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REMARKS ON TWO RECENT RESULTS ABOUT POLYNOMIALS WITH PRESCRIBED ZEROS

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Abstract. We make some observations about the results contained in a paper published in this journal in the year 2011.

1. Introduction

For a polynomial $p(z)$, let $M(p, \rho) := \max_{|z|=\rho} |p(z)|$.

The following result was proved by Aziz [1].

Theorem A. *If $p(z)$ is a polynomial of degree n, which does not vanish in $|z| < k$, $k \geq 1$, then*

$$M(p, R) \leq \frac{R^n + k^n}{1 + k^n} M(p, 1) \ \text{ for } \ R > k^2, \tag{1}$$

provided $|p'(k^2 z)|$ and $|p'(z)|$ attain the maximum at the same point on $|z| = 1$. The result is best possible with equality for $p(z) = z^n + k^n$.

The next result appears in [2].

Theorem B. *If $p(z)$ is a polynomial of degree n, which does not vanish in $|z| < k$, $k < 1$, then*

$$M(p, r) \geq \frac{r^n + k^n}{1 + k^n} M(p, 1) \ \text{ for } \ 0 < k < r < 1, \tag{2}$$

provided $|p'(z)|$ and $|q'(z)|$ attain the maximum at the same point on $|z| = 1$, where $q(z) = z^n \overline{p(1/\overline{z})}$. The result is best possible and equality holds for $p(z) = z^n + k^n$.

In a paper published in this journal in the year 2011, which is quoted as item number [3] in the list of references, Dewan and Hans make the following statements (see Theorems 1 and 2 of [3]).

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Theorem 1. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu < n$ *is a polynomial of degree* n*, which does not vanish in* $|z| < k$, $k \ge 1$*, then*

$$M(p\,,\,R) \le \frac{R^n + k^n(1 + k^{n-\mu+1}) - k^{2n}}{1 + k^{n-\mu+1}}\, M(p\,,\,1) \quad \textit{for} \quad R > k^2\,, \tag{3}$$

provided $|p'(k^2 z)|$ *and* $|p'(z)|$ *attain the maximum at the same point on* $|z| = 1$.

Theorem 2. *If* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\mu} z^{n-\mu}$, $1 \le \mu < n$ *is a polynomial of degree* n*, which does not vanish in* $|z| < k$, $k < 1$*, then*

$$M(p\,,\,r) \ge \left(\frac{r^{n-\mu+1} + k^{n-\mu+1}}{\lambda^{n-\mu+1} + k^{n-\mu+1}}\right)^{\frac{n}{n-\mu+1}} M(p\,,\,\lambda) \quad \textit{for} \quad 0 < k < r < \lambda \le 1\,, \tag{4}$$

provided $|p'(z)|$ *and* $|q'(z)|$ *attain maximum at the same point on* $|z| = 1$*, where* $q(z) = z^n\overline{p(1/\bar{z})}$*. The result is best possible and equality holds for* $p(z) = \left(z^{n-\mu+1} + k^{n-\mu+1}\right)^{\frac{n}{n-\mu+1}}$.

2. Some remarks on Theorems 1 and 2

Remark 1. Theorem 1 is much ado about nothing. In the case where $\mu = 1$, Theorem 1 is the same as Theorem A. The authors acknowledge this in [3] (see Remark 1 on page 12). For $1 < \mu < n$, their result is weaker than Theorem A which it is supposed to refine. The upper bound for $M(p\,,\,R)$ given in (1) is smaller than the one given in inequality (3). This can be seen as follows.

