KANNAN TYPE MAPPING IN TVS-VALUED CONE METRIC SPACES AND THEIR APPLICATION TO URYSOHN INTEGRAL EQUATIONS

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Abstract. We obtain sufficient conditions for the existence of a common fixed point of three mappings satisfying Kannan type conditions in TVS valued cone metric spaces. We also give an application by finding the solution for a system of two Urysohn integral equations. Our results generalize several well-known recent results in the literature.

1. Introduction and preliminaries

A system $x = T_i x$ ($i \in \Omega$), of operator equations has one or more simultaneous solutions obtained by using the common fixed point technique. Recently Beg et al [5, 3, 8, 11, 12], studied common fixed points of a pair of maps on topological vector space (TVS) valued cone metric spaces. The class of TVS cone metric spaces is larger than class of cone metric spaces, used in [1, 2, 9, 10, 14, 15, 16, 17]. In this paper we obtain common fixed points and points of coincidence of three mappings in TVS-valued cone metric spaces without the assumption of normality. As an application we prove the existence of the unique solution of a system of two Urysohn integral equations. Our results improve and generalize several contemporary and recent results in the literature (e.g., see [1, 6, 9, 13, 15, 19]).

Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is called a contraction [4] if there exists $\lambda \in [0, 1)$ such that

$$d (Tx, Ty) \leq \lambda d (x, y),$$

for all $x, y \in X$. Mapping $T$ is called Kannan [13] if there exists $\alpha \in \left[0, \frac{1}{2}\right)$ such that

$$d (Tx, Ty) \leq \alpha [d (x, Tx) + d (y, Ty)],$$

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for all \( x, y \in X \). The main difference between contraction and Kannan mappings is that “contractions are always continuous where as Kannan mappings are not necessarily continuous. The Banach contraction theorem \([4]\) is an extremely dynamic tool in mathematical analysis. However, the Kannan fixed point theorem \([13]\) is imperative because it characterizes completeness of metric spaces \([18]\), while Banach theorem cannot characterize the metric completeness of \( X \) \([7]\). the Banach type contractive condition (i.e. 
\[
d(Sx, Ty) \leq k d(x, y)
\]
for a pair \( S, T : X \to X \) of mappings implies that both \( S \) and \( T \) are equal, whereas, the condition
\[
d(Sx, Ty) \leq k_1 [d(x, Sx) + d(y, Ty)]
\]
does not assert that \( S = T \). Thus Kannan type conditions are useful to find common fixed point of a pair of nonlinear operators. An other type of contractive condition, due to Chatterjea \([6]\), is based on an assumption analogous to Kannan condition (2): there exists \( \alpha \in [0, \frac{1}{2}] \) such that
\[
d(Tx, Ty) \leq \alpha [d(x, Ty) + d(y, Tx)]
\]
for all \( x, y \in X \). It is well-known that the Banach contractions, Kannan mappings and Chatterjea mappings are independent in general.

Let \((E, \tau)\) be a topological vector space (TVS) and \( P \) a subset of \( E \). Then, \( P \) is called a cone whenever

(i) \( P \) is closed, non-empty and \( P \neq \{\theta\} \),
(ii) \( ax + by \in P \) for all \( x, y \in P \) and non-negative real numbers \( a, b \),
(iii) \( P \cap (-P) = \{\theta\} \).

For a given cone \( P \subseteq E \), we can define a partial ordering \( \preceq \) with respect to \( P \) by \( x \preceq y \) if and only if \( y - x \in P \). \( x \prec y \) will stand for \( x \preceq y \) and \( x \neq y \), while \( x \ll y \) will stand for \( y - x \in intP \), where \( intP \) denotes the interior of \( P \). A cone \( P \) is called solid if \( intP \) is nonempty.

**Definition 1.** \([3, 5]\) Let \( X \) be a nonempty set. Suppose the mapping \( d : X \times X \to E \) satisfies

\((d_1)\) \( \theta \preceq d(x, y) \) for all \( x, y \in X \) and \( d(x, y) = \theta \) if and only if \( x = y \),
\((d_2)\) \( d(x, y) = d(y, x) \) for all \( x, y \in X \),
\((d_3)\) \( d(x, y) \preceq d(x, z) + d(z, y) \) for all \( x, y, z \in X \).

