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GAPS IN THE PAIRS (BORDER RANK, SYMMETRIC RANK) FOR SYMMETRIC TENSORS

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Abstract. Fix integers $m \geq 2$, $s \geq 5$ and $d \geq 2s+2$. Here we describe the possible symmetric tensor ranks $\leq 2d+s-7$ of all symmetric tensors (or homogeneous degree d polynomials) in $m+1$ variables with border rank s.

1. Introduction

An important practical question concerning symmetric tensors (e.g. in Signal Processing, Statistics and Data Analysis) is their " minimal " decomposition as a sum of pure symmetric tensors (see e.g. [11], [16], [6], [10], [15], [4], [14] and references therein). This problem may be translated into the following problem for homogeneous polynomials in $m+1$ variables: for any degree d homogeneous polynomial $f \in \mathbb{K}[x_0, \dots, x_m]$ find the minimal integer r such that $f = \sum_{i=1}^{r} {L_i}^d$, where each L_i is a homogeneous degree 1 polynomial. The latter problem is translated in the following way into a problem concerning Veronese embeddings of $\mathbb{P}^m$.

Let $\nu_d : \mathbb{P}^m \to \mathbb{P}^{n_{m,d}}$, $n_{m,d} := \binom{m+d}{m} - 1$, denote the degree d Veronese embedding of $\mathbb{P}^m$. Set $X_{m,d} := \nu_d(\mathbb{P}^m)$. We often write n instead of $n_{m,d}$. For any subset or closed subscheme A of a projective space $\mathbb{P}^k$ let $\langle A \rangle$ denote its linear span. For any integer $s > 0$ the s-secant variety $\sigma_s(X_{m,d})$ is the closure in $\mathbb{P}^n$ of the union of all linear spaces spanned by s points of $X_{m,d}$. Fix $P \in \mathbb{P}^n$. The symmetric rank $sr(P)$ of P is the minimal cardinality of a finite set $S \subset X_{m,d}$ such that $P \in \langle S \rangle$. The border rank $br(P)$ of P is the minimal integer $s > 0$ such that $P \in \sigma_s(X_{m,d})$. There is another notion of rank of P (the cactus rank $cr(P)$ ([7], [5]), but we do not need to define it, because in the range $br(P) \leq d+1$ we always have $cr(P) = br(P)$ (Remark 1). For any fixed $s \geq 2$ one would like to have the stratification by the

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symmetric rank of $\sigma_s(X_{m,d}) \setminus \sigma_{s-1}(X_{m,d})$, i.e. to know what are the ranks of the homogeneous degree d polynomials with border rank s. This is due to Sylvester if $m = 1$, i.e. for binary forms ([9], [15], Theorem 4.1, [4]). For general m this is known if $s = 2, 3$ ([4]) and if $s = 4$ ([1]). For all positive integers a, b set

$$\sigma_{a,b}(X_{m,d}) := \{P \in \mathbb{P}^n : br(P) = a,\ sr(P) = b\}.$$

Notice that $\sigma_{a,b}(X_{m,d}) = \{P \in \sigma_a(X_{m,d}) \setminus \sigma_{a-1}(X_{m,d}) : sr(P) = b\}$ if $a \geq 2$, that $\sigma_{a,a}(X_{m,d})$ contains a non-empty open subset of $\sigma_a(X_{m,d})$ if $\sigma_{a-1}(X_{m,d}) \neq \mathbb{P}^n$ and that $\sigma_{a,b}(X_{m,d}) = \emptyset$ if $b < a$. If either $m = 1$ and s is very low ($s = 2, 3$ in [4], $s = 4$ in [1]), then for fixed s and large d near s several integers are not the symmetric rank of any $P \in \sigma_s(X_{m,d}) \setminus \sigma_{s-1}(X_{m,d})$. Here we show that this is the case for arbitrary s, d not too small, but for low ranks, i.e. if we assume $r \leq 2d + s - 7$. We prove the following result.

Theorem 1. *Fix integers m, s, d, r such that $m \geq 2$, $s \geq 5$, $d \geq 2s + 2$ and $s \leq r \leq 2d + s - 7$.*

Then $\sigma_{s,r}(X_{m,d}) \neq \emptyset$ if and only if one of the following conditions is satisfied:

- *$r = s$;*
- *$d + 2 - s \leq r \leq d + s - 2$ and $r + s \equiv d \pmod 2$;*
- *$2d + 2 - s \leq r \leq 2d + s - 7$.*

