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Local regularity result for very weak solutions of obstacle problems

Li Juan and Gao Hongya (P. R. China)

Abstract. This paper gives a local regularity result for very weak solutions to obstacle problems of the A–harmonic equation $divA(x, \bigtriangledown u(x)) = 0$, $|A(x, \xi)| \approx |\xi|^{p-1}$, where $1 < p < n$ and the obstacle function satisfies $\psi \geq 0$.

1. Introduction and statement of result

Let Ω be a bounded regular domain in $R^n, n \geq 2$. By a regular domain we understand any domain of finite measure for which the estimates for the Hodge decomposition in (3) and (4) are satisfied. See [1]. A Lipschitz domain, for example, is a regular domain. We consider the second order degenerate elliptic equation (A–harmonic equation)

$$divA(x, \bigtriangledown u(x)) = 0, \tag{1}$$

where $A(x, \xi) : \Omega \times R^n \to R^n$ is a Carathéodory function satisfying the following conditions:

(i) $\langle A(x, \xi), \xi \rangle \geq \alpha |\xi|^p$
(ii) $|A(x, \xi)| \leq \beta |\xi|^{p-1}$
(iii) $(A(x, \xi_1) - A(x, \xi_2)) \cdot (\xi_1 - \xi_2) > 0$

where $p > 1$ and $0 < \alpha \leq \beta < \infty$. The prototype of equation (1) is the p–harmonic equation

$$div(|\bigtriangledown u|^{p-2} \bigtriangledown u) = 0.$$

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Suppose that ψ is an arbitrary function in Ω with values in $R \cup \{\infty\}$, and $\theta \in W^{1,r}(\Omega)$, $\max\{1, p-1\} < r \leq p$. Let

$$K^r_{\psi,\theta}(\Omega) = \{v \in W^{1,r}(\Omega) : v \geq \psi, \text{ a.e.; } v - \theta \in W^{1,r}_0(\Omega)\},$$

the function ψ is an obstacle and θ determines the boundary values.

We introduce the Hodge decomposition for $| \bigtriangledown (v-u)|^{r-p} \bigtriangledown (v-u) \in L^{\frac{r}{r-p+1}}(\Omega)$, see [1],

$$| \bigtriangledown (v-u)|^{r-p} \bigtriangledown (v-u) = \bigtriangledown \phi_{v,u} + h_{v,u}, \tag{2}$$

where $\phi_{v,u} \in W^{1,\frac{r}{r-p+1}}(\Omega)$ and $h_{v,u} \in L^{\frac{r}{r-p+1}}(\Omega)$ is divergence free vector. The following estimates hold

$$\| \bigtriangledown \phi_{v,u}\|_{\frac{r}{r-p+1}} \leq c \| \bigtriangledown (v-u)\|_r^{r-p+1}, \tag{3}$$

$$\|h_{v,u}\|_{\frac{r}{r-p+1}} \leq c(p-r) \| \bigtriangledown (v-u)\|_r^{r-p+1}. \tag{4}$$

Definition. A very weak solution to the $K^r_{\psi,\theta}$–obstacle problem is a function $u \in K^r_{\psi,\theta}$ such that

$$\int_\Omega \langle A(x, \bigtriangledown u), | \bigtriangledown (v-u)|^{r-p} \bigtriangledown (v-u)\rangle dx \geq \int_\Omega \langle A(x, \bigtriangledown u), h_{v,u}\rangle dx, \tag{5}$$

whenever $v \in K^r_{\psi,\theta}$ and $h_{v,u}$ comes from the Hodge decomposition (2).

Meyers and Elcat [2] first considered the higher integrability for solutions of (1) in 1975, see also [3]. The local and global higher integrability of the derivatives in obstacle problems was first considered by Li and Martio [4] in 1994, using the so-called reverse Hölder inequality. Recently, regularity theory for very weak solutions of A-harmonic equations have been considered [1], but the regularity theory for very weak solutions of obstacle problems have not been explored. This paper gives a local regularity result in this direction. The main result is the following theorem.