Then \( d \) is called a TVS-valued cone metric on \( X \) and \((X, d)\) is called a TVS-valued cone metric space.

If \( E \) is a real Banach space then \((X, d)\) is called cone metric space \([9]\).

**Definition 2.** \([5]\) Let \((X, d)\) be a TVS-valued cone metric space, \( x \in X \) and \( \{x_n\}_{n \geq 1} \) a sequence in \( X \). Then
(i) \( \{x_n\}_{n \geq 1} \) converges to \( x \) whenever for every \( c \in E \) with \( \theta \ll c \) there is a natural number \( N \) such that \( d(x_n, x) \ll c \) for all \( n \geq N \). We denote this by \( \lim_{n \to \infty} x_n = x \) or \( x_n \to x \).

(ii) \( \{x_n\}_{n \geq 1} \) is a Cauchy sequence whenever for every \( c \in E \) with \( \theta \ll c \) there is a natural number \( N \) such that \( d(x_n, x_m) \ll c \) for all \( n, m \geq N \).

(iii) \((X, d)\) is a complete TVS valued cone metric space if every Cauchy sequence is convergent.

A pair \((F, T)\) of self-mappings on \( X \) is said to be weakly compatible if \( FTx = TFx \) whenever \( Fx = Tx \). A point \( y \in X \) is called point of coincidence of a family \( T_j, j \in J \), of self-mappings on \( X \) if there exists a point \( x \in X \) such that \( y = T_jx \) for all \( j \in J \).

**Lemma 3.** [2] Let \( X \) be a nonempty set and the mappings \( S, T, F : X \to X \) have a unique point of coincidence \( v \in X \). If \((S, F)\) and \((T, F)\) are weakly compatible, then \( S, T \) and \( F \) have a unique common fixed point.

## 2. Common fixed point

**Theorem 4.** Let \((X, d)\) be a complete TVS-valued cone metric space, \( P \) be a solid cone, and mappings \( S, T, F : X \to X \) satisfy:

\[
d(Sx, Ty) \leq Ad(Fx, Sx) + Bd(Fy, Ty),
\]

for all \( x, y \in X \), where \( A, B \) are non-negative real numbers with \( A + B < 1 \). If

\[
S(X) \cup T(X) \subseteq F(X),
\]

and \( F(X) \) or \( S(X) \cup T(X) \) is a complete subspace of \( X \), then \( S, T \) and \( F \) have a unique point of coincidence. Moreover if \((S, F)\) and \((T, F)\) are weakly compatible, then \( S, T \) and \( F \) have a unique common fixed point.

**Proof.** We shall first show that, if \( S, T \) and \( F \) have a point of coincidence, then it is unique. For this, assume that there exist two distinct points of coincidence \( v, v^* \) of mappings \( S, T \) and \( F \) in \( X \). It follows that there exists \( u, u^* \in X \) such that

\[
v = Fu = Su = Tu,
\]

and

\[
v^* = Fu^* = Su^* = Tu^*.
\]

From (4), we obtain

\[
d(v, v^*) = d(Su, Tu^*) \\
\leq Ad(Fu, Su) + Bd(Fu, Tu) \\
\leq (A + B) d(v, v^*),
\]
it implies that
\[ v = v^*, \] a contradiction.

