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INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. We consider inequalities of the form $\sum_{i=0}^{m} a_i\varphi(b_i) \geq 0$, and we give necessary and sufficient conditions on the nodes $b_0, b_1, \ldots, b_m$, and the weights a_i for such an inequality to be true for every real convex function φ. In the case the nodes are integers with b_0 the smallest of them, then $\sum_{i=0}^{m} a_i\varphi(b_i) \geq 0$ if and only if $x^{-b_0}\sum_{i=0}^{m} a_i x^{b_i}/(x-1)^2$ is a polynomial with positive coefficients.

1. Introduction

A real function φ is convex if and only if $\frac{\varphi(v)-\varphi(u)}{v-u} \leq \frac{\varphi(w)-\varphi(v)}{w-v}$ whenever $u < v < w$ are in its domain. The last inequality can be replaced by the equivalent form

$$(w-v)\varphi(u) + (u-w)\varphi(v) + (v-u)\varphi(w) \geq 0. \tag{1}$$

Convex functions are extremely useful in proving inequalities mainly because of Jensen's inequality, a finite form of which states that if φ is a real convex function, if the numbers $x_1, x_2, \ldots, x_n$ are in its domain, if the weights a_i are positive, then $\varphi\left(\frac{\sum_{i=1}^{n} a_i x_i}{\sum_{i=1}^{n} a_j}\right) \leq \frac{\sum_{i=1}^{n} a_i\varphi(x_i)}{\sum_{i=1}^{n} a_j}$. Another inequality for convex functions is the so called Karamata's inequality. Let $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$ be in the domain of a convex function φ. Suppose that $\sum_{i=k}^{n} y_i \leq \sum_{i=k}^{n} x_i$ for $k = 2, 3, \ldots, n$ and that $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i$. Then Karamata's inequality is that $\sum_{i=1}^{n}\varphi(x_i) \geq \sum_{i=1}^{n}\varphi(y_i)$. The reader can find further information on Karamata's and related inequalities in [1], which contains an extensive bibliography on the subject.

The conclusions of Jensen's and Karamata's inequalities are of the form $\sum_{i=0}^{m} a_i\varphi(b_i) \geq 0$ for every appropriate convex function. Because 1 and -1 are convex, $\sum_{i=0}^{m} a_i = 0$ and because x and $-x$ are convex, $\sum_{i=0}^{m} a_i b_i = 0$.

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In this paper we show that such an inequality is true for every real convex function φ provided it holds for the $m+1$ convex functions

$$g_k(x) = \begin{cases} 0 & x < b_k \\ x - b_k & x \geq b_k \end{cases}.$$

Then we use that result to prove Jensen's and Karamata's inequalities. Next we present a simple characterization when the nodes are integers (or if the spacing between nodes are integer multiples of some h). The result in this case is that $\sum_{i=0}^m a_i\varphi(b_i) \geq 0$ for every real convex function if and only if $x^{-b_0}\sum_{i=0}^m a_i x^{b_i} / \, (x-1)^2$ is a polynomial with positive coefficients. Two examples are given to illustrate the "effectiveness" of this characterization. We finish the paper with a brief discussion of inequalities for n convex functions.

2. Necessary and sufficient conditions

If f is a function, then we denote by $[f : u, v, w]$ the operator

$$[f : u, v, w] = (w-v)f(u) + (u-w)f(v) + (v-u)f(w).$$

From (1) it follows that φ is convex if and only if $[\varphi : u, v, w] \geq 0$ whenever $u < v < w$ are in its domain. One obvious property of this operator is that $[cx + d : u, v, w] = 0$. This property will be used in the proof of Theorem 2 below.