2. Main result

Theorem. *There exists $r_1 \in (p-1, p)$, such that for arbitrary $0 \leq \psi \in W^{1,s}_{loc}(\Omega)$, $r < s < n$ and $r_1 < r < \min(p, n)$, a solution u to the $K^r_{\psi,\theta}$–obstacle problem belongs to $L^{s^*}_{loc}(\Omega)$, where s^* satisfies $\frac{1}{s^*} = \frac{1}{s} - \frac{1}{n}$.*

Remark. Notice that we have restricted ourselves to the case $r < n$, because when $r \geq n$, every function in $W^{1,r}_{loc}(\Omega)$ is trivially in $L^t_{loc}(\Omega)$ for every $t > 1$ by the Sobolev theorem. We remark also that the method used in this paper is also different from that of [4].

We now introduce some symbols used in this paper. If $x_0 \in \Omega$ and $t > 0$, then B_t denotes the ball of radius t centered at x_0. For the function $u(x)$ and $k > 0$, let $A_k = \{x \in \Omega : |u(x)| > k\}$, $A_{k,t} = A_k \bigcap B_t$. Moreover, if $m < n, m^*$ is always the real number satisfying $\frac{1}{m^*} = \frac{1}{m} - \frac{1}{n}$. Let $T_k(u)$ be the usual truncation of u at level $k > 0$, that is,

$$T_k(u) = \max\big\{ -k, \min\{k, u\}\big\}. \tag{6}$$

We recall two lemmas that we use in the proof of the theorem.

Lemma 1[5]**.** *Let $u \in W^{1,r}_{loc}(\Omega), \varphi_0 \in L^p_{loc}(\Omega)$, where $1 < r < n$ and p satisfies*

$$1 < p < \frac{n}{r}.$$

Assume that the following integral estimate holds:

$$\int_{A_{k,\tau}} |\nabla u|^r dx \le C_0 \left[\int_{A_{k,t}} \varphi_0 dx + (t-\tau)^{-\alpha} \int_{A_{k,t}} |u|^r dx \right] \tag{7}$$

for every $k \in N$ and $R_0 \le \tau < t \le R_1$, where C_0 is a positive constant depends only on N, $p, r, R_0, R_1, |\Omega|$ and α is a real positive constant. Then $u \in L^s_{loc}(\Omega)$, where $s = (pr)^*$.

Lemma 2[6]**.** *Let $f(\tau)$ is a non-negative bounded function defined for $0 \le R_0 \le t \le R_1$. Suppose that for $R_0 \le \tau < t \le R_1$ we have*

$$f(\tau) \le A(t-\tau)^{-\alpha} + B + \theta f(t), \tag{8}$$

where A, B, α, θ are non-negative constants, and $\theta < 1$. Then there exists a constant c, depending only on α and θ, such that for every $\rho, R, R_0 \le \rho < R \le R_1$ we have

$$f(\rho) \le c\,[A(R-\rho)^{-\alpha} + B]. \tag{9}$$

Proof of the theorem. Let u be a very weak solution to the $K^r_{\psi,\theta}$–obstacle problem. By Lemma 1, it is sufficient to prove that u satisfies the inequality (7) with $\alpha = r$. Let $B_{R_1} \subset\subset \Omega$ and $0 \le R_0 \le \tau < t \le R_1$ be arbitrarily fixed. Fix a cut-off function $\phi \in C_0^\infty(B_{R_1})$ such that

$$\text{supp}\phi \subset B_t,\ \ 0 \le \phi \le 1,\ \ \phi = 1 \ \text{ in } B_\tau,\ \ |\bigtriangledown \phi| \le 2(t-\tau)^{-1}$$

consider the function

$$v = u - T_k(u) - \phi^r(u-\psi),$$

where $T_k(u)$ is the usual truncation of u at level $k \geq 0$ defined in (6). Now $v \in K^r_{\psi - T_k(u), \theta - T_k(u)}$; indeed,

$$v - \big(\theta - T_k(u)\big) = u - \theta - \phi^r(u-\psi) \in W_0^{1,r}(\Omega),$$

since $\phi \in C_0^\infty(\Omega)$ and

$$v - \big(\psi - T_k(u)\big) = (u-\psi) - \phi^r(u-\psi) = (1-\phi^r)\cdot(u-\psi) \geq 0$$

a.e.in Ω. Let

$$\begin{aligned} E(v,u) = {} & |\phi^r \bigtriangledown u|^{r-p}\phi^r \bigtriangledown u \\ & + |\bigtriangledown (v - u + T_k(u))|^{r-p} \bigtriangledown \big(v - u + T_k(u)\big). \end{aligned} \tag{10}$$

