REMARKS ON TWO RECENT RESULTS ABOUT POLYNOMIALS WITH PRESCRIBED ZEROS

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Abstract. We make some observations about the results contained in a paper published in this journal in the year 2011.

1. Introduction

For a polynomial \( p(z) \), let \( M(p, \rho) := \max_{|z|=\rho} |p(z)|. \)

The following result was proved by Aziz [1].

**Theorem A.** If \( p(z) \) is a polynomial of degree \( n \), which does not vanish in \( |z| < k, k \geq 1 \), then

\[
M(p, R) \leq \frac{R^n + k^n}{1 + k^n} M(p, 1) \quad \text{for} \quad R > k^2,
\]

provided \( |p'(k^2z)| \) and \( |p'(z)| \) attain the maximum at the same point on \( |z| = 1 \). The result is best possible with equality for \( p(z) = z^n + k^n \).

The next result appears in [2].

**Theorem B.** If \( p(z) \) is a polynomial of degree \( n \), which does not vanish in \( |z| < k, k < 1 \), then

\[
M(p, r) \geq \frac{r^n + k^n}{1 + k^n} M(p, 1) \quad \text{for} \quad 0 < k < r < 1,
\]

provided \( |p'(z)| \) and \( |q'(z)| \) attain the maximum at the same point on \( |z| = 1 \), where \( q(z) = z^n p(1/z) \). The result is best possible and equality holds for \( p(z) = z^n + k^n \).

In a paper published in this journal in the year 2011, which is quoted as item number [3] in the list of references, Dewan and Hans make the following statements (see Theorems 1 and 2 of [3]).

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Theorem 1. If \( p(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu} z^{\nu}, \) \( 1 \leq \mu < n \) is a polynomial of degree \( n \), which does not vanish in \( |z| < k, k \geq 1 \), then

\[
M(p, R) \leq \frac{R^n + k^n(1 + k^{n-\mu+1}) - k^{2n}}{1 + k^{n-\mu+1}} M(p, 1) \text{ for } R > k^2, \tag{3}
\]

provided \( |p'(k^2 z)| \) and \( |p'(z)| \) attain the maximum at the same point on \( |z| = 1 \).

Theorem 2. If \( p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\mu}, \) \( 1 \leq \mu < n \) is a polynomial of degree \( n \), which does not vanish in \( |z| < k, k < 1 \), then

\[
M(p, r) \geq \left( \frac{r^{n-\mu+1} + k^{n-\mu+1}}{\lambda^{n-\mu+1} + k^{n-\mu+1}} \right)^{n-\mu+1} M(p, \lambda) \text{ for } 0 < k < r < \lambda \leq 1, \tag{4}
\]

provided \( |p'(z)| \) and \( |q'(z)| \) attain maximum at the same point on \( |z| = 1 \), where \( q(z) = z^n p(1/z) \). The result is best possible and equality holds for \( p(z) = \left( z^{n-\mu+1} + k^{n-\mu+1} \right)^{n-\mu+1} \).

2. Some remarks on Theorems 1 and 2

Remark 1. Theorem 1 is much ado about nothing. In the case where \( \mu = 1 \), Theorem 1 is the same as Theorem A. The authors acknowledge this in [3] (see Remark 1 on page 12). For \( 1 < \mu < n \), their result is weaker than Theorem A which it is supposed to refine. The upper bound for \( M(p, R) \) given in (1) is smaller than the one given in inequality (3). This can be seen as follows.

Pay attention to the fact that \( k > 1 \) and \( R > k^2 \). Clearly,

\[
\frac{R^n + k^n}{1 + k^n} < \frac{R^n + k^n(1 + k^{n-\mu+1}) - k^{2n}}{1 + k^{n-\mu+1}}
\]

if and only if

\[
(R^n + k^n)(1 + k^{n-\mu+1}) < (R^n + k^n(1 + k^{n-\mu+1}) - k^{2n})(1 + k^n)
\]

which holds if and only if

\[
k^n (1 + k^{n-\mu+1}) - k^n (1 + k^{n-\mu+1})(1 + k^n) + k^{2n} (1 + k^n) < R^n (k^n - k^{n-\mu+1}),
\]

i.e., if and only if

\[
-k^{2n} (1 + k^{n-\mu+1}) + k^{2n} (1 + k^n) < R^n (k^n - k^{n-\mu+1}),
\]

which in turn holds if and only if

\[
k^{2n} (k^n - k^{n-\mu+1}) < R^n (k^n - k^{n-\mu+1}),
\]

i.e., if and only if \( R > k^2 \), being given that \( k > 1 \).
In [3], the authors consider what can be seen as a subclass of polynomials satisfying the conditions of Theorem A and then end up with a conclusion that is weaker than the one given in Theorem A.

Remark 2. Going through the proof of Theorem 2 we notice that it uses the following statement which the authors call Lemma 1 (see [3, page 13]).

Lemma 1. Let $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$ be a polynomial of degree $n$, having no zero in $|z| < k$, $k \leq 1$ and $q(z) = z^n p(1/z)$. If $|p'(z)|$ and $|q'(z)|$ attain maximums at the same point on $|z| = 1$, then

$$M(p', 1) \leq \frac{n}{1 + k^{n-\mu+1}} M(p, 1).$$ (5)

This “so-called lemma” is invalid for $1 < \mu < n$. In order to see this, consider the polynomial

$$p(z) = p(\varepsilon, z) = z^n + \varepsilon z^{n-\mu} + \ell^n,$$

where $\ell \in (0, 1)$ and $\varepsilon > 0$. The zeros of a polynomial are continuous functions of the coefficients (see [4, p. 10, Theorem 1.3.1]). Hence, for small values of $\varepsilon$, the polynomial $p(\varepsilon, z)$ has no zeros in $|z| < k = k(\varepsilon)$ for some $k \in (0, 1)$, where $k(\varepsilon) \rightarrow \ell$ as $\varepsilon \rightarrow 0$. Clearly,

$$M(p, 1) = \max_{|z|=1} |p(z)| = 1 + \varepsilon + \ell^n$$

and

$$M(p', 1) = \max_{|z|=1} |p'(z)| = n + (n-\mu)\varepsilon.$$

Thus, if (5) was true, then we would have

$$\frac{n + (n-\mu)\varepsilon}{1 + \varepsilon + \ell^n} \leq \frac{n}{1 + k(\varepsilon)^{n-\mu+1}}, \quad \varepsilon > 0.$$ (6)

Letting $\varepsilon \rightarrow 0$ in (6), we obtain

$$\frac{n}{1 + \ell^n} \leq \frac{n}{1 + \ell^{n-\mu+1}},$$

i.e., $1 \leq \ell^{\mu-1}$, which is a contradiction, since $\ell \in (0, 1)$ and $1 < \mu < n$ by hypothesis. Thus, Lemma 1 is invalid.

Since Lemma 1 is invalid, so is Theorem 2 because the proof of Theorem 2 depends on Lemma 1.

Remark 3. The last sentence in the statement of Theorem 2 is: “The result is best possible and equality holds for $p(z) = (z^{n-\mu+1} + k^{n-\mu+1}) \frac{n}{n-\mu+1}$.”

The authors seem to believe that $p(z) = (z^{n-\mu+1} + k^{n-\mu+1}) \frac{n}{n-\mu+1}$ is a polynomial. There is absolutely no mention of the fact that for $p(z) = (z^{n-\mu+1} + k^{n-\mu+1}) \frac{n}{n-\mu+1}$ to be a polynomial, $\frac{n}{n-\mu+1}$ must be an integer.
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