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Some remarks on semirings

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Abstract. In this paper we prove the following results. (1) If R is a right M–semiring then every prime right ideal of R is finitely generated if and only if R is right noetherian. (2) If R is a right P–semiring then every prime right ideal of R is principal if and only if every right ideal of R is principal. (3) $(Z^+, +, .)$ is a noetherian semiring; thus we give a short and elementary proof of the main theorem of Allen and Dale [3]. (4) Let R be a strongly euclidean semiring. If I is a principal ideal of R then I is a partitioning ideal of R.

All semirings in this paper have an identity element. We will use the terminology of [1] and [5]. Z^+ will denote the set of all non–negative integers. A right ideal I of a semiring R is called subtractive (= k–right ideal) if $a, a+b \in I$, $b \in R$ then $b \in I$. A semiring R is called right IS (= right k–semiring) if all its right ideals are subtractive. An ideal I of a semiring R is called a partitioning ideal (= Q–ideal) if there exists a subset Q of R such that

1. $R = U\{q + I : q \in Q\}$.
2. If $q_1, q_2 \in Q$ then $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ if and only if $q_1 = q_2$.

A right ideal I of a semiring R is called prime if $AB \subseteq I, AI \subseteq I$ then $A \subseteq I$ or $B \subseteq I$ where A and B are right ideals of R. A semiring R is called right noetherian (resp., right artinian) if there exists no infinite properly ascending (resp., descending) sequence of right ideals $I_1 \subset I_2 \subset I_3 \ldots$ (resp., $I_1 \supset I_2 \supset I_3 \supset \ldots$) of R.

The following statements for a semiring R are equivalent:

(1) R is right noetherian.
(2) Every right ideal of R is finitely generated.

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(3) Any non–empty collection of right ideals of R has a maximal element ([5, Theorem 1.1]).

A semiring R is called a right M–semiring if I is a right ideal of R which is not finitely generated then there exists a subtractive right ideal J of R such that $I \subseteq J$ and J is maximal with respect to not being finitely generated (see [2]).

Proportion 1. A right IS–semiring is a right M–semiring.

Proof. Let I be a right ideal of R which is not finitely generated. Let α be the class of all right ideals of R containing I and which are not finitely generated. Clearly α is non–empty. If α' is any totally ordered subset of α then $UJ_{J \in \alpha'}$ is a right ideal of R containing I that is not finitely generated. Hence $UJ_{J \in \alpha'} \in \alpha$. By Zorn's Lemma α has a maximal element which by hypothesis is a subtractive right ideal of R. Therefore R is a right M–semiring.

Now we generalize the Theorem 11 of [2] for semirings that are not necessarily be commutative.

Theorem 2. *If R is a right M–semiring then every prime right ideal of R is finitely generated if and only if R is right noetherian.*

Proof. Let $\alpha = \{I : I$ is a right ideal of R which is not finitely generated$\}$. Suppose $\alpha \neq \emptyset$. Let $I \in \alpha$. Since R is right M–semiring, there exists a subtractive right ideal J in R such that $I \subseteq J$ and J is maximal with respect to not being finitely generated. If J is a prime right ideal then by hypothesis J is finitely generated which is impossible. Hence J is not a prime right ideal. Thus there exist right ideals A, B in R such that $AB \subseteq J$ and $AJ \subseteq J$ but $A \not\subseteq J$ and $B \not\subseteq J$.

Let $a \in A$ such that $a \notin J$. Then clearly $J \subset J + aR$. If $J + aR$ is not finitely generated then there exists a subtractive right ideal K in R such that $J + aR \subseteq K$ and K is maximal with respect to not being finitely generated since R is a right M–semiring. But then $I \subseteq J \subset J + aR \subseteq K$. Now by maximality of J, $J = K = J + aR$, a contradiction. Hence $J + aR$ is finitely generated.

Let $J + aR = \sum_{i=1}^{n} a_i R$. Let $(a : J) = \{x \in R : ax \in J\}$. Then $J + B \subseteq (a : J)$, since $AJ \subseteq J$ and $AB \subseteq J$. Moreover $J \subset (a : J)$. It is easy to verify that $(a : J)$ is a subtractive right ideal of R. Hence $(a : J)$ is finitely generated.

Let $(a : J) = \sum_{i=1}^{m} b_iR$. Let $a_i = k_i + ar_i$ for some $k_i \in J$ and $r_i \in R$. Clearly $\sum_{i=1}^{n} k_iR + a(a : J) \subseteq J$. On the other hand, let $u \in J$. Then

$$u = \sum_{i=1}^{n} a_iu_i = \sum_{i=1}^{n}(k_i + ar_i)u_i = \sum_{i=1}^{n} k_iu_i + a\sum_{i=1}^{n} r_iu_i$$

for some $u_1, u_2, \ldots, u_n \in R$. Since u, $\sum_{i=1}^{n} k_iu_i \in J$ and J is a subtractive right ideal, hence $a\sum_{i=1}^{n} r_iu_i \in J$. This implies that $\sum_{i=1}^{n} r_iu_i \in (a : J)$. So $u \in \sum_{i=1}^{n} k_iR + a(a : J)$. Now

$$J = \sum_{i=1}^{n} k_iR + a(a : J) = \sum_{i=1}^{n} k_iR + a\left(\sum_{i=1}^{m} b_iR\right) = \sum_{i=1}^{n} k_iR + \sum_{i=1}^{m} ab_iR.$$

Hence J is finitely generated, a contradiction. Hence $\alpha = \emptyset$. Thus every right ideal of R is finitely generated.