Proposition 1. *If $\sum_{i=0}^m a_i = 0$ and $\sum_{i=0}^m a_i b_i = 0$ where $b_0 < b_1 < \cdots < b_m$, then there are numbers $\alpha_0, \ldots, \alpha_{m-2}$ such that for every function f we have $\sum_{i=0}^m a_i f(b_i) = \sum_{j=0}^{m-2} \alpha_j [f : b_j, b_{j+1}, b_{j+2}]$.*

Proof. Let $\alpha_0 = a_0/(b_2 - b_1)$; then we can write $\sum_{i=0}^m a_i f(b_i) - \alpha_0[f : b_0, b_1, b_2]$ as $\sum_{i=1}^m a_i' f(b_i)$. Notice that $\sum_{i=1}^m a_i' = \sum_{i=0}^m a_i - \alpha_0((b_2 - b_1) + (b_0 - b_2) + (b_1 - b_0)) = 0$ and similarly $\sum_{i=1}^m a_i' b_i = \sum_{i=0}^m a_i b_i - \alpha_0((b_2 - b_1)b_0 + (b_0 - b_2)b_1 + (b_1 - b_0)b_2) = 0$. Next let $\alpha_1 = a_1'/(b_3 - b_2)$; then

$$\sum_{i=0}^m a_i f(b_i) - \alpha_0[f : b_0, b_1, b_2] - \alpha_1[f : b_1, b_2, b_3] = \sum_{i=1}^m a_i' f(b_i) - \alpha_1[f : b_1, b_2, b_3] = \sum_{i=2}^m a_i'' f(b_i)$$

with $\sum_{i=2}^m a_i'' = 0$ and $\sum_{i=2}^m a_i'' b_i = 0$. Continuing this way we obtain $\sum_{i=0}^m a_i f(b_i) - \sum_{j=0}^{m-2} \alpha_j [f : b_j, b_{j+1}, b_{j+2}] = cf(b_{m-1}) + df(b_m)$ with $c + d = 0$ and $cb_{m-1} + db_m = 0$. But $c + d = 0$ and $cb_{m-1} + db_m = 0$ if and only if $c = d = 0$. ☐

Now we are ready to prove our main result.

Theorem 2. *Suppose that* $\sum_{i=0}^m a_i = 0$ *and* $\sum_{i=0}^m a_i b_i = 0$, *then the inequality* $\sum_{i=0}^m a_i\varphi(b_i) \geq 0$ *holds for every real convex function with the nodes,* b_i, *in its domain if and only if it holds for*

$$g_k(x) = \begin{cases} 0 & x < b_k \\ x - b_k & x \geq b_k \end{cases}$$

for $k = 0, 1, \ldots, m$.

Proof. Since $g_k(x)$ are convex functions, we need only to prove the $\Longleftarrow$ part. We can assume that all the nodes are distinct and that $b_0 < b_1 < \cdots < b_m$. With this additional assumption, it is enough to assume that $\sum_{i=0}^m a_i g_k(b_i) \geq 0$ for $k = 1, \ldots, m-1$. Let $\alpha_0, \alpha_1, \ldots, \alpha_{m-2}$ be from Proposition 1. Then for each $k = 1, \ldots, m-1$.

$$\begin{aligned}\sum_{i=0}^m a_i g_k(b_i) &= \sum_{j=0}^{m-2} \alpha_j [g_k : b_j, b_{j+1}, b_{j+2}] \\ &= \alpha_{k-1}[g_k : b_{k-1}, b_k, b_{k+1}] = \alpha_{k-1}(b_k - b_{k-1})(b_{k+1} - b_k) \geq 0\end{aligned}$$

if and only if $\alpha_{k-1} \geq 0$. (The second equality follows from $[cx+d : u, v, w] = 0$.) Thus $\alpha_k \geq 0$ for $k = 0, \ldots, m-2$. Now let φ be any real convex function with the nodes in its domain. Then $\sum_{i=0}^m a_i\varphi(b_i) = \sum_{j=0}^{m-2} \alpha_j[\varphi : b_j, b_{j+1}, b_{j+2}] \geq 0$ since each $\alpha_j \geq 0$ and φ is convex. □

Since $\sum_{i=0}^m a_i g_k(b_i) = \sum_{b_i > b_k} a_i(b_i - b_k)$, an immediate consequence of Theorem 2 is the following result.