Now, we prove the existence of a point of coincidence of the mappings \( S, T \) and \( F \). Let \( x_0 \) be an arbitrary point in \( X \). Choose a point \( x_1 \) in \( X \) such that \( Fx_1 = Tx_0 \). This can be done since \( S(X) \cup T(X) \subseteq F(X) \). Similarly choose a point \( x_2 \) in \( X \) such that \( Fx_2 = Sx_1 \). Continuing this process having chosen \( x_n \) in \( X \), we obtain \( x_{n+1} \) in \( X \) such that

\[
Fx_{2n+1} = Tx_{2n} \\
Fx_{2n+2} = Sx_{2n+1}, \quad n \geq 0.
\]

Suppose there exists \( n \) such that \( Fx_{2n} = Fx_{2n+1} \). Then \( Fx_{2n} = Tx_{2n} \) and from (4)

\[
d(Fx_{2n}, Sx_{2n}) = d(Fx_{2n+1}, Sx_{2n}) \\
= d(Tx_{2n}, Sx_{2n}) \\
\leq Ad(Fx_{2n}, Sx_{2n}) + Bd(Fx_{2n}, Tx_{2n}) \\
\leq Ad(Fx_{2n}, Sx_{2n}) + Bd(Fx_{2n}, Fx_{2n+1}) \\
\leq Ad(Fx_{2n}, Sx_{2n}),
\]

which yields \( Fx_{2n} = Sx_{2n} \) and so, \( Fx_{2n} = Sx_{2n} = Tx_{2n} = y \) (say) is the required unique point of coincidence of \( F, S \) and \( T \). Similarly, if \( Fx_{2n+1} = Fx_{2n+2} \) for some \( n \). Then \( Fx_{2n+1} = Sx_{2n+1} = Tx_{2n+1} = y \) is the required point. Thus in this sequel of proof we can suppose that \( Fx_n \neq Fx_{n+1} \) for all \( n \). From (4)

\[
d(Fx_{2n}, Fx_{2n+1}) = d(Sx_{2n-1}, Tx_{2n}) \\
\leq Ad(Fx_{2n-1}, Sx_{2n-1}) + Bd(Fx_{2n-1}, Tx_{2n}) \\
\leq Ad(Fx_{2n-1}, Fx_{2n}) + Bd(Fx_{2n}, Fx_{2n+1}) \\
\leq \frac{A}{1 - B} d(Fx_{2n-1}, Fx_{2n}) \\
\leq \max \left\{ \frac{B}{1 - A}, \frac{A}{1 - B} \right\} d(Fx_{2n-1}, Fx_{2n}),
\]

and

\[
d(Fx_{2n-1}, Fx_{2n}) = d(Tx_{2n-2}, Sx_{2n-1}) \\
\leq A(Fx_{2n-1}, Sx_{2n-1}) + Bd(Fx_{2n-2}, Tx_{2n-2}) \\
\leq A(Fx_{2n-1}, Fx_{2n}) + Bd(Fx_{2n-2}, Fx_{2n-1}) \\
\leq \frac{B}{1 - A} d(Fx_{2n-2}, Fx_{2n-1})
\]
\[ \leq \max\left\{ \frac{B}{1-A}, \frac{A}{1-B} \right\} d(Fx_{2n-2}, Fx_{2n-1}) \cdot \]

It implies that

\[ d(Fx_{2n}, Fx_{2n+1}) \leq \lambda d(Fx_{2n-1}, Fx_{2n}), \]

where \( \lambda = \max\left\{ \frac{B}{1-A}, \frac{A}{1-B} \right\} \). As \( Fx_n \neq Fx_{n+1} \) and \( A + B < 1 \), therefore \( 0 < \lambda < 1 \), and for all \( n \),

\[ d(Fx_n, Fx_{n+1}) \leq \lambda d(Fx_{n-1}, Fx_n) \leq \cdots \leq \lambda^n d(Fx_0, Fx_1), \]

Now for any \( m > n \),

\[ d(Fx_m, Fx_n) \leq d(Fx_n, Fx_{n+1}) + d(Fx_{n+1}, Fx_{n+2}) + \cdots + d(Fx_{m-1}, Fx_m) \]

\[ \leq \left[ \lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1} \right] d(Fx_0, Fx_1) \]

\[ \leq \left[ \frac{\lambda^n}{1-\lambda} \right] d(Fx_0, Fx_1). \]