Pay attention to the fact that $k > 1$ and $R > k^2$. Clearly,

$$\frac{R^n + k^n}{1 + k^n} < \frac{R^n + k^n(1 + k^{n-\mu+1}) - k^{2n}}{1 + k^{n-\mu+1}}$$

if and only if

$$(R^n + k^n)(1 + k^{n-\mu+1}) < \left(R^n + k^n(1 + k^{n-\mu+1}) - k^{2n}\right)(1 + k^n)$$

which holds if and only if

$$k^n\,(1 + k^{n-\mu+1}) - k^n\,(1 + k^{n-\mu+1})(1 + k^n) + k^{2n}\,(1 + k^n) < R^n\,(k^n - k^{n-\mu+1})\,,$$

i.e., if and only if

$$-k^{2n}\,(1 + k^{n-\mu+1}) + k^{2n}\,(1 + k^n) < R^n\,(k^n - k^{n-\mu+1})\,,$$

which in turn holds if and only if

$$k^{2n}\,(k^n - k^{n-\mu+1}) < R^n\,(k^n - k^{n-\mu+1})\,,$$

i.e., if and only if $R > k^2$, being given that $k > 1$.

In [3], the authors consider what can be seen as a subclass of polynomials satisfying the conditions of Theorem A and then end up with a conclusion that is weaker than the one given in Theorem A.

Remark 2. Going through the proof of Theorem 2 we notice that it uses the following statement which the authors call Lemma 1 (see [3, page 13]).

Lemma 1. *Let* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu < n$ *be a polynomial of degree* n, *having no zero in* $|z| < k$, $k \le 1$ *and* $q(z) = z^n \overline{p(1/\bar{z})}$. *If* $|p'(z)|$ *and* $|q'(z)|$ *attain maximums at the same point on* $|z| = 1$, *then*

$$M(p', 1) \le \frac{n}{1 + k^{n-\mu+1}} M(p, 1). \tag{5}$$

This "so-called lemma" is invalid for $1 < \mu < n$. In order to see this, consider the polynomial

$$p(z) = p(\varepsilon, z) = z^n + \varepsilon z^{n-\mu} + \ell^n,$$

where $\ell \in (0, 1)$ and $\varepsilon > 0$. The zeros of a polynomial are continuous functions of the coefficients (see [4, p. 10, Theorem 1.3.1]). Hence, for small values of ε, the polynomial $p(\varepsilon, z)$ has no zeros in $|z| < k = k(\varepsilon)$ for some $k \in (0, 1)$, where $k(\varepsilon) \to \ell$ as $\varepsilon \to 0$. Clearly,

$$M(p, 1) = \max_{|z|=1} |p(z)| = 1 + \varepsilon + \ell^n \quad \text{and} \quad M(p', 1) = \max_{|z|=1} |p'(z)| = n + (n-\mu)\varepsilon.$$

Thus, if (5) was true, then we would have

$$\frac{n + (n-\mu)\varepsilon}{1 + \varepsilon + \ell^n} \le \frac{n}{1 + k(\varepsilon)^{n-\mu+1}}, \ \varepsilon > 0. \tag{6}$$

Letting $\varepsilon \to 0$ in (6), we obtain

$$\frac{n}{1 + \ell^n} \le \frac{n}{1 + \ell^{n-\mu+1}},$$

i.e., $1 \le \ell^{\mu-1}$, which is a contradiction, since $\ell \in (0, 1)$ and $1 < \mu < n$ by hypothesis. Thus, Lemma 1 is invalid.

Since Lemma 1 is invalid, so is Theorem 2 because the proof of Theorem 2 depends on Lemma 1.

Remark 3. The last sentence in the statement of Theorem 2 is: "The result is best possible and equality holds for $p(z) = \left(z^{n-\mu+1} + k^{n-\mu+1}\right)^{\frac{n}{n-\mu+1}}$". The authors seem to believe that $p(z) = \left(z^{n-\mu+1} + k^{n-\mu+1}\right)^{\frac{n}{n-\mu+1}}$ is a polynomial. There is absolutely no mention of the fact that for $p(z) = \left(z^{n-\mu+1} + k^{n-\mu+1}\right)^{\frac{n}{n-\mu+1}}$ to be a polynomial, $\frac{n}{n-\mu+1}$ must be an integer.

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