\in\mathbb{N}$, where

$$\beta_n := \frac{2 + \left(\lambda_n^-\right)^2 + \left(\lambda_n^+\right)^2 + \sqrt{4 - 8\lambda_n^+\lambda_n^- + \left(\left(\lambda_n^-\right)^2 + \left(\lambda_n^+\right)^2\right)^2}}{2\left(\lambda_n^- + \lambda_n^+\right)}.$$

Note that $\beta_n \to +\infty$, $n\to+\infty$. The completion $\left(\mathcal{B}, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}}\right)$ of $\left(\mathcal{D}(\widehat{B}), (\widehat{B}\,\cdot\,,\,\cdot\,)_2\right)$ is isomorphic to the space $\widehat{\mathcal{B}} \subset \mathbb{C}^{\mathbb{N}}$ defined by

$$(\xi_n)_{n\in\mathbb{N}} \in \widehat{\mathcal{B}} \iff \left(\sqrt{\beta_n}\,\xi_{2n-1}\right)_{n\in\mathbb{N}} \in \ell^2 \quad \text{and} \quad \left(\frac{\xi_{2n}}{\sqrt{\beta_n}}\right)_{n\in\mathbb{N}} \in \ell^2. \tag{5.16}$$

Since $\lim_{n\to+\infty} \beta_n\left(\lambda_n^- + \lambda_n^+\right) = 2$, the relation (5.16) is equivalent to

$$(\xi_n)_{n\in\mathbb{N}} \in \widehat{\mathcal{B}} \iff \left(\frac{\xi_{2n-1}}{\sqrt{\lambda_n^- + \lambda_n^+}}\right)_{n\in\mathbb{N}} \in \ell^2 \text{ and } \left(\sqrt{\lambda_n^- + \lambda_n^+}\,\xi_{2n}\right)_{n\in\mathbb{N}} \in \ell^2. \tag{5.17}$$

Comparing (5.17) and (5.15) we conclude that we would need to choose $\mu_n = \left(\lambda_n^- + \lambda_n^+\right)^2$ to obtain the non–canonical completion of part (I). But this choice violates (5.6).

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Neprekidna ulaganja, upotpunjenja i komplementarnost u Kreinovim prostorimau

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Sadržaj

Neka je Kreinov prostor $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ neprekidno uložen u Kreinov prostor $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$. Koristeći operator koji je adjungiran operatoru ulaganja prostora $\mathcal{A}$ u prostor $\mathcal{K}$, jedinstven hermitski operator A u $\mathcal{K}$ je pridružen prostoru $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$. Tada je Kreinov prostor $(\mathcal{A}, [\,\cdot\,,\cdot\,]_{\mathcal{A}})$ upotpunjenje prostora $\mathcal{R}(A)$ snabdjevenog sa A–scalarnim produktom. Općenito ovo upotpunjenje nije jedinstveno određeno. Ako je ulaganje prostora $\mathcal{A}$ u prostor $\mathcal{K}$ još i t–neprekidno, onda je operator A definitizabilan u $\mathcal{K}$ i prostor $\mathcal{R}(A)$ snabdjeven sa A–scalarnim produktom ima jedinstveno upotpunjenje do Kreinovog prostora. U ovom slučaju spektralna funkcija operatora A daje određene informacije o ulaganju prostora $\mathcal{A}$ u $\mathcal{K}$. Rezulatati su primjenjeni na de Branges–ovu teoriju komplementiranja.