Corollary 3. *If* $\sum_{i=0}^m a_i = 0$ *and* $\sum_{i=0}^m a_i b_i = 0$, *then* $\sum_{i=0}^m a_i\varphi(b_i) \geq 0$ *for every real convex function,* φ, *with the nodes in its domain if and only if* $\sum_{b_i > b_k} a_i(b_i - b_k) \geq 0$ *for* $k = 0, 1, \ldots, m$. *The inequality* $\sum_{b_i > b_k} a_i(b_i - b_k) \geq 0$ *can be replaced with* $\sum_{b_i < b_k} a_i(b_i - b_k) \leq 0$.

The second part follows from the fact that $\sum a_i(b_i - b_k) = 0$.

3. Proofs of Jensen's and Karamata's inequalities

In this section we will give proofs of Jensen's and Karamata's inequalities based on our main result.

Corollary 4 (Jensen's inequality)**.** *If* φ *is a real convex function defined on* $[c, d]$, $x_1, x_2, \ldots, x_n$ *in its domain, the weights* a_i *positive, then*

$$\sum_{i=1}^n a_i\varphi(x_i) - \Big(\sum_{j=1}^n a_j\Big)\varphi\left(\frac{\sum_{i=1}^n a_i x_i}{\sum_{j=1}^n a_j}\right) \geq 0.$$

Proof. Let $\overline{x} = \dfrac{\sum_{i=1}^n a_i x_i}{\sum_{j=1}^n a_j}$. Since $c \le x_i \le d$ and $a_i > 0$, it follows that $c \le \overline{x} \le d$. If we write Jensen's inequality in the form $\sum c_i \varphi(d_i) \ge 0$, then it is clear that $\sum c_i = 0$ and $\sum c_i d_i = 0$ so that Corollary 3 applies. Let $z \in \{\overline{x}, x_1, \dots, x_n\}$. If $z \ge \overline{x}$, by Corollary 3 first part we need to check validity of $\sum_{x_i > z} a_i (x_i - z) \ge 0$ while if $z < \overline{x}$, by the second part of Corollary 3 we need to check validity of $\sum_{x_i < z} a_i (x_i - z) \le 0$. Both of these inequalities are true since by assumption each a_i is positive. □

Before we prove Karamata's inequality we will prove the following lemma on majorized sequences which is interesting in its own right.

Lemma 5. *Let $x_1 \le x_2 \le \cdots \le x_n$ and $y_1 \le y_2 \le \cdots \le y_n$. If $\sum_{i=r}^n y_i \le \sum_{i=r}^n x_i$ for $r = 1, 2, 3, \dots, n$, then for every real number z, $\sum_{x_i \ge z} (x_i - z) \ge \sum_{y_i \ge z} (y_i - z)$.*

Proof. The case $z > x_n$ is trivial. If $x_n \ge z \ge y_n$, then $\sum_{x_i \ge z} (x_i - z) - \sum_{y_i \ge z} (y_i - z) = \sum_{x_i \ge z} (x_i - z) \ge 0$. It remains to verify the case $z < y_n$. Let $0 \le k \le n-1$ denote the number of $x's$ that are less than or equal z, and let $0 \le r \le n-1$ denote the number of $y's$ that are less than or equal z. Since both sequences are increasing

$$\sum_{x_i \ge z} (x_i - z) - \sum_{y_i \ge z} (y_i - z) = \sum_{i=k+1}^{n} (x_i - z) - \sum_{i=r+1}^{n} (y_i - z).$$

