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Coincidence points and best approximations in p–normed spaces

A. Bano (Pakistan), A.R. Khan and A. Latif (Saudi Arabia)

Abstract. Using a coincidence points theorem of H. Kaneko [5] our results generalize some known results of Jungck and Sessa [4], Latif and Tweddle [9] and Singh [15].

1. Introduction

Brosowski [1], Meinardus [11] and Singh [15] established some interesting results on invariant approximation in normed spaces using fixed point theory. Jungck and Sessa [4] have also obtained some results on approximation theory in the setting of normed spaces. Their work has been extended, generalized and unified by many authors; for example, see Habiniak [2], Hicks and Humphries [3], Latif and Bano [8], Narang [12], Smoluk [17]. Singh [16], in particular presented his work for locally convex spaces in unified form.

The main objective of this paper is to extend, unify and discuss the work carried out by different investigators, like Singh [15], Sahab, Khan and Sessa [14] and Lami Dozo [7] in the setting of p–normed spaces.

Some useful notions and results

Let X be a linear space over the field of real numbers. A p–norm on X is a real valued function $\|.\|_p$ on X with $0 < p \leq 1$ [6], satisfying the following conditions:

(1) $\|x\|_p \geq 0$ and $\|x\|_p = 0 \Leftrightarrow x = 0$
(2) $\|\alpha x\|_p = |\alpha|^p \|x\|_p$
(3) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$

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for all $x, y \in X$ and all scalars α. The pair $(X, \|.\|_p)$ is called a p–normed space. It is a metric space with $d_p(x, y) = \|x - y\|_p$ for all $x, y \in X$ defining a translation invariant metric d_p on X. If $p = 1$, we obtain a normed linear space. It is well known that the topology of every Hausdorff locally bounded topological linear space is given by some p–norm, $0 < p \leq 1$ [6]. Note that a p–normed space is not necessarily a locally convex space.

Let L_p, $0 < p \leq 1$ be the space of all measurable functions $f(t)$ on $I = [a, b]$ with $\int_a^b |f(t)|^p dt < \infty$ (we identify functions which are equal almost everywhere). For all $f \in L_p$, $0 < p \leq 1$, let the function $\|f\|_p$ be defined by

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{\frac{1}{p}}. \tag{1.1}$$

This expression is an example of a quasi–norm on a topological linear space [6].

Let M be a subset of a p–normed space. Then M is said to be star-shaped with respect to a point $q \in M$ if for each $x \in M$, the segment joining x to q is contained in M (that is, $\lambda x + (1 - \lambda) q \in M$ for each $x \in M$, and real λ with $0 \leq \lambda \leq 1$).

$M \subset X$ is said to be starshaped if it is starshaped with respect to one of its element. A convex set is obviously starshaped, but starshaped need not be convex.

Let X be a p–normed space and M a nonempty subset of X. We denote by $2^X, CB(X), K(X)$ the collection of all nonempty, nonempty closed bounded and nonempty compact subsets of X, respectively. The Hausdorff metric H_p on $CB(X)$ induced by the p–norm of X is defined by

$$H_p(A, B) = \max \left\{ \sup_{a \in A} d_p(a, B), \sup_{b \in B} d_p(A, b) \right\} \text{ for all } A, B \in CB(X)$$

where $d_p(x, B) = \inf\{\|x - y\|_p : y \in B\}$ for each $x \in X$. Let M be a subset of p–normed space X for each $x \in X$, define

$P_M(x) = \{z \in M : \|x - z\|_p = d_p(x, M)\}$, the set of the best M–approximants to x.

The set $P_M(x)$ is always a bounded subset of X and it is closed or convex if M is closed or convex (see [1]).

Let $f : M \to X$ be a single–valued map. A multivalued map $T : M \to CB(X)$ is said to be a f–contraction if for a fixed constant k, $0 \leq k \leq 1$ and for all $x, y \in X$,

$$H_p(T(x), T(y)) \leq k\|f(x) - f(y)\|_p.$$

If $k = 1$ in the above inequality then T is called f–nonexpansive. Indeed, if $f = I$ (the identity map on X), then each f–contraction is a contraction. An element $x \in X$ is called a fixed point of multivalued map T if

$$x \in T(x).$$

An element $x \in X$ is called a coincidence point of f and T if

$$f(x) \in T(x).$$

We denote by $F(T)$ the set of fixed points of T and by $C(f \cap T)$ the set of coincidence points of f and T.

