RADOVI MATEMATIČKI
Vol. 12 (2003), 5–11

Decomposition theorems for periodic near rings

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Abstract. The purpose of this paper is to obtain the structure of certain near rings satisfying the following conditions:
(i) $xy = x^m y^n p(x)$; (ii) $xy = y^m x^n p(x)$; (iii) $xy = x^m p(y)x^n$; (iv) $xy = y^m p(x)y^n$; where $m = m(x,y) \geq 1$; $n = n(x,y) \geq 1$; and $p(x)$ denotes an element of a near ring which is the finite sum of powers of x^k for $k \geq 2$ and additive inverses of such powers.

1. Introduction

The property $x^n = x$ has been among the favorites of many ring theorists over the last few decades since Jacobson [11] first studied the commutativity of rings satisfying this condition in order to generalize the classical Wedderburn theorem [17]. Later this property has been weakened by Searcoid and MacHale who proved that a ring satisfying the property $(xy)^{n(xy)} = xy$ must be commutative. Ligh and Luh [12] established that a ring R satisfying the above condition is a direct sum of a J-ring and a zero ring. Further, Bell and Ligh [8] studied the direct sum decomposition of rings satisfying the related properties like $xy = (xy)^2 p(x,y)$ or $xy = (yx)^2 p(x,y)$, for $p(X,Y) \in \mathbb{Z}(X,Y)$, the ring of polynomials in two non-commuting indeterminates and remarked that in case of near rings the analogous hypotheses do not quite yield a direct sum decomposition. The authors have defined a weaker notion of orthogonal sum. We say that a near ring R is an orthogonal sum of subnear rings A and B denoted by $R = A \uplus B$ if $AB = BA = \{0\}$ and each element of R has a unique representation in the form $a + b$ with $a \in A$ and $b \in B$. In this paper we continue the study and obtain some decomposition theorems for a near ring R satisfying the following conditions :

2000 Mathematics Subject Classification: 16W25, 16Y30.
Key words and phrases: Distributively generated (d-g)-near ring, D-near ring, zero-symmetric near ring and zero-commutative near ring.

(P_1) For every pair of elements x, y in R, there exist positive integers $m = m(x,y) \geq 1$; $n = n(x,y) \geq 1$, such that $xy = y^m x^n p(x)$,

(P_2) For every pair of elements x, y in R, there exist positive integers $m = m(x,y) \geq 1$; $n = n(x,y) \geq 1$, such that $xy = x^m y^m p(x)$,

(P_3) For every pair of elements x, y in R, there exist positive integers $m = m(x,y) \geq 1$; $n = n(x,y) \geq 1$, such that $xy = y^m p(x) y^n$,

(P_4) For every pair of elements x, y in R, there exist positive integers $m = m(x,y) \geq 1$; $n = n(x,y) \geq 1$, such that $xy = x^m p(y) x^n$.

2. Preliminaries

Throughout, R is a left near ring with multiplicative center $Z(R)$. We shall denote by N the set of nilpotent elements and by P the set of potent elements of R that is, the set $\{x \in R \mid x^{n(x)} = x$, for some positive integer $n(x) > 1\}$. The set of commutators is denoted by C. An element $x \in R$ is said to be distributive if $(y+z)x = yx+zx$ for all $y, z \in R$. If every element of R is distributive, then R is said to be a distributive near ring. A near ring R is said to be distributively generated (d-g) if it contains a multiplicative subsemigroup of distributive elements which generates the additive group $(R, +)$. A near ring R is called zero-commutative if $xy = 0$ implies $yx = 0$ for all $x, y \in R$ and if for all $x \in R$, $0x = 0$, R is called zero-symmetric (we may recall that left distributivity in R yields $x0 = 0$). A near ring R is called periodic if for every $x \in R$ there exist distinct positive integers $m = m(x)$; $n = n(x)$ such that $x^m = x^n$. A sufficient condition for a ring R to be periodic is Chacron's criterion: For each $x \in R$ there exists an integer $m = m(x) \geq 1$ and a polynomial $f(X) \in Z(X)$ such that $x^m = x^{m+1} f(x)$ ([9]).

An ideal of a near ring R is defined to be a normal subgroup I of $(R, +)$ such that

(i) $RI \subseteq I$.

(ii) $(x+i)y - xy \in I$ for all $x, y \in R$ and $i \in I$.

A near ring R is called a D-near ring if every nonzero homomorphic image T of R satisfies the following conditions:

(i) T has a nonzero right distributive element.

(ii) The additive group $(T, +)$ of T is Abelian implies that T is a ring.

It is evident by definition that all distributive near rings are examples of D–near rings. However example 2.5 # 6 of [10] illustrates that the class of D–near rings is larger than the class of $d - g$ near rings.