If $k = r$, then the last equality reduces to $\sum_{i=r+1}^n x_i - \sum_{i=r+1}^n y_i$ which is positive by our assumption. If $k > r$, then $\sum_{i=k+1}^n (x_i - z) - \sum_{i=r+1}^n (y_i - z)$ can be written as $\sum_{i=r+1}^n (x_i - y_i) - \sum_{i=r+1}^k (x_i - z)$. By our assumption $\sum_{i=r+1}^n (x_i - y_i) \ge 0$, while $\sum_{i=r+1}^k (x_i - z) \le 0$ by the choice of k. Thus in this case $\sum_{x_i \ge z} (x_i - z) \ge \sum_{y_i \ge z} (y_i - z)$. If $k < r$, then $\sum_{i=k+1}^n (x_i - z) - \sum_{i=r+1}^n (y_i - z) = \sum_{i=r+1}^n (x_i - y_i) + \sum_{i=k+1}^r (x_i - z)$ which again is positive by our assumption and the choice of k. □

Corollary 6 (Karamata's inequality)**.** *Let $x_1 \le x_2 \le \cdots \le x_n$ and $y_1 \le y_2 \le \cdots \le y_n$ be in the domain of a real convex function φ. Suppose that $\sum_{i=k}^n y_i \le \sum_{i=k}^n x_i$ for $k = 2, 3, \dots, n$ and $\sum_{i=1}^n y_i = \sum_{i=1}^n x_i$. Then $\sum_{i=1}^n \varphi(x_i) \ge \sum_{i=1}^n \varphi(y_i)$.*

Proof. If we write Karamata's inequality $\sum_{i=1}^n \varphi(x_i) \ge \sum_{i=1}^n \varphi(y_i)$ as $0 \le \sum_{i=1}^n \varphi(x_i) - \sum_{i=1}^n \varphi(y_i) = \sum_{i=1}^{2n} c_i \varphi(d_i)$, then clearly $\sum c_i = 0$ while $\sum c_i d_i = \sum_{i=1}^n x_i - \sum_{i=1}^n y_i = 0$ by our assumption. Now by Corollary 3 we have to prove that if $z \in \{x_i, y_i\}_{i=1}^n$, then $\sum_{x_i > z} (x_i - z) - \sum_{y_i > z} (y_i - z) \ge 0$. But by Lemma 5 the last inequality is true for any z. □

The next result states that these type of inequalities remain true if the nodes are transformed by a linear function.

Corollary 7. *Let $c \neq 0$ and d be real numbers. The nodes $\{b_i\}_{i=0}^m$ and the corresponding weights $\{a_i\}_{i=0}^m$ satisfy $\sum_{i=0}^m a_i\varphi(b_i) \geq 0$ for every real convex function φ with the nodes in its domain, if and only if $\sum_{i=0}^m a_i\psi(cb_i+d) \geq 0$ for every real convex function ψ with the nodes $\{cb_i + d\}_{i=0}^m$ in its domain.*

Proof. First suppose that $\sum_{i=0}^m a_i\varphi(b_i) \geq 0$ for every real convex function. As was pointed out earlier this assumption implies that $\sum_{i=0}^m a_i = 0$ and $\sum_{i=0}^m a_ib_i = 0$. Consequently $\sum_{i=0}^m a_i(cb_i + d) = 0$. Assume that for each $i = 0, 1, \dots, m$, cb_i+d is in the domain of a convex function ψ. By Corollary 3 we have to verify the inequality $\sum_{cb_j+d>cb_k+d} a_j((cb_j + d) - (cb_k + d)) \geq 0$ for each $k = 0, 1, \dots, m$. We have

$$\sum_{cb_j+d>cb_k+d} a_jc(b_j - b_k) = \begin{cases} c\sum_{b_j>b_k} a_j(b_j - b_k) & \text{if } c \geq 0 \\ c\sum_{b_j<b_k} a_j(b_j - b_k) & \text{if } c < 0 \end{cases} \geq 0$$

again by Corollary 3 applied to φ. The converse follows from this case applied to the pair $\frac{1}{c}$ and $\frac{-d}{c}$. □