A complete p–normed space is said to satisfy Opial's property [13] if for every sequence $\{x_n\}$ in X weakly convergent to $x \in X$, the inequality

$$\lim_{n\to\infty} \inf \|x_n - x\|_p < \lim_{n\to\infty} \inf \|x_n - y\|_p$$

holds for all $y \neq x$.

A complete p–normed space satisfying Opial's property is called an Opial space. Every Hilbert space satisfies Opial's property [13]. The spaces $l_p(1 < p < \infty)$ satisfy Opial's property [7]. However there are Banach spaces which do not satisfy Opial's property, e.g. $L_p[0, 2\pi](p \neq 2)$.

Let $T : M \subseteq X \to CB(X)$. A best approximation z in M to an element x_0 in X such that $T(x_0) = \{x_0\}$ is called an invariant approximation in X to x_0 if $z \in T(z)$.

A multi–valued map $T : M \to 2^X$ is said to be demiclosed if for every sequence $\{x_n\} \subset M$ and $y_n \in T(x_n)$, $n = 1, 2, 3, \ldots$ such that $x_n \xrightarrow{w} x$ and $y_n \to y$, we have $x \in M$ and $y \in T(x)$.

Here and throughout the paper $\xrightarrow{w}$ and $\to$ denote weak and strong convergence respectively.

Example 1.1. We will show that the p–th power of the quasi–norm $\|f\|_p$ in L_p defined by (1.1) is a p–norm on L_p.

For each $f \in L_p$ the p–th power of the quasi-norm in L_p is defined by

$$\|f\|_p^p = \int_a^b |f(t)|^p \, dt. \tag{1.2}$$

The norm defined by (1.2) is a p–norm on L_p.

(1) For each $f \in L_p$, $\|f\|_p^p \geq 0$. If $\|f\|_p^p = 0$ then $f(t) = 0$ almost everywhere,

(2) $\|af\|_p^p = \int_a^b |af(t)|^p \, dt = |a|^p \int_a^b |f(t)|^p \, dt = |a|^p \|f\|_p^p$ for all scalars a and all $f \in L_p$,

(3) $\|f+g\|_p^p = \int_a^b |f(t)+g(t)|^p\,dt \le \int_a^b |f(t)|^p\,dt + \int_a^b |g(t)^{\,p}\,dt \le \|f\|_p^p + \|g\|_p^p$ for all $f, g \in L_p$.

Thus all the properties of p–norm, $0 < p \le 1$, are satisfied. Hence the p–th power of the quasi-norm $\|f\|_p$ in L_p is a p–norm on L_p.

In [5] Kaneko has proved the following coincidence point theorem for complete metric spaces.

Theorem 1.2. *Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ and $f : X \to X$ be a continuous mapping which commutes with T and $T(X) \subseteq f(X)$. Suppose that there exists $k \in (0, 1)$ such that*

$$H\big(T(x), T(y)\big) \le kd\,\big(f(x), f(y)\big) \text{ for each } x, y \in X.$$

Then there exists an $x_0 \in X$ such that $f(x_0) \in T(x_0)$.

In [10], Latif and Tweddle proved the following result which is an extension of Lami Dozo [7].

Lemma 1.3. *Let M be a nonempty weakly compact subset of a Banach space X satisfying Opial's property. Let $f : M \to X$ be a weakly continuous map and $T : M \to K(X)$ an–nonexpansive map. Then $f - T$ is demiclosed.*

In [9] Latif and Tweddle obtained the following coincidence point result in the setting of Banach spaces.

Theorem 1.4. *Let X be a Banach space and M be a weakly compact subset which is starshaped with respect to $q \in M$. Let $f : M \to M$ be a weakly continuous map such that $f(M) = M$ and $f(q) = q$ and let $T : M \to K(M)$ be a multivalued f–nonexpansive map which commutes with f. Then $C(f \cap T) \neq \emptyset$ provided that one of the following condition holds: either*

(1) *$(f - T)$ is demiclosed or*
(2) *X satisfies Opial's property.*

2. Main results

The main objective of this section is to extend and unify the work of Lami Dozo [7] and Singh [15] to a multivalued f–nonexpansive map in the setting of p–normed spaces. A corollary in the setting of a complete p–normed space satisfying Opial's property is also obtained.