If $\sigma_s(X_{m,d}) \supsetneq \sigma_{s-1}(X_{m,d})$, then a general $P \in \sigma_s(X_{m,d})$ satisfies $br(P) = s = sr(P)$. Hence in the set-up of Theorem 1 only the pairs (s, r) with $2 \leq s < r$ need to be checked. The statement of Theorem 1 is of the form " if and only if ". However, the proofs of both implications use similar tools. The key technical tool used in almost all our lemmas is an inductive method to handle cohomology groups (vanishing and non-vanishing) often called the Horace Method. The starting observation is that for every s as in Theorem 1 there is a zero-dimensional scheme $A \subset \mathbb{P}^m$ such that $\deg(A) = s$ and $P \in \langle \nu_d(A) \rangle$ (Remark 1). Moreover if $sr(P) > br(P)$, then there is a finite set $B \subset \mathbb{P}^m$ such that $\sharp(B) = sr(P)$, $P \in \langle \nu_d(A) \rangle \cap \langle \nu_d(B) \rangle$ and the scheme $A \cup B$ has two strong properties: $h^1(\mathcal{I}_{A \cup B}(d)) > 0$ and there is a line or a conic (say D) such that $\deg((A \cup B) \cap D) \geq \deg(D) \cdot d + 2$ and $B \setminus B \cap D = A \setminus A \cap D$ (Lemmas 1 and 5). In this way it is easy to get the non-existence part of Theorem 1. The existence part in the range $s + r \leq d + 2$ (i.e. with $r = d + 2 - s$) is done taking A and B contained in a line L. To cover the case $r + s \equiv d \pmod 2$ and $d + 2 - s < r \leq d + s - 2$ we take $A = (A \cap L) \sqcup E$ and $B = (B \cap L) \sqcup E$ with $E \subset \mathbb{P}^m \setminus L$, $\sharp(E) = (s + r - d - 2)/2$, $B \cap A \cap L = \emptyset$ and $\deg(A \cap L) + \sharp(B \cap L) = d + 2$. To cover the case in which $r + s$ is even and $2d + 2 \leq r + s$ we use a smooth conic $C \subset \mathbb{P}^m$ and take $A = (A \cap C) \sqcup E$,

$B = (B \cap C) \sqcup E$ with $A \cap B \cap C = \emptyset$, $\deg(A \cap C) + \sharp(B \cap C) = 2d + 2$ and $\sharp(E) = (r + s - 2d - 2)/2$. To cover the case $2d + 3 - s \le r \le 2d + s - 7$ and $r + s$ odd we use a reducible conic instead of C; we need two different constructions according to the parity of the integer $(2d+3+s-r)/2$ (Lemmas 8 and 9).

In all cases the delicate part is the proof that there is no finite set $S \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(S) \rangle$ and $\sharp(S) < \sharp(B) = r$. In all cases we again use Lemmas 1 and 5.

We work over an algebraically closed base field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

2. The proof

For any sheaf $\mathcal{F}$ on $\mathbb{P}^m$ and any integer $i \ge 0$ set $h^i(\mathcal{F}) := \dim(H^i(\mathbb{P}^m, \mathcal{F}))$. For any scheme X, any effective Cartier divisor D of X and any closed subscheme $Z \subset X$ let $\mathrm{Res}_D(Z)$ denote the residual scheme of Z with respect to D, i.e. the closed subscheme of X with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. For any $R \in \mathrm{Pic}(X)$ we have the following exact sequence of coherent sheaves (called the *residual exact sequence*):

$$0 \to \mathcal{I}_{\mathrm{Res}_{D(Z)}} \otimes R(-D) \to \mathcal{I}_Z \otimes R \to \mathcal{I}_{D \cap Z, D} \otimes (R|D) \to 0 \qquad (1)$$

We need the following lemma (see [12] for the case in which the scheme Z is reduced, [4], Lemma 34, for the case $z \le 2d + 1$, and [13] for a strong tool to prove much more in $\mathbb{P}^2$).

Fix positive integers m, d, any $P \in \mathbb{P}^{n_{m,d}}$ and any finite set $B \subset \mathbb{P}^m$. We say that B evinces $sr(P)$ if $P \in \langle \nu_d(B) \rangle$ and $\sharp(B) = sr(P)$.

Lemma 1. *Fix integers m, d, z such that $m \ge 2$ and $0 < z < 3d$. Let $Z \subset \mathbb{P}^m$ be a zero-dimensional scheme such that $\deg(Z) = z$. If $m > 2$, then assume $\deg(Z) - \deg(Z_{red}) \le d$. We have $h^1(\mathcal{I}_Z(d)) > 0$ if and only if either there is a line $L \subset \mathbb{P}^m$ such that $\deg(L \cap Z) \ge d + 2$ or there is a conic $T \subset \mathbb{P}^m$ such that $\deg(T \cap Z) \ge 2d + 2$.*

Proof. Since Z is zero-dimensional, the restriction map $H^0(Z, \mathcal{O}_Z(d)) \to H^0(W, \mathcal{O}_W(d))$ is surjective for any $W \subseteq Z$. Hence for any $W \subseteq Z$ we have $h^1(\mathcal{I}_W(d)) \le h^1(\mathcal{I}_Z(d))$. Since $h^0(L, \mathcal{O}_L(d)) = d + 1$ for any line L and $h^0(T, \mathcal{O}_T(d)) = 2d + 1$ for any conic T, we get the " if " part. Now assume $h^1(\mathcal{I}_Z(d)) > 0$.