4. Nodes whose differences are integers

If $b_i - b_0$ is an integer for each $i = 1, 2, \dots, m$, then for any function f we can write $\sum_{i=0}^m a_if(b_i) = \sum_{j=0}^R a_jf(b_0 + j)$ where $R = b_m - b_0$, and $a_j = \begin{cases} a_i & \text{if } b_0 + j = b_i \\ 0 & \text{otherwise} \end{cases}$. If in addition, $b_0 < b_1 < \dots < b_m$, $\sum_{i=0}^m a_i = 0$ and $\sum_{i=0}^m a_ib_i = 0$, the statement of Proposition 1, for the function $f(t) = x^t$, takes on the simple form

$$\begin{aligned}\sum_{i=0}^m a_ix^{b_i} &= \sum_{i=0}^m a_if(b_i) = \sum_{j=0}^R a_jf(b_0 + j) \\ &= \sum_{j=0}^{R-2} \alpha_j[x^t : b_0 + j, b_0 + j + 1, b_0 + j + 2] = x^{b_0}(x - 1)^2(\sum_{j=0}^{R-2} \alpha_jx^j)\end{aligned}$$

where the last equality follows from $[x^t : u, u+1, u+2] = x^u(x^2-1)$. Recall that the numbers α_j from the statement of Proposition 1 depend only on the nodes and the weights. In particular if φ is any convex function defined on $[c, d]$ that contains all of the nodes, then for the same α_js we also have

$$\sum_{i=0}^m a_i\varphi(b_i) = \sum_{j=0}^{R-2} \alpha_j[\varphi : b_0 + j, b_0 + j + 1, b_0 + j + 2].$$

Now it is easy to modify the proof of Theorem 2 to obtain our next result.

Theorem 8. *Suppose that the nodes are integers, b_0 the smallest of them and $\{b_0, b_1, \dots, b_m\} \subseteq [p, q]$. Then $\sum_{i=0}^m a_i\varphi(b_i) \geq 0$ for every real convex function φ defined on $[p, q]$ if and only if $x^{-b_0} \sum_{i=0}^m a_i x^{b_i}/(x-1)^2$ is a polynomial with positive coefficients.*

Proof. First notice that if $h(x) = x^{-b_0} \sum_{i=0}^m a_i x^{b_i}$, then $\sum_{i=0}^m a_i = 0$ and $\sum_{i=0}^m a_i b_i = 0$ if and only if $h(1) = h'(1) = 0$ if and only if $h(x)/(x-1)^2$ is a polynomial. Thus under either condition $\sum_{i=0}^m a_i\varphi(b_i) \geq 0$ for every real convex function φ or $x^{-b_0} \sum_{i=0}^m a_i x^{b_i}/(x-1)^2$ is a polynomial, we have $\sum_{i=0}^m a_i = 0$ and $\sum_{i=0}^m a_i b_i = 0$. As in the proof of Theorem 2 we may assume that $b_0 < b_1 < \cdots < b_m$. By Proposition 1 there are numbers $\alpha_0, \alpha_1, \dots, \alpha_{R-2}$ such that

$$\sum_{i=0}^{m} a_i\varphi(b_i) = \sum_{j=0}^{R-2} \alpha_j[\varphi : b_0 + j, b_0 + j + 1, b_0 + j + 2] \tag{2}$$

and

$$\sum_{i=0}^{m} a_i x^{b_i} = x^{b_0}(x-1)^2(\sum_{j=0}^{R-2} \alpha_j x^j). \tag{3}$$

Suppose each α_j is positive and let φ be convex. Then for each $j = 0, 1, \dots, R-2$, $[\varphi : b_0 + j, b_0 + j + 1, b_0 + j + 2] \geq 0$ and hence each term of the first equation (2) is positive. Thus $\sum_{i=0}^m a_i\varphi(b_i) \geq 0$. This proves the $\Longleftarrow$ part. To prove the $\Longrightarrow$ part, for $1 \leq k \leq R-1$, we consider the convex functions

$$\varphi(x) = g_k(x) = \begin{cases} 0 & x < b_0 + k \\ x - (b_0 + k) & x \geq b_0 + k \end{cases}.$$