A semiring R will be called a right P–semiring if for any right ideal I of R which is not principal there exists a subtractive right ideal J of R such that $I \subseteq J$ and J is maximal with respect to not being principal.

Theorem 3. *If R is a right P–semiring then every prime right ideal of R is principal if and only if every right ideal of R is principal.*

Proof. Let $\alpha = \{I : I \text{ is a right ideal of } R \text{ which is not principal}\}$. Suppose $\alpha \neq \emptyset$. Let $I \in \alpha$. Since R is right P–semiring, there exists a subtractive right ideal J of R such that $I \subseteq J$ and J is maximal with respect to not being principal. It is clear that J is not a prime right ideal. Hence there exist right ideals A and B of R such that $AB \subseteq J$ and $AJ \subseteq J$ but $A \nsubseteq J$ and $B \nsubseteq J$. Let $a \in A$ be such that $a \notin J$. Then clearly $J \subset J + aR$. Using the definition of right P–semiring, it can be shown that $J + aR$ is a principal right ideal. Hence $J + aR = bR$ for some $b \in R$.

Let $b = j_0 + ar_0$ for some $j_0 \in J$ and $r_0 \in R$. Let $x \in J \subseteq bR$. Therefore $x = br$ for some $r \in R$. Now $br = x \in J$ implies that $r \in (b : J)$. Therefore $x = br \in b(b : J)$. Hence $J \subseteq b(b : J)$. Also $b(b : J) \subseteq J$. Now $J = b(b : J)$. We claim that $J = (b : J)$. It is clear that $J \subseteq (b : J)$. If $J \neq (b : J)$ then $(b : J) = dR$ for some $d \in R$, since $(b : J)$ is a subtractive right ideal of R. Thus $J = b(b : J) = bdR$, which gives a contradiction. Hence $(b : J) = J$. Since $b = j_0 + ar_0$, it is easy to verify that $J \subseteq (ar_0 : J)$. Since $ar_0B \subseteq AB \subseteq J$, we have $B \subseteq (ar_0 : J)$. Thus $(ar_0 : J) \neq J$, for if $(ar_0 : J) = J$ then $B \subseteq J$, a contradiction.

Let $x \in (b : J)$. Then $j_0x + ar_0x = (j_0 + ar_0)x = bx \in J$. Hence $ar_0x \in J$, since $j_0x \in J$ and J is a subtractive right ideal. So $x \in (ar_0 : J)$. Thus $(b : J) \subseteq (ar_0 : J)$.

On the other hand, let $x \in (ar_0 : J)$. Then $bx = (j_0 + ar_0)x = j_0x + ar_0x \in J$. Hence $x \in (b : J)$. So $(ar_0 : J) \subseteq (b : J)$. Now $(ar_0 : J) = (b : J) = J$, a contradiction. Hence $\alpha = \emptyset$. Thus every right ideal of R is principal.

An Ideal I of a commutative semiring R is called semiregular if for each $a_1, a_2 \in I$, there exist $r_1, r_2 \in I$ such that $a_1 + r_1 + a_1r_1 + a_2r_2 = a_2 + r_2 + a_1r_2 + a_2r_1$. The sum of all semiregular ideals of R is called the Jacobson–Bourne radical of R (see [4]).

Now we give a short and elementary proof of the main theorem of [3].

Theorem 4. *If $R = (Z^+, +, \cdot)$ then R is a noetherian semiring and the Jacobson–Bourne radical of R is* 0.

Proof. Let I be a non–zero ideal of R. Choose a smallest positive integer n in I. For each fixed $r \in R$ where $0 \leq r < n$, let A_r be the set of all elements a of I where $a = nq_a + r$ for some non–negative integer q_a. Let $\alpha = \{A_r : A_r \text{ is non–empty}\}$. After reindexing we can write this class as $\alpha = \{A_{r1} = A_0, A_{r2}, A_{r3}, \ldots, A_{rm}\}$. For each $j, 1 \leq j \leq m$, choose the smallest positive integer q_{r_j} such that $nq_{r_j} + r_j \in A_{r_j}$. We claim that I is generated by the finite set $B = \{n, nq_{r_2} + r_2, \ldots, nq_{r_m} + r_m\}$. Let $0 \neq a \in I$. By the division algorithm we get $a = nq + r$ where $q, r \in R$, $0 \leq r < n$. If $r = 0$ then the result is clear. Suppose $r \neq 0$. Then $a \in A_r$. Hence $A_r = A_{r_j}$ for some r_j. Obviously $r = r_j$ and $q_{r_j} \leq q$. Hence $a = n(q - q_{jr}) + (nq_{jr} + r_j)$. So I is finitely generated.