From the elementary formula $[7, P_{271},(4.1)]$

$$||X|^{-\varepsilon}X - |Y|^{-\varepsilon}Y| \leq 2^\varepsilon \frac{1+\varepsilon}{1-\varepsilon}|X-Y|^{1-\varepsilon}, \ \ 0 \leq \varepsilon < 1, \tag{11}$$

and $\bigtriangledown v = \bigtriangledown(u - T_k(u)) - \phi^r \bigtriangledown (u-\psi) - r\phi^{r-1}\bigtriangledown \phi(u-\psi)$, we can derive that

$$|E(v,u)| \leq 2^{p-r}\frac{p-r+1}{r-p+1}|\phi^r \bigtriangledown u - \phi^r \bigtriangledown (u-\psi) - r\phi^{r-1}\bigtriangledown \phi(u-\psi)|^{r-p}. \tag{12}$$

From (10), we get that

$$\begin{aligned} & \int_{A_{k,t}} \langle A(x,\bigtriangledown u), |\phi^r \bigtriangledown u|^{r-p}\phi^r \bigtriangledown u\rangle\, dx = \\ & = \int_{A_{k,t}} \langle A(x,\bigtriangledown u), E(v,u)\rangle\, dx - \int_{A_{k,t}} \langle A(x,\bigtriangledown u), |\bigtriangledown (v-u)|^{r-p}\bigtriangledown (v-u)\rangle\, dx. \end{aligned} \tag{13}$$

Now, we estimate the left-hand side of (13)

$$\begin{aligned} & \int_{A_{k,t}} \langle A(x,\bigtriangledown u), |\phi^r \bigtriangledown u|^{r-p}\phi^r \bigtriangledown u\rangle\, dx \\ & \qquad \geq \int_{A_{k,\tau}} \langle A(x,\bigtriangledown u), |\bigtriangledown u|^{r-p}\bigtriangledown u\rangle\, dx \\ & \qquad \geq \alpha \int_{A_{k,\tau}} |\bigtriangledown u|^r dx. \end{aligned} \tag{14}$$

Using the Hodge decomposition (2), we get

$$|\bigtriangledown(v-u+T_k(u))|^{r-p}\bigtriangledown(v-u+T_k(u))=D\phi+h,$$

by (4) we have that

$$||h||\leq c(p-r)||\bigtriangledown\big(v-u+T_k(u)\big)|^{r-p}||_r^{r-p+1}. \tag{15}$$

Since $u-T_k(u)$ is a very weak solution to the $K^r_{\psi-T_k(u),\theta-T_k(u)}$-obstacle problem, we derive, by the definition, that

$$\int_\Omega\langle A(x,\bigtriangledown\big(u+T_k(u)\big),|\bigtriangledown(v-u+T_k(u))|^{r-p}\bigtriangledown(v-u+T_k(u))\rangle\,dx$$
$$\geq\int_\Omega\langle A(x,\bigtriangledown(u+T_k(u)),h\rangle\,dx$$

that is

$$\int_{A_{k,t}}\langle A(x,\bigtriangledown u),|\bigtriangledown(v-u)|^{r-p}\bigtriangledown(v-u)\rangle\,dx$$
$$\geq\int_{A_{k,t}}\langle A(x,\bigtriangledown u),h\rangle\,dx\,. \tag{16}$$