As in the proof of Theorem 2 the sum

$$\sum_{i=0}^{m} a_i\varphi(b_i) = \sum_{j=0}^{R-2} \alpha_j[\varphi : b_0 + j, b_0 + j + 1, b_0 + j + 2]$$

reduces to

$$\alpha_{k-1}[\varphi : b_0 + k - 1, b_0 + k, b_0 + k + 1] = \alpha_{k-1} \geq 0.$$

Thus $\alpha_0, \alpha_1, \dots, \alpha_{R-2}$ are all positive, and hence from (3) it follows that $x^{-b_0} \sum_{i=0}^m a_i x^{b_i}/(x-1)^2$ is a polynomial with positive coefficients. □

If there is an h such that $b_k - b_0$ is an integer multiple of h, (which is the case if all the nodes are rational numbers,) then Corollary 7 and Theorem 8 produce the following result.

Corollary 9. *Suppose there is an h such that for $b_0 < b_1 < \cdots < b_m$ we have $b_k - b_0$ is an integer multiple of h for $k = 1, \ldots, m$. Then $\sum_{i=0}^m a_i\varphi(b_i) \geq 0$ for every real convex function φ defined on $[b_0, b_m]$ if and only if $\sum_{i=0}^m a_i x^{\frac{b_i-b_0}{h}}/(x-1)^2$ is a polynomial with positive coefficients.*

Proof. First we apply Corollary 7 with $c = \frac{1}{h}$ and $d = -\frac{b_0}{h}$. Thus $\sum_{i=0}^m a_i\varphi(b_i) \geq 0$ for every real convex function φ defined on $[b_0, b_m]$ if and only if $\sum_{i=0}^m a_i\psi(\frac{b_i-b_0}{h}) \geq 0$ for every real convex function ψ defined on $[0, \frac{b_m-b_0}{h}]$. Since by the assumption $\frac{b_i-b_0}{h}$ are integers, from Theorem 8 we obtain that $\sum_{i=0}^m a_i\psi(\frac{b_i-b_0}{h}) \geq 0$ for every real convex function defined on $[0, \frac{b_m-b_0}{h}]$ if and only if $\sum_{i=0}^m a_i x^{\frac{b_i-b_0}{h}}/(x-1)^2$ is a polynomial with positive coefficients. □

We finish this section with another application of Theorem 8. For every real convex function φ on real line

$$43\varphi(5) - 82\varphi(4) + 63\varphi(3) - 51\varphi(2) + 26\varphi(1) + \varphi(0) \geq 0.$$

This can be verified by Karamata's inequality but it is much easier to apply Theorem 8. All we need to check is that

$$(43x^5 - 82x^4 + 63x^3 - 51x^2 + 26x + 1)/(x-1)^2$$

is a polynomial with all positive coefficients. Indeed

$$43x^5 - 82x^4 + 63x^3 - 51x^2 + 26x + 1 = (x-1)^2(1 + 28x + 4x^2 + 43x^3).$$

On the other hand it is not true that

$$43\varphi(5) - 87\varphi(4) + 73\varphi(3) - 56\varphi(2) + 26\varphi(1) + \varphi(0) \geq 0$$