Let I be a non–zero semiregular ideal of R. Without loss of generality we can assume that there exists $a_1, a_2 \in I$ such that $a_1 + 3 \leq a_2$. By definition of semiregular ideal there exist $r_1, r_2 \in I$ such that $a_1 + r_1 + a_1r_1 + a_2r_2 = a_2 + r_2 + a_1r_2 + a_2r_1$. Hence $(a_2 - a_1 - 1)r_2 = (a_2 - a_1) + (a_2 - a_1 - 1)r_1$. This yields a contradiction. Hence 0 is the only semiregular ideal of R. Hence the Jacobson–Bourne radical of R is 0.

The semiring R is not artinian since if $T_n = \{0, n, n+1, n+2, \ldots\}$ then $\{T_n\}_{n \geq 0}$ is an infinite properly descending sequence of ideals of R.

Now we prove the following theorem.

Theorem 5. *If $R = (Z^+ \cup \{\infty\}, \max, \min)$ then R is an artinian semiring and the Jacobson–Bourne radical of R is R.*

Proof. Clearly R is a commutative semiring with identity element ∞. We claim that I is an ideal of R if and only if $I = \{0, 1, 2, \ldots, t\} = I_t$ for some $t \in Z^+$ or $I = Z^+$ or $I = R$. Let I be an ideal of R. Suppose $I \neq Z^+$, $I \neq R$. Then there exists an element $x \neq \infty$ such that $x \in Z^+$ but $x \notin I$. Let $S = \{a \in Z^+ : a \notin I\}$. Clearly $S \neq \emptyset$. By the well ordering principle S has a minimal element say r. Clearly the non–negative integer $t = r - 1 \in I$. Now by definition of ideal in R, it is clear that $I = \{0, 1, 2, \ldots, t\}$.

Let $I^{(1)} \supseteq I^{(2)} \supseteq I^{(3)} \supseteq \dots$ be a descending sequence of ideals of R. From the above claim $I^{(1)} = I_t = \{0, 1, 2, \dots, t\}$ for some $t \in Z^+$ or $I^{(1)} = Z^+$ or $I^{(1)} = R$. In any case there is no infinite properly descending sequence of ideals of R.

Let I be an ideal of R and let $a_1, a_2 \in I$. Then we can choose $r_1 = r_2 = \max\{a_1, a_2\} \in I$ such that $a_1 + r_1 + a_1 r_1 + a_2 r_2 = a_2 + r_2 + a_1 r_2 + a_2 r_1$. Hence every ideal of R is semiregular. Thus the Jacobson–Bourne radical of R is R.

The semiring R is not noetherian since $I_0 \subset I_1 \subset I_2 \subset \dots$ is an infinite properly ascending sequence of ideals of R where $I_t = \{0, 1, 2, \dots, t\}$ for $t \in Z^+$.

A commutative semiring R will be called strongly euclidean if there exists a function $d : R - \{0\} \to Z^+$ such that
(1) $d(ab) \geq d(a)$ for all $a, b \in R - \{0\}$ and
(2) if $a, b \in R$ with $b \neq 0$
then there exist unique elements $q, r \in R$ such that $a = bq + r$ where either $r = 0$ or $d(r) < d(b)$.

Theorem 6. *Let R be a strongly euclidean semiring. If I is a principal ideal of R then I is a partitioning ideal of R.*

Proof. Let $I = \langle a \rangle$. If $a = 0$ then I is a partitioning ideal of R where $Q = R$. Suppose $a \neq 0$. Let $Q = \{q \in R : d(q) < d(a)\} \cup \{0\}$. Let $b \in R$. Then there exist unique elements $p, q \in R$ such that $b = ap + q$ where $q = 0$ or $d(q) < d(a)$. Clearly $b \in \cup\{q + I : q \in Q\}$. Hence $R = \cup\{q + I : q \in Q\}$.

For $q_1, q_2 \in Q$, suppose that $(q_1 + I) \cap (q_2 + I) \neq \emptyset$. Let $x \in (q_1 + I) \cap (q_2 + I)$. Then $q_1 + ax_1 = x = q_2 + ax_2$ for some $x_1, x_2 \in R$. But this representation is unique. Hence $q_1 = q_2$. Therefore I is a partitioning ideal of R.

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Neke napomene o poluprstenovima

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Sadržaj

U radu se dokazuju slijedeći rezultati: (1) Ako je R desni M–poluprsten tada je svaki prosti ideal od R konačno generiran ako i samo ako je R desni neterian. (2) Ako je R desni P–poluprsten tada je svaki prosti desni ideal od R glavni ako i samo ako je svaki desni ideal od R glavni. (3) $(Z^+, +, .)$ je neterian poluprsten; prema tome se daje kratak i elementarni dokaz glavnog teorema Allen and Dale [3]. (4) Neka je R jaki euklidski poluprsten. Ako je I glavni ideal od R tada je I djelomični ideal od R.