First we prove Lemma 1.3 in the setting of p–normed spaces.

Lemma 2.1. *Let M be weakly compact subset of a complete p–normed space X satisfying Opial's property. Let $f : M \to X$ be weakly continuous* *map and* $T : M \to K(X)$ *be an* f*–nonexpansive multivalued map. Then* $f-T$ *is demiclosed.*

Proof. Let $\{x_n\} \subset M$ and $y_n \in (f-T)\,x_n$ be such that $x_n \xrightarrow{w} x$ and $y_n \to y$. It is obvious that $x \in M$ and $f(x_n) \xrightarrow{w} f(x)$. Since $y_n \in f(x_n) - T(x_n)$, we get

$$y_n = f(x_n) - u_n \text{ for some } u_n \in T(x_n). \tag{2.1}$$

Since $T(x)$ is compact set, there is a $v_n \in T(x)$ such that

$$\|u_n - v_n\|_p \le H_p\big(T(x_n), T(x)\big).$$

So, by using the f-nonexpansiveness of T, we have

$$H_p\big(T(x_n), T(x)\big) \le \|f(x_n) - f(x)\|_p.$$

Thus

$$\|u_n - v_n\|_p \le \|f(x_n) - f(x)\|_p. \tag{2.2}$$

From (2.1) and (2.2), passing to the limit with respect to n, we obtain

$$\begin{aligned} \lim_{n\to\infty} \inf \|f(x_n) - f(x)\|_p &\ge \lim_{n\to\infty} \inf \|u_n - v_n\|_p \\ &\ge \lim_{n\to\infty} \inf \|f(x_n) - y_n - v_n\|_p. \end{aligned} \tag{2.3}$$

$T(x)$ is compact and $y_n \to y$, so for a convenient subsequence still denoted by $\{v_n\}$, we have $v_n \to v \in T(x)$. So from (2.3) we have

$$\lim_{n\to\infty} \inf \|f(x_n) - f(x)\|_p \ge \lim_{n\to\infty} \inf \|f(x_n) - y - v\|_p.$$

Since X satisfies Opial's property and $f(x_n) \xrightarrow{w} f(x)$, we obtain $f(x) = y+v$. Thus $y = f(x) - v \in f(x) - T(x) = (f-T)\,x$, which proves that $f-T$ is demiclosed.

We have the following coincidence point result for a complete p–normed space.

Theorem 2.2. *Let* X *be a complete* p*–normed space and* M *be a weakly compact subset which is starshaped with respect to* $q \in M$*. Let* $f : M \to M$ *be a weakly continuous and affine map such that* $f(M) = M$ *and* $f(q) = q$ *and let* $T : M \to K(M)$ *be a multivalued* f*–nonexpansive map which commutes with* f*. If* $(f-T)$ *is demiclosed on* X *or* X *satisfies Opial's property then* $C(f \cap T) \ne \emptyset$*.*

Proof. Consider a sequence $\{k_n\}$ of real numbers such that $0 < k_n < 1$ and $k_n \to 1$ as $n \to \infty$. Define a mapping T_n by setting

$$T_n(x) = k_n T(x) + (1 - k_n)\, q$$

for all $x \in M$. Since T maps M into $K(M)$, we observe that for each $n \geq 1$ T_n maps M into $K(M)$, since M is starshaped with respect to $q \in M$ and $T(M) \subseteq M$. Also $T_n(M) \subseteq f(M)$. Let $x, y \in M$. Then

$$H_p\big(T_n(x), T_n(y)\big) = k_n H_p\big(T(x), T(y)\big).$$

By the f–nonexpansiveness of T we have

$$H_p\big(T_n(x), T_n(y)\big) \leq k_n \| f(x) - f(y) \|_p$$

showing that T_n is an f–contraction. Now we show that T_n commutes with f.