for every convex function because this time

$$43x^5 - 87x^4 + 73x^3 - 56x^2 + 26x + 1 = (x-1)^2(1 + 28x - x^2 + 43x^3).$$

5. Inequalities for n convex functions

In this section we briefly discuss inequalities for n convex functions. Convexity can be described via divided differences. If u, v, and w are three distinct points, then $[u, v, w : f] = \frac{f(u)}{(u-v)(u-w)} + \frac{f(v)}{(v-u)(v-w)} + \frac{f(w)}{(w-u)(w-v)}$ is called the divided difference of f at points u, v, and w. A function is convex if and only if $[u, v, w : f] \geq 0$ for any three distinct points u, v, and w from its domain. If we consider a set V of $n+1$ distinct points, then we say that f is n convex if $[V : f] = \sum_{u \in V} \frac{f(u)}{\prod_{v \neq u}(u-v)} \geq 0$ for any such set V from the domain of f. Thus being convex is equivalent to being 2 convex. One can see that increasing and 1 convex are equivalent concepts and the same is true for nonnegative and 0 convex. An interested reader can find more information about n convex functions in [2]. Proposition 1 was instrumental in obtaining our results for convex functions. For n convex functions this proposition takes on the following form.

Proposition 10. *Suppose that* $\sum_{i=0}^{m} a_i b_i^k = 0$ *for* $k = 0, 1, \ldots, n-1$. *If* $b_0 < b_1 < \cdots < b_m$, *then there are numbers* $\alpha_0, \ldots, \alpha_{m-n}$ *such that for every function* f *we have* $\sum_{i=0}^{m} a_i f(b_i) = \sum_{j=0}^{m-n} \alpha_j [b_j, b_{j+1}, \ldots, b_{j+n} : f]$.

Now in the case of increasing functions (the case $n = 1$) the role of the functions $g_k(x)$ in the statement of Theorem 2 are played by increasing functions $g_k(x) = \begin{cases} 0 & x < b_k \\ 1 & x \geq b_k \end{cases}$ and Corollary 3 takes on the following form.

Theorem 11. *Suppose that* $\sum_{i=0}^{m} a_i = 0$. *If* $b_0 \leq b_1 \leq \cdots \leq b_m$; *then* $\sum_{i=0}^{m} a_i f(b_i) \geq 0$ *for every increasing function* f *if and only if* $\sum_{i=k}^{m} a_i \geq 0$ *for* $k = 1, 2, \ldots, m$.

Unfortunately the inequalities for n convex for $n \geq 3$ are not as nice as those for convex functions. For example in the case of integer nodes only the easy implication of Theorem 8 is true.

Theorem 12. *Let* m, n *be integers with* $m \geq n+1$. *Suppose that* $p = b_0 < b_1 < \cdots < b_m = q$ *are integers. If* $x^{-p} \sum_{i=0}^{m} a_i x^{b_i}/(x-1)^n$ *is a polynomial with positive coefficients, then* $\sum_{i=0}^{m} a_i \varphi(b_i) \geq 0$ *for every real* n *convex function* φ *defined on* $[p, q]$.

We omit the proofs since they are very similar to the proofs of the corresponding results for convex functions. But the converse fails for $n \geq 3$ as the following example shows. We will show that

$$-5f(0) + 16f(1) - 22f(2) + 20f(3) - 13f(4) + 4f(5) \geq 0 \tag{4}$$

for every 3 convex function defined on $[0, 5]$. Since

$$-5 + 16x - 22x^2 + 20x^3 - 13x^4 + 4x^5 = (x-1)^3(5 - x + 4x^2)$$

this will be a counterexample to the converse of Theorem 12. Let $g(x) = f(x/2)$; then g is 3 convex on $[0, 10]$ and

$$-5g(0) + 16g(2) - 22g(4) + 20g(6) - 13g(8) + 4g(10) \geq 0 \tag{5}$$

since by Theorem 12 the corresponding polynomial

$$-5 + 16x^2 - 22x^4 + 20x^6 - 13x^8 + 4x^{10} = (x^2-1)^3(5 - x^2 + 4x^4)$$

$$= (x-1)^3(5 + 15x + 14x^2 + 2x^3 + x^4 + 11x^5 + 12x^6 + 4x^7).$$

Now the inequality (4) follows from (5).

References

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