Combining the inequalities (13), (14), (16), we obtain

$$\begin{aligned}\alpha\int_{A_{k,\tau}}|\bigtriangledown u|^r dx &\leq\int_{A_{k,t}}\langle A(x,\bigtriangledown u),E(v,u)\rangle dx-\int_{A_{k,t}}\langle A(x,\bigtriangledown u),h\rangle\,dx\\
&\leq\beta\frac{2^{p-r}(p-r+1)}{r-p+1}\int_{A_{k,t}}|\bigtriangledown u|^{p-1}|\phi^r\bigtriangledown\psi\\
&\quad-r\phi^{r-1}\bigtriangledown\phi(u-\psi)|^{r-p+1}dx+\beta\int_{A_{k,t}}|\bigtriangledown u|^{p-1}|h|\,dx\\
&\leq\beta\frac{2^{p-r}(p-r+1)}{r-p+1}\int_{A_{k,t}}|\bigtriangledown u|^{p-1}|\phi^r\bigtriangledown\psi|^{r-p+1}\,dx\\
&\quad+\beta\frac{2^{p-r}(p-r+1)}{r-p+1}\int_{A_{k,t}}|\bigtriangledown u|^{p-1}|r\phi^{r-1}\bigtriangledown\phi(u-\psi)|^{r-p+1}\,dx\\
&\quad+\beta\int_{A_{k,t}}|\bigtriangledown u|^{p-1}|h|\,dx\\
&\leq\beta\frac{2^{p-r}(p-r+1)}{r-p+1}\left(\int_{A_{k,t}}|\bigtriangledown u|^r dx\right)^{\frac{p-1}{r}}\\
&\quad\left(\int_{A_{k,t}}|\bigtriangledown\psi|^r dx\right)^{\frac{r-p+1}{r}}+\beta\frac{2^{p-r}(p-r+1)}{r-p+1}\end{aligned}$$

$$\left(\int_{A_{k,t}}|\bigtriangledown u|^r dx\right)^{\frac{p-1}{r}}\left(\int_{A_{k,t}}|r\phi^{r-1}\bigtriangledown\phi(u-\psi)|^r dx\right)^{\frac{r-p+1}{r}}$$
$$+\beta\left(\int_{A_{k,t}}|\bigtriangledown u|^r dx\right)^{\frac{p-1}{r}}\left(\int_{A_{k,t}}|h|^{\frac{r}{r-p+1}}dx\right)^{\frac{r-p+1}{r}}. \qquad (17)$$

Let $c_1=\frac{2^{p-r}(p-r+1)}{r-p+1}$, by (15) and Young's inequality

$$ab\le\varepsilon a^{p'}+c_2(\varepsilon,p)b^p,\quad \frac{1}{p}+\frac{1}{p'}=1,\quad a,b\ge 0,\quad \varepsilon\ge 0,\quad p\ge 1. \qquad (18)$$

We can derive that

$$\begin{aligned}\alpha\int_{A_{k,\tau}}|\bigtriangledown u|^r dx\le{}&\beta c_1\varepsilon\int_{A_{k,t}}|\bigtriangledown u|^r dx\\&+c_1c_2(\varepsilon,p)\beta\int_{A_{k,t}}|\bigtriangledown\psi|^r dx+\beta c_1\varepsilon\int_{A_{k,t}}|\bigtriangledown u|^r dx\\&+\beta c_1c_2(\varepsilon,p)\int_{A_{k,t}}|r\phi^{r-1}\bigtriangledown\phi(u-\psi)|^r dx\\&+\varepsilon\beta c_3(p-r)\int_{A_{k,t}}|\bigtriangledown u|^r dx\\&+c_2(\varepsilon,p)c_3\beta(p-r)\int_{A_{k,t}}|\bigtriangledown(v-u)|^r dx\\\le{}&\beta\varepsilon(c_1+c_3p-c_3r)\int_{A_{k,t}}|\bigtriangledown u|^r dx\\&+\beta c_1c_2(\varepsilon,p)\int_{A_{k,t}}|\bigtriangledown\psi|^r dx\\&+\beta c_1c_2(\varepsilon,p)\int_{A_{k,t}}|r\phi^{r-1}\bigtriangledown\phi(u-\psi)|^r dx\\&+c_2(\varepsilon,p)c_3\beta(p-r)\int_{A_{k,t}}|\bigtriangledown(v-u)|^r dx, \qquad (19)\end{aligned}$$

where $c_1=\frac{2^{p-r}(p-r+1)}{r-p+1}$ and $c_2,c_3\ge 0$. By equality

$$\bigtriangledown v=\bigtriangledown(u-T_k(u))-\phi^r\bigtriangledown(u-\psi)-r\phi^{r-1}\bigtriangledown\phi(u-\psi)$$