Let \( \theta \ll c \) be given, choose a symmetric neighborhood \( V \) of \( \theta \) such that \( c + V \subseteq \text{int}P \). Also, choose a natural number \( N_1 \) such that \( \frac{\lambda^n}{1-\lambda} d(Fx_1, Fx_0) \ll c \), for all \( n \geq N_1 \). Thus,

\[ d(Fx_m, Fx_n) \leq \left[ \frac{\lambda^n}{1-\lambda} \right] d(Fx_0, Fx_1) \ll c, \]

for all \( m > n \). Therefore, \( \{Fx_n\}_{n\geq1} \) is a Cauchy sequence. Since \( FX \) is complete, there exist \( u \in X, v \in FX \) such that \( Fx_n \to v = Fu \) (this holds also if \( S(X) \cup T(X) \) is complete with \( v \in S(X) \cup T(X) \)). Choose a natural number \( N_2 \) such that for all \( n \geq N_2 \)

\[ d(Fx_{n+1}, Fx_n) \ll \frac{c(1-B)}{2A} \text{ and } d(Fx_{n+1}, Fu) \ll \frac{c(1-B)}{2}. \]

Then for all \( n \geq N_2 \)

\[ d(Fu, Tu) \leq d(Fu, Fx_{2n+2}) + d(Fx_{2n+2}, Tu) \]

\[ \leq d(Fu, Fx_{2n+2}) + d(Sx_{2n+1}, Tu) \]

\[ \leq d(Fu, Fx_{2n+2}) + Ad(Fx_{2n+1}, Fx_{2n+2}) + Bd(Fu, Tu) \]

\[ \leq \frac{1}{1-B} d(Fu, Fx_{2n+2}) + \frac{A}{1-B} d(Fx_{2n+1}, Fx_{2n+2}) \]

\[ \ll \frac{c}{2} + \frac{c}{2} = c. \]

Thus

\[ d(Fu, Tu) \ll \frac{c}{m}, \text{ for all } m \geq 1. \]
So, \( c_m - d(Fu, Tu) \in P \), for all \( m \geq 1 \). Since \( \frac{c_m}{m} \rightarrow \theta \) (as \( m \rightarrow \infty \)) and \( P \) is closed, \( -d(Fu, Su) \in P \). But \( d(Fu, Tu) \in P \), therefore, \( d(Fu, Tu) = \theta \). Hence
\[
v = Fu = Tu,
\]
and
\[
d(Fu, Su) = d(Tu, Su) \leq Ad(Fu, Su) + Bd(Fu, Tu) = Ad(Fu, Su),
\]
implies that \( v \) is a unique point of coincidence of \( F, S \) and \( T \). If \((S, F)\) and \((T, F)\) are weakly compatible, then by Lemma 3, \( v \) is a unique common fixed point of \( S, T \) and \( F \). □

**Example 5.** Let \( X = 1, 2, 3 \) and \( E \) be the set of all real valued functions on \([0, 1]\) which also have continuous derivatives on \( X \). Then \( E \) is a vector space over \( \mathbb{R} \) under the following operations:

\[
(x + y)(t) = x(t) + y(t), \quad (\alpha x)(t) = \alpha x(t),
\]
for all \( x, y \in E, \alpha \in \mathbb{R} \). Let \( \tau \) be the strongest vector (locally convex) topology on \( E \). Then \((X, \tau)\) is a topological vector space which is not normable and is not even metrizable. Define \( d : X \times X \rightarrow E \) as follows:

\[
d(x, y)(t) = \begin{cases} 
0 & \text{if } x = y \\
\exp(\ln \frac{4}{7} + t) & \text{if } x \neq y \text{ and } x, y \in X - \{2\} \\
\exp(t) & \text{if } x \neq y \text{ and } x, y \in X - \{3\} \\
\exp(t - \ln 2) & \text{if } x \neq y \text{ and } x, y \in X - \{1\}.
\end{cases}
\]

Let \( P = \{ x \in E : x(t) \geq 0 \text{ for all } t \} \). Then \((X, d)\) is a TVS-valued cone metric space. Define a mappings \( F, T : X \rightarrow X \) as follows:

\[
T(x) = \begin{cases} 
3 & \text{if } x \neq 2 \\
1 & \text{if } x = 2.
\end{cases}, \quad F(x) = x
\]