$$\begin{aligned} T_n\big(f(x)\big) &= k_n T\, f(x) + (1 - k_n) f q \\ &= k_n\, f T(x) + (1 - k_n) f q \\ &= f\big(k_n T(x) + (1 - k_n)\, q\big) \\ &= f\, T_n(x) \end{aligned}$$

for each $x \in M$. The above holds since $fq = qf$, $fT = Tf$ and f is an affine. Thus $T_n f = f\, T_n$ for each $n \geq 1$. Thus all the conditions of Theorem 1.2 are satisfied and hence there is an $x_n \in M$ such that $f(x_n) \in T_n(x_n)$. So by the definition of $T_n(x_n)$, there is a $u_n \in T(x_n)$ such that

$$\begin{aligned} f(x_n) &= k_n u_n + (1 - k_n)\, q \\ f(x_n) - u_n &= (1 - k_n)(q - u_n) \\ &= \left(\frac{1}{k_n} - 1\right) \big(q - f(x_n)\big) \quad \textit{and} \\ \| f(x_n) - u_n \| &\leq \left(\frac{1}{k_n} - 1\right)^p \big\{ \|g\|_p + \|f(x_n)\|_p \big\}. \end{aligned}$$

Since $T(M) \subseteq M$ is bounded and $f(x_n) \in T_n(x_n) \subseteq M$ we have $\|f(x_n)\|_p$ is bounded so by the fact that $k_n \to 1$, we have $\|f(x_n) - u_n\|_p \to 0$. Since M is weakly compact, there is a subsequence $\{x_{n_i}\}$ of sequence which converges weakly to $x_0 \in M$. As $f(x_{n_i}) - u_{n_i} \in (f - T)\, x_{n_i}$, if $f - T$ is demiclosed, then $0 \in (f - T)\, x_0$ and hence $f(x_0) \in T(x_0)$. If X satisfies Opial's property, then it follows from Lemma 1.3 that $f - T$ is demiclosed and hence f and T have a coincidence point $x_0 \in M$ as in the previous case.

Remark 2.3. Theorem 2.2 extends Theorem 1.4 for p–normed spaces and if $f = I$, the identity map on M and $p = 1$, then we have a generalization of the earlier stated results of Jungck and Sessa [4] and Lami Dozo [7] for multivalued nonexpansive maps. As an application of Theorem 2.2 we have the following result on best approximation.

Theorem 2.4. *Let X be a complete p–normed space. Let $f : X \to X$ be a weakly continuous map and let $T : X \to CB(X)$ be an f–nonexpansive map such that $T(x_0) = \{x_0\}$ for some $x_0 \in X$. Let M be a nonempty T–invariant subset of X and $x_0 \in F(T) \cap F(f)$. Assume that $P_M(x_0)$ is nonempty, weakly compact and starshaped with respect to $q \in F(f)$. Further assume that f is continuous and affine map on $P_M(x_0)$ with $f\big(P_M(x_0)\big) = P_M(x_0)$. If $f - T$ is demiclosed on $P_M(x_0)$, then $P_M(x_0) \cap F(f) \cap F(T) \neq \emptyset$.*

Proof. Let $D = P_M(x_0)$ and let $u \in D$. Then $u \in M$ and

$$\|x_0 - u\|_p = d_p(x_0, M).$$

Let $v \in T(u) \subset M$. Then we have

$$\|v - x_0\|_p \leq H_p\big(T(u), T(x_0)\big)$$

so, by using the f–nonexpansiveness of T, we get

$$H_p\big(T(u), T(x_0)\big) \leq \|f(u) - f(x_0)\,\|_p$$

$$\|v - x_0\|_p \leq \|f(u) - f(x_0)\,\|_p.$$

As $f(x_0) = x_0$ and $f(u) \in P_M(x_0)$, we have

$$\|v - x_0\|_p \leq \|f(u) - x_0\| = d_p(x_0, M),$$

which gives that $v \in D$ and thus $T(u) \subset D$. Therefore T carries D into $CB(D)$. Now let q be the starcenter of D. Then for each $x \in D$ and any k $(0 < k < 1)$,

$$(1 - k)\,q + kx \in D.$$

Take a $\{k_n\}$ sequence of real numbers such that $0 < k_n < 1$ and $k_n \to 1$ as $n \to \infty$. Now for each n define a multivalued map T_n by setting

$$T_n(x) = (1 - k)\,q + k_n T(x), \ \text{ for all } \ x \in D.$$

Clearly, each T_n maps D into $CB(D)$. Since f is continuous, affine and commute with T we have

$$\begin{aligned} T_n f(x) &= (1-k_n) fq + k_n T f(x) \\ &= (1-k_n) fq + k_n f T(x) \\ &= f\big((1-k_n)\, q + k_n T(x)\big) \\ &= f T_n(x). \end{aligned}$$