$$\begin{aligned}\int_{A_{k,t}}|\bigtriangledown(v-u)|^r dx={}&\int_{A_{k,t}}|\phi^r\bigtriangledown(u-\psi)+r\phi^{r-1}\bigtriangledown\phi(u-\psi)|^r dx\\\le{}&\int_{A_{k,t}}|\phi^r\bigtriangledown(u-\psi)|^r dx+\int_{A_{k,t}}|r\phi^{r-1}\bigtriangledown\phi(u-\psi)|^r dx\\\le{}&\int_{A_{k,t}}|\bigtriangledown u|^r dx+\int_{A_{k,t}}|\bigtriangledown\psi|^r dx\\&+\int_{A_{k,t}}|r\phi^{r-1}\bigtriangledown\phi(u-\psi)|^r dx. \qquad (20)\end{aligned}$$

Finally we obtain that

$$\left|\int_{A_{k,\tau}} |\bigtriangledown u|^r dx\right| \leq \frac{\beta\varepsilon(c_1 + c_3 p - c_3 r) + \beta c_2(\varepsilon,p)c_3(p-r)}{\alpha}\int_{A_{k,t}} |\bigtriangledown u|^r dx$$
$$+\frac{\beta c_1 c_2(\varepsilon,p) + \beta c_2(\varepsilon,p)c_3(p-r)}{\alpha}\int_{A_{k,t}} |\bigtriangledown \psi|^r dx$$
$$+\frac{\beta c_1 c_2 + r\beta c_2(\varepsilon,p)c_3(p-r)}{\alpha(t-\tau)^r}\int_{A_{k,t}} |u|^r dx.$$

Now we want to eliminate the first term in the right-hand side containing $\bigtriangledown u$. Choosing ε and r_1 such that

$$\theta = \frac{\beta\varepsilon(c_1 + c_3 p - c_3 r) + c_2(\varepsilon,p)c_3\beta(p-r)}{\alpha} < 1$$

and let ρ, R be arbitrarily fixed with $R_0 \leq \rho < R \leq R_1$. Thus, from (21), we deduce that for every t and τ such that $\rho \leq \tau < t \leq R$, we have

$$\int_{A_{k,\tau}} |\bigtriangledown u|^r dx \leq \theta\int_{A_{k,t}} |\bigtriangledown u|^r dx + \frac{c_4}{\alpha}\int_{A_{k,R}} |\bigtriangledown \psi|^r dx + \frac{c_5}{\alpha(t-\tau)^r}\int_{A_{k,R}} |u|^r dx, \tag{22}$$

where $c_4 = c_1 c_2(\varepsilon,p)\beta + c_2(\varepsilon,p)c_3\beta(p-r)$ and $c_5 = \beta c_1 c_2 + r c_2 c_3 \beta(p-r)$.

Applying Lemma 2 in (22) we conclude that

$$\int_{A_{k,\rho}} |\bigtriangledown u|^r dx \leq \frac{cc_4}{\alpha}\int_{A_{k,R}} |\bigtriangledown \psi|^r dx + \frac{cc_5}{\alpha(R-\rho)^r}\int_{A_{k,R}} |u|^r dx,$$

where c is the constant given by Lemma 2. Thus u satisfies inequality (7) with $\varphi_0 = |\bigtriangledown \psi|^r$ and $\alpha = r$. Now the theorem follows from Lemma 1.

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Li Juan
Department of Applied Mathematics
Shanghai Jiaotong University
Shanghai 200030
P.R.China
College of Mathematics and Computer Science
Hebei University
Baoding, Hebei 071002
P.R.China
e–mail: juanjuan-li@163.com

Gao Hongya
College of Mathematics and Computer Science
Hebei University
Baoding, Hebei 071002
P.R.China

Rezultat lokalne regularnosti za za vrlo slaba rješenja "obstacle" problema

Li Juan i Gao Hongya

Sadržaj

U radu je dan jedan rezultat lokalne regularnosti za vrlo slaba rješenja "obstacle" problema A–harmonične jednadžbe $divA(x, \bigtriangledown u(x)) = 0$, $|A(x,\xi)| \approx |\xi|^{p-1}$, gdje je $1 < p < n$ i gdje "obstacle" funkcija zadovoljava $\psi \geq 0$.