Note that, for all \( t \in [0, 1] \) and for \( \alpha < \frac{1}{4} \)
\[
d(T(3), T(2))(t) = d(3, 1)(t) = e^{\ln \frac{4}{7} + t},
\]
\[
\alpha [d(F(3), T(3))(t) + d(F(2), T(2))(t)] \\
= \alpha [d(3, T(3))(t) + d(2, T(2))(t)] \\
= \alpha [d(2, 1)(t)] = \alpha e^t \\
< d(T(3), T(2))(t).
\]

Therefore the previous relevant results on fixed points [9, 13, 15, 19] and on common fixed points [1] are not applicable to obtain fixed point of \( T \) and
common fixed point of $F$ and $T$. In order to apply Theorem 4, consider the mapping $Sx = 3$ for each $x \in X$. Then,

$$d(Sx, Ty)(t) = \begin{cases} 0 & \text{if } y \neq 2 \\ e^{(\ln \frac{4}{7} + t)} & \text{if } y = 2 \end{cases}$$

and for $B = \frac{4}{7}$

$$Bd(Fy, Ty)(t) = \frac{4}{7}e^t \text{ if } y = 2.$$

It follows that $F, S$ and $T$ satisfy all conditions of Theorem 4 for $A = 0, B = \frac{4}{7}$ and we obtain $F(3) = T(3) = S(3) = 3$.

In the following we use a Chatterjea type condition to obtain point of coincidence and common fixed point of three mappings on a TVS-valued cone metric space.

**Theorem 6.** Let $(X, d)$ be a complete TVS-valued cone metric space, $P$ be a solid cone, and mappings $S, T, F : X \to X$ satisfy:

$$d(Sx, Ty) \leq Cd(Fy, Sx) + Dd(Fx, Ty), \quad (5)$$

for all $x, y \in X$, where $C, D$ are non-negative real numbers with $C + D < 1$. If $S(X) \cup T(X) \subseteq F(X)$, and $F(X)$ or $S(X) \cup T(X)$ is a complete subspace of $X$, then $S, T$ and $F$ have a unique point of coincidence. Moreover if $(S, F)$ and $(T, F)$ are weakly compatible, then $S, T$ and $F$ have a unique common fixed point.

**Proof.** It can be easily seen that if $S, T$ and $F$ have a point of coincidence, then it is unique. Let $x_0$ be an arbitrary point in $X$. Choose a point $x_1$ in $X$ such that $Fx_1 = Tx_0$. This can be done since $S(X) \cup T(X) \subseteq F(X)$. Similarly choose a point $x_2$ in $X$ such that $Fx_2 = Sx_1$. Continuing this process having chosen $x_n$ in $X$, we obtain $x_{n+1}$ in $X$ such that

$$Fx_{2n+1} = Tx_{2n} \quad Fx_{2n+2} = Sx_{2n+1}, \quad n \geq 0.$$ 

Suppose there exists $n$ such that $Fx_{2n} = Fx_{2n+1}$. Then using (5), we obtain $Fx_{2n} = Sx_{2n} = Tx_{2n} = y$ (say) is the required unique point of coincidence of $F, S$ and $T$. Similarly, if $Fx_{2n+1} = Fx_{2n+2}$ for some $n$. Then $Fx_{2n+1} = Sx_{2n+1} = Tx_{2n+1} = y$ is the required point. Thus in this sequel of proof we
can suppose that $F_{x_n} \neq F_{x_{n+1}}$. From (5), we obtain