Thus each T_n commutes with f for each n and $T_n(D) \subseteq D = f(D)$. Let $x, y \in D$. Then by the definition of T_n and the f–nonexpansiveness of T, we have

$$H_p\big(T_n(x), T_n(y)\big) = k_n H_p\big(T(x), T(y)\big) \leq k_n \| f(x) - f(y) \|,$$

which proves that each T_n is an f–contraction map. Also since D is a complete metric space, it follows from Theorem 1.2 that for each $n \geq 1$, there exists $x_n \in D$ such that $f(x_n) \in T_n(x_n)$. This implies that there is a $y_n \in T(x_n)$ such that

$$\begin{aligned} f(x_n) &= (1-k_n)\, q + k_n y_n \\ f(x_n) - y_n &= (1-k_n)\, q - (1-k_n) y_n \\ &= (1-k_n)(q - y_n) \\ &= \left(\frac{1}{k_n} - 1\right)\big(q - f(x_n)\big) \qquad \text{and} \end{aligned}$$

$$\| f(x_n) - y_n \|_p = \left(\frac{1}{k_n} - 1\right)^p \| q - f(x_n) \|_p \to 0 \quad \text{as } n \to \infty.$$

Since D is weakly compact, there exists a subsequence of $\{x_n\}$, still denoted by $\{x_n\}$, and we have $x_n \xrightarrow{w} z \in D$. Now as $f(x_n) - y_n \in (f-T)\, x_n$ and $f - T$ is demiclosed, we conclude that $0 \in (f-T)\, z$ and hence $f(z) \in T(z)$.

Now by Lemma 2.1, we have the following result on invariant approximation.

Corollary 2.5. *Let X be a complete p–normed space satisfying Opial's property. Let $f : X \to X$ be a weakly continuous map and $T : X \to K(X)$ be an f–nonexpansive map such that $T(x_0) = \{x_0\}$ for some $x_0 \in X$. Let M be a nonempty T– invariant subset of X and $x_0 \in F(T) \cap F(f)$. Assume that $P_M(x_0)$ is nonempty, weakly compact and starshaped with respect to $q \in F(f)$. Further, if f is continuous and an affine map on $P_M(x_0)$ with $f\big(P_M(x_0)\big) = P_M(x_0)$ then $P_M(x_0) \cap F(f) \cap F(T) \neq \emptyset$.*

Proof. Let $D = P_M(x_0)$ and let $u \in D$. Then $u \in M$ and

$$\| x_0 - u \|_p = d_p(x_0, M).$$

Let $v \in T(u) \subset M$. Then we have that

$$\|v - x_o\|_p \leq H_p\big(T(u), T(x_0)\big)$$

so, by using the f–nonexpansiveness of T, we get

$$H_p\big(T(u), T(x_0)\big) \leq \|f(u) - f(x_{})\|_p.$$

Thus

$$\|v - x_0\|_p \leq \|f(u) - f(x_0)\|_p = d_p(x_0, M),$$

we have $v \in D$ and thus $T(u) \subset D$. Therefore T carries $D = P_M(x_0)$ into $K(D)$. Thus by Lemma 2.1, $f - T$ is demiclosed on D. Now the result follows from Theorem 2.4.

Remark 2.6. Theorem 2.4 extends Theorem 3 [8] and thus contains the result of Singh [14] as a special case.

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A.R. Khan
Department of Mathematical Sciences
King Fahad University of Petroleum and Minerals
Dhahran–31261
Saudi Arabia
e-mail: arhaim@kfupm.edu.sa

A. Bano
Department of Mathematics
Gomal University
Dera Ismail Khan
Pakistan
e-mail: arjamandbano2002@yahoo.com

A. Latif
Department of Mathematics
King Abdul Aziz University
P.O. Box 80203, Jeddah-21589
Saudi Arabia
e-mail: abdul_latifn@hotmail.com

Koincidentne tačke i najbolje aproksimacije u p–normiranim prostorima

A.Bano, A.R. Khan and A.Latif

Sadržaj

U ovom radu, koristeći teorem koincidentnih tačaka H. Kaneka [5], autori generaliziraju neke poznate rezultate autora Jungck i Sessa [4], Latif i Tweddle [9] i Singh [15].