For each element x of a near ring R, the subnear ring generated by x will be denoted by $< x >$. Note that if x is a distributive element, then the element $t(x) \in < x >$ (or of $< 1, x >$ if R has unity 1) may be assumed to be a finite sum of powers of x.

3. Decomposition theorems

Theorem 3.1. *Let R be a D–near ring satisfying condition (P_1). Then $R = P \uplus N$, where P is a subring and N is a subnear ring with trivial multiplication.*

Theorem 3.2. *Let R be a D–near ring satisfying condition (P_2). If idempotent elements of R are central, then $R = P \uplus N$, where P is a subring and N is a subnear ring with trivial multiplication.*

Theorem 3.3. *Let R be a zero symmetric D–near ring satisfying condition (P_3). If idempotent elements of R are central, then $R = P \uplus N$, where P is a subring and N is a subnear ring with trivial multiplication.*

Theorem 3.4. *Let R be a zero commutative D–near ring satisfying condition (P_4). If idempotent elements of R are central, then $R = P \uplus N$, where P is a subring and N is a subnear ring with trivial multiplication.*

The results proved in [1], [8], [12], [13] and [14] become corollaries of the above Theorems.

The following Lemmas can essentially be found in [2], [5], [8] and [15], respectively.

Lemma 3.1. *Let R be a zero symmetric near ring satisfying the following properties:*
(i) *For each x in R, there exists a positive integer $n(x) > 1$, such that $x^{n(x)} = x$.*
(ii) *Every nontrivial homomorphic image of R contains a nonzero central idempotent.*
Then $(R, +)$ is commutative.

Lemma 3.2. *Let R be a near ring with unity 1. Then for every $x \in R$, $< x >= x < 1, x >$.*

Lemma 3.3. *If R is a zero commutative periodic near ring, then $R = P + N$.*

Lemma 3.4. *Let R be a near ring in which idempotents are multiplicatively central. If e and f are any idempotents, there exists an idempotent g such that $ge = e$ and $gf = f$.*

Lemma 3.5. *If R is a zero commutative near ring, then N is an ideal of R.*

Now we prove the following:

Lemma 3.6. *Let R be a D–near ring satisfying $x^n = x^n p(x)$ or $x^n = x^n p(x)x^n$, where $n = n(x)$ a positive integer and $p(x) \in < x >$. If $N \subseteq Z(R)$, then R/N is a periodic and commutative ring.*

Proof. Since $N \subseteq Z(R)$, N is an ideal. Consider the near ring R/N. Since R/N can be written as a subdirect product of near rings without zero divisors, so we may assume R/N has no nonzero divisors. Let d be a nonzero distributive element of R/N. Note that $d = dt(d)$ for some $t(d) \in < d >$ by property $x^n = x^n p(x)$. It follows that $e = t(d)$ is a nonzero idempotent. Since $e(er - r) = 0$ for every $r \in R/N$, hence e is the left identity in R/N. Considering arbitrary $x, y \in R/N$ and using the fact that e commutes with d and $d = de$, we have $0 = (x+y)de - (xde - yde) = [(x+y)e - (xe - ye)]d$. Since R/N has no nonzero divisors, e is the distributive element of R/N. Hence e is multiplicative identity in R/N. If x is an arbitrary nonzero element of R/N, then $x = x(t)$ where $t(x) \in < x >$. Using Lemma 3.2, we can write $t(x) = xt'(x)$ for some $t'(x) \in < 1, x >$. Now $x = x^2 t'(x)$, yields that $xt'(x) = 1$ i.e. R/N is a division near ring. Thus R/N is additively commutative and is a ring. Hence R/N is periodic by Chacron's criterion [9] and a commutative ring by [5, Theorem 2].

The proof of the Lemma runs on the same parallel lines if we replace the property $x^n = x^n p(x)$ by $x^n = x^n p(x)x^n$.

Lemma 3.7. *Let R be a near ring satisfying either of the conditions (P_1) or (P_2). Then $RN = NR = \{0\}$.*

Proof. Notice that R satisfying condition (P_1) is zero commutative. Indeed if $xy = 0$, then there exist positive integers $m' = m'(y, x) \geq 1$ and $n' = n'(y, x) \geq 1$, such that $yx = x^{m'} y^{n'} p(y) = 0$. Replacing y by x in (P_1), we find that

$$x^2 = x^{r+s} h(x);\ \text{for } r + s \geq 2. \tag{3.1}$$

If $u \in N$, then by repeated use of (3.1), we get $u^2 = 0$. Now for any $u \in N$ by condition (P_1), we have $ux = x^l u^t g(u) = 0$, for $l = l(u, x) \geq 1$, $t = t(u, x) \geq 1$. Zero commutativity in R yields that $xu = 0$, for $u \in N$ and $x \in R$. Hence $NR = RN = \{0\}$.