$$d(F_{x_{2n}}, F_{x_{2n+1}}) = d(S_{x_{2n-1}}, T_{x_{2n}})$$

$$\leq Cd(F_{x_{2n}}, S_{x_{2n-1}}) + Dd(F_{x_{2n-1}}, T_{x_{2n}})$$

$$\leq D [d(F_{x_{2n-1}}, F_{x_{2n}}) + d(F_{x_{2n}}, F_{x_{2n+1}})]$$

$$\frac{D}{1-D}d(F_{x_{2n-1}}, F_{x_{2n}}),$$

and

$$d(F_{x_{2n-1}}, F_{x_{2n}}) = d(T_{x_{2n-2}}, S_{x_{2n-1}})$$

$$\leq Cd(F_{x_{2n-2}}, S_{x_{2n-1}}) + Dd(F_{x_{2n-1}}, T_{x_{2n-2}})$$

$$\leq Cd(F_{x_{2n-2}}, F_{x_{2n}})$$

$$\leq C [d(F_{x_{2n-2}}, F_{x_{2n-1}}) + d(F_{x_{2n-1}}, F_{x_{2n}})]$$

$$\leq \frac{C}{1-C}d(F_{x_{2n-2}}, F_{x_{2n-1}}).$$

It follows that

$$d(F_{x_{2n}}, F_{x_{2n+1}}) \leq \frac{D}{1-D}d(F_{x_{2n-1}}, F_{x_{2n}})$$

$$\leq \frac{D}{1-D} \frac{C}{1-C}d(F_{x_{2n-2}}, F_{x_{2n-1}})$$

$$\leq \left[ \frac{D}{1-D} \frac{C}{1-C} \right]^n d(F_{x_{o}}, F_{x_{1}}),$$

and

$$d(F_{x_{2n+1}}, F_{x_{2n+2}}) \leq \frac{C}{1-C}d(F_{x_{2n}}, F_{x_{2n+1}})$$

$$\leq \frac{C}{1-C} \left[ \frac{D}{1-D} \frac{C}{1-C} \right]^n d(F_{x_{o}}, F_{x_{1}}).$$

Let

$$\alpha = \frac{C}{1-C}, \quad \beta = \frac{C}{1-C},$$

then, as $F_{x_n} \neq F_{x_{n+1}}$ and $C + D < 1$,

$$0 < \alpha \beta = \frac{C}{1-C} \frac{D}{1-D} = \frac{D}{1-C} \frac{C}{1-D} < 1.$$
Now for $p < q$ we have,
\[
d(Fx_{2p+1}, Fx_{2q+1}) \leq d(Fx_{2p+1}, Fx_{2p+2}) + d(Fx_{2p+2}, Fx_{2p+3}) + \cdots + d(Fx_{2q}, Fx_{2q+1}) \\
\leq \alpha [\alpha \beta]^p d(Fx_0, Fx_1) + [\alpha \beta]^{p+1} d(Fx_0, Fx_1) + \cdots + [\alpha \beta]^q d(Fx_0, Fx_1) \\
\leq \left[ \alpha \sum_{i=p}^{q-1} (\alpha \beta)^i + \sum_{i=p+1}^{q} (\alpha \beta)^i \right] d(Fx_0, Fx_1) \\
\leq \left[ \frac{\alpha (\alpha \beta)^{p}[1-\alpha \beta]^{q-p}}{1-\alpha \beta} + \frac{(\alpha \beta)^{p+1}[1-\alpha \beta]^{q-p}}{1-\alpha \beta} \right] d(Fx_0, Fx_1) \\
\leq \left[ \frac{\alpha (\alpha \beta)^{p}}{1-\alpha \beta} + \frac{(\alpha \beta)^{p+1}}{1-\alpha \beta} \right] d(Fx_0, Fx_1) \\
\leq (1 + \beta) \left[ \frac{\alpha (\alpha \beta)^{p}}{1-\alpha \beta} \right] d(Fx_0, Fx_1),
\]
\[
d(Fx_{2p}, Fx_{2q+1}) \leq (1 + \alpha) \left[ \frac{(\alpha \beta)^{p}}{1-\alpha \beta} \right] d(Fx_0, Fx_1),
\]
\[
d(Fx_{2p}, Fx_{2q}) \leq (1 + \alpha) \left[ \frac{(\alpha \beta)^{p}}{1-\alpha \beta} \right] d(Fx_0, Fx_1),
\]
and
\[
d(Fx_{2p+1}, Fx_{2q}) \leq (1 + \beta) \left[ \frac{\alpha (\alpha \beta)^{p}}{1-\alpha \beta} \right] d(Fx_0, Fx_1).
\]
Hence, for $0 < n < m$
\[
d(Fx_n, Fx_m) \leq \left[ \frac{2(\alpha \beta)^{p}}{1-\alpha \beta} \right] d(Fx_0, Fx_1)
\]
where $p$ is the integer part of $n \frac{1}{2}$. Let $\theta \ll c$ be given, choose a symmetric neighborhood $V$ of $\theta$ such that $c + V \subseteq \text{int} P$. Since
\[
\lim_{p \to \infty} \left[ \frac{2(\alpha \beta)^{p}}{1-\alpha \beta} \right] d(Fx_0, Fx_1) = \theta,
\]
there exists a natural number $N_1$ such that
\[
\left[ \frac{2(\alpha \beta)^{p}}{1-\alpha \beta} \right] d(Fx_0, Fx_1) \in V,
\]
for all $p \geq N_1$ and so
\[
\left[ \frac{2(\alpha \beta)^{p}}{1-\alpha \beta} \right] d(Fx_0, Fx_1) \ll c, \text{ for all } p \geq N_1.
Consequently, for all \( n, m \in \mathbb{N} \), with \( 2N_1 < n < m \), we have

\[
d(F_{x_{n}}, F_{x_{m}}) \ll c.
\]

Therefore, \( \{F_{x_{n}}\}_{n \geq 1} \) is a Cauchy sequence. Since \( FX \) is complete, there exist \( u \in X, v \in FX \) such that \( F_{x_{n}} \to v = Fu \) (this hold also if \( S(X) \cup T(X) \) is complete with \( v \in S(X) \cup T(X) \)). Choose a natural number \( N_2 \) such that for all \( n \geq N_2 \)

\[
d(F_{x_{n+1}}, Fu) \ll \frac{c}{2M},
\]

where \( M = \max \left\{ \frac{1 + C}{1 - D}, \frac{D}{1 - D} \right\} \). Then for all \( n \geq N_2 \)

\[
d(Fu, Tu) \leq d(Fu, F_{x_{2n+2}}) + d(F_{x_{2n+2}}, Tu)
\leq d(Fu, F_{x_{2n+2}}) + d(S_{x_{2n+1}}, Tu)
\leq d(Fu, F_{x_{2n+2}}) + Cd(Fu, S_{x_{2n+1}}) + Dd(F_{x_{2n+1}}, Tu)
\leq d(Fu, F_{x_{2n+2}}) + Cd(Fu, F_{x_{2n+2}}) + Dd(F_{x_{2n+1}}, Tu)
\leq d(Fu, F_{x_{2n+2}}) + Cd(Fu, F_{x_{2n+2}}) + Dd(F_{x_{2n+1}}, Tu)
\leq \left( \frac{1 + C}{1 - D} \right) d(Fu, F_{x_{2n+2}}) + \frac{D}{1 - D} d(F_{x_{2n+1}}, Fu)
\leq Md(Fu, F_{x_{2n+2}}) + Md(F_{x_{2n+1}}, Fu) \ll \frac{c}{2} + \frac{c}{2} = c.
\]

By a similar argument \( Fu = Tu = Su \), which implies that \( v \) is a unique point of coincidence of \( F, S \) and \( T \). If \( (S, F) \) and \( (T, F) \) are weakly compatible, then by Lemma 3, \( v \) is a unique common fixed point of \( S, T \) and \( F \).

\( \square \)

The following example shows that the above theorem is an improvement and a real generalization of results [1, 6, 9, 13, 15, 19].