The proof of the Lemma follows similarly if R satisfies condition (P_2).

Proceeding on the same lines we can prove the following:

Lemma 3.8. *Let R be a zero symmetric near ring satisfying condition (P_3). Then $RN = NR = \{0\}$.*

Lemma 3.9. *Let R be a zero commutative near ring satisfying condition (P_4). Then $RN = NR = \{0\}$.*

Lemma 3.10. *Let R be a D-near ring satisfying condition (P_1). Then idempotent elements of R are central.*

Proof. Let e be an idempotent and $x \in R$. Then by condition (P_1) there exist positive integers $l' = l'(x,e) \geq 1$ and $t' = t'(x,e) \geq 1$ such that $xe = ex^{t'}p(x)$. Multiplying by e on the left we get $exe = xe$. Application of Lemma 3.6 yields that $C \subseteq N \subseteq Z(R)$ and we have $e(xe - ex) = 0$, for all $x \in R$. Hence $ex = xe$, for all $x \in R$.

Proof of Theorem 3.1. In view of Lemma 3.6 for each $x \in R$, there exist distinct positive integers $m = m(x)$ and $n = n(x)$ such that $x^m - x^n \in N$. Hence using Lemma 3.7, we have $x^{m+1} = x^{n+1}$, for each $x \in R$ and R is periodic.

Now we show that P is a subring. Let $a, b \in P$ and choose integers $p = p(a) > 1$ and $q = q(b) > 1$ such that $a^p = a$ and $b^q = b$. Let $r = (p-1)q - (p-2) = (q-1)p - (q-2)$. Then it is clear that $a^r = a$ and $b^r = b$. Note that $e = a^{r-1}$ and $f = b^{r-1}$ are idempotents with $ea = a$ and $fb = b$. By Lemma 3.6 and Lemma 3.7 $C \subseteq N \subseteq Z(R)$ and $a^2b = aba$, for all, $a, b \in P$. Obviously $(ab)^r = a^r b^r = (ab)^r$, hence $ab \in P$, for all $a, b \in P$. Moreover since R/N has $x^l = x$ property we have an integer $j > 1$ such that

$$(a-b)^j = a - b + u; \;\; u \in N. \tag{3.2}$$

Using Lemma 3.4 we can choose an idempotent g for which $ge = e$ and $gf = f$. Therefore, $ga = a$ and $gb = b$. Now multiplying (3.2) by g we have $(a-b)^j = a - b$ i.e. $a - b \in P$. Also by Lemma 3.1 $(P, +)$ is abelian. Hence P is a subring.

Trivially $P \cap N = \{0\}$. Let $a + u = b + v$, where $a, b \in P$ and $u, v \in N$. Then $a - b = v - u \in P \cap N = \{0\}$, which yields that $a = b$ and $v = u$. Hence $R = P + N$.

The proof of Theorem 3.2, Theorem 3.3 and Theorem 3.4 runs on the same parallel lines.

The example 2.5#29 of [10] suggests that one cannot get a direct sum decomposition under the hypotheses of the above theorems.

Example 3.1. Consider the non–abelian additive group $(R, +)$, isomorphic to symmetric group S_3 and define the multiplication in R as follows:

$\cdot$	0	a_1	a_2	a_3	a_4	a_5
0	0	0	0	0	0	0
a_1	0	a_1	a_1	a_1	0	0
a_2	0	a_1	a_1	a_1	0	0
a_3	0	a_1	a_1	a_1	0	0
a_4	0	0	0	0	0	0
a_5	0	0	0	0	0	0

Then $(R, +, \cdot)$ is a commutative near ring satisfying $(ba)^2 = a^2b^2 = ab$, for all $a, b \in R$. However, $M = \{0, a_1\}$ is not an ideal of R.

Acknowledgement. The authors are indebted to the referee for his valuable suggestions.

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(Received: November 30, 2002)
(Revised: May 29, 2003)

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Teoremi dekompozicije za periodične skoro prstenove

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Sadržaj

Svrha ovog rada je da se dobije struktura izvjesnih skoro prstenova koji zadovoljavaju slijedeće uvjete:
(i) $xy = x^m y^n p(x)$; (ii) $xy = y^m x^n p(x)$; (iii) $xy = x^m p(y) x^n$; (iv) $xy = y^m p(x) y^n$; gdje su $m = m(x, y) \geq 1$; $n = n(x, y) \geq 1$; a $p(x)$ označava elemenat skoro prstena koji je konačna suma eksponenata od x^k za $k \geq 2$ i aditivnih inverzera takvih eksponenata.