**Example 7.** Let \( (X, d) \) be the TVS-valued cone metric space of Example 5. Define a mappings \( F, T : X \to X \) as follows:

\[
T(x) = \begin{cases} 
1 & \text{if } x \neq 2 \\
3 & \text{if } x = 2 
\end{cases}, \quad F(x) = x.
\]

Note that, for all \( t \in [0, 1] \) and for \( \alpha < \frac{1}{2} \)

\[
d(T(3), T(2) (t)) = d(1, 3) (t) = e^{(\ln \frac{3}{2} + t)}.
\]

\[
\alpha [d(F(3), T(2)) (t) + d(F(2), T(3)) (t)]
= \alpha [d(3, T(2)) (t) + d(2, T(3)) (t)] = \alpha e^t
< d(T(3), T(2)) (t),
\]
Therefore the previous relevant results on fixed points \([6, 9, 15, 19]\) and on common fixed points \([1]\) are not applicable to obtain fixed point of \(T\) and common fixed point of \(F\) and \(T\). In order to apply the Theorem 6, consider the mapping \(Sx = 1\) for each \(x \in X\). Then,

\[
    d(Sx, Ty)(t) = \begin{cases} 
    0 & \text{if } y \neq 2 \\
    e^{(\ln \frac{4}{7} + t)} & \text{if } y = 2 
    \end{cases}
\]

and for \(D = \frac{4}{7}\)

\[
    Dd(Fx, Ty)(t) = \frac{4}{7}e^t \text{ if } y = 2.
\]

It follows that \(S\) and \(T\) satisfy all conditions of Theorem 6 and we obtain \(F(1) = T(1) = S(1) = 1\).

3. Application

In this section we prove an existence theorem for the common solution for two Urysohn integral equations. Let \(X = C([a, b], \mathbb{R}^n), E\) is a topological vector space of Example 5

\[
    P = \{ x \in E : x(t) \geq 0 \text{ for all } t \in [0, 1] \},
\]

and \(d : X \times X \to E\) is defined as follows:

\[
    d(x, y)(t) = (\|x - y\|_\infty) e^t.
\]

It is easily seen that \((X, d)\) is a complete TVS-valued cone metric space.

**Theorem 8.** Consider the Urysohn integral equations

\[
    x(t) = \int_a^b K_1(t, s, x(s)) ds + g(t), \quad (6)
\]

\[
    x(t) = \int_a^b K_2(t, s, x(s)) ds + h(t), \quad (7)
\]

where \(t \in [a, b] \subset \mathbb{R}, x, g, h \in X\).

Suppose that \(K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n\) are such that \(F_x, G_x \in X\) for each \(x \in X\), where

\[
    F_x(t) = \int_a^b K_1(t, s, x(s)) ds, \quad G_x(t) = \int_a^b K_2(t, s, x(s)) ds \text{ for all } t \in [a, b].
\]

If there exist \(0 < h < 1\) such that for every \(x, y \in X\)

\[
    (\|F_x(t) - G_y(t) + g(t) - h(t)\|_\infty) e^t \leq hM(x, y) e^t,
\]

for all \(x, y \in X\), where

\[
    M(x, y) = \max \left\{ \|F_x(t) + g(t) - x(t)\|_\infty, \|G_y(t) + h(t) - y(t)\|_\infty, \|F_x(t) + g(t) - x(t)\|_\infty, \|G_y(t) + h(t) - y(t)\|_\infty \right\}.
\]
Then the system of integral equations (6) and (7) have a unique common solution.

**Proof.** Define $F, S, T : X \rightarrow X$ by

\[ F(x) = x, \quad S(x) = F_x + g, \quad T(x) = G_x + h. \]

If

\[ M(x, y) = \|F_x(t) + g(t) - x(t)\|_\infty, \]

it is easily seen that

\[ (\|S - T\|_\infty)e^t \leq h(\|S(x) - x\|_\infty)e^t \]

for every $x, y \in X$. By Theorem 4 if $A = h, B = 0$, the Urysohn integral equations (6) and (7) have a unique common solution. If

\[ M(x, y) = \|G_y(t) + h(t) - y(t)\|_\infty, \]

then

\[ (\|S - T\|_\infty)e^t \leq h(\|T(y) - y\|_\infty)e^t \]

for every $x, y \in X$. Again by Theorem 4 if $A = 0, B = h$, the Urysohn integral equations (6) and (7) have a unique common solution. Similarly in other cases the result follows by Theorem 6. □

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**References**


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