(a) First assume $m = 2$. Apply [13], Remarques (i) at page 116.

(b) Now assume $m \ge 3$. We use induction on m. Let $H_1 \subset \mathbb{P}^m$ be a hyperplane such that $\deg(H_1 \cap Z)$ is maximal. Set $Z_0 := Z$, $Z_1 := \mathrm{Res}_{H_1}(Z_0)$ and $w_1 := \deg(Z_0 \cap H_1)$. As in the proof of [1], Proposition 12, we define recursively the hyperplanes $H_i \subset \mathbb{P}^m$, $i \ge 2$, the schemes $Z_i \subseteq Z_{i-1}$, and the integers w_i, $i \ge 1$, in the following way. Let H_i be any hyperplane such that

$\deg(Z_{i-1} \cap H_i)$ is maximal. Set $Z_i := \mathrm{Res}_{H_i}(Z_{i-1})$ and $w_i := \deg(H_i \cap Z_{i-1})$. Any zero-dimensional scheme $F \subset \mathbb{P}^m$ with $\deg(F) \le m$ is contained in a hyperplane. Hence if $w_i \le m-1$, then $w_{i+1} = 0$ and $Z_i = \emptyset$. Since $z < 3d$, we get $w_i = 0$ for all $i \ge d$ and $Z_d = \emptyset$. For any integer $i > 0$ the residual sequence (1) gives the following exact sequence:

$$0 \to \mathcal{I}_{Z_i}(d-i) \to \mathcal{I}_{Z_{i-1}}(d-i+1) \to \mathcal{I}_{H_i \cap Z_{i-1}, H_i}(d-i+1) \to 0 \tag{2}$$

Since $h^1(\mathcal{I}_Z(d)) > 0$ and $Z_d = \emptyset$, (2) gives the existence of integer x such that $1 \le x \le d-1$ and $h^1(H_x, \mathcal{I}_{H_x \cap Z_{x-1}, H_x}(d-x+1)) > 0$. We call e the minimal such an integer x. First assume $e = 1$, i.e. assume $h^1(H_1, \mathcal{I}_{Z \cap H_1}(d)) > 0$. Since $\deg(Z \cap H_1) \le \deg(Z) < 3d$, the inductive assumption on m gives that either there is a line $L \subset H_1$ such that $\deg(L \cap Z) \ge 2$ or there is a conic $T \subset H_1$ such that $\deg(T \cap Z) \ge 2d+2$. From now on we assume $e \ge 2$. First assume $w_e \ge 2(d-e+1)+2$. Since $w_i \ge w_e$ for all $i < e$, we get $z \ge 2e(d-e+1)+2e$. Since $2 \le e \le d-1$ and $z < 3d$, we get a contradiction. Hence $w_e \le 2(d-e+1)+1$. Since $h^1(H_e, \mathcal{I}_{H_e \cap Z_{e-1}, H_e}(d-e+1)) > 0$ and $w_e \le 2(d-e+1)+1$, there is a line $L \subset H_e$ such that $\deg(L \cap Z_{e-1}) \ge d-e+3$ ([4], Lemma 34). Since $Z_{e-1} \ne \emptyset$, Z_{e-2} spans $\mathbb{P}^m$. Hence there is a hyperplane $M \subset \mathbb{P}^m$ such that $M \supset L$ and $\deg(M \cap Z_{e-2}) \ge \deg(Z_{e-2} \cap L) + m - 2 \ge d-e+m+1$. Hence $w_i \ge d-e+m+1$ for all $i < e$. Hence $z \ge e(d-e+3) + (e-1)(m-2)$.

First assume $e \ge 3$. Since $3d > z \ge e(d-e+3) + (e-1)(m-2)$ and $3 \le e \le d$, we get a contradiction.

Now assume $e = 2$. We have $\deg(L \cap Z) \ge d+1$. If $\deg(L \cap Z) \ge d+2$, then we are done. Hence we may assume $\deg(L \cap Z) = d+1$. Set $W_0 := Z$. Let $M_1 \subset \mathbb{P}^m$ be a hyperplane containing L and with $m_1 := \deg(M_1 \cap W_0)$ maximal among the hyperplanes containing L. We define recursively the hyperplanes $M_i \subset \mathbb{P}^m$, $i \ge 2$, the schemes $W_i \subseteq W_{i-1}$, and the integers m_i, $i \ge 1$, in the following way. Let M_i be any hyperplane such that $\deg(W_{i-1} \cap M_i)$ is maximal. Set $W_i := \mathrm{Res}_{M_i}(W_{i-1})$ and $m_i := \deg(H_i \cap W_{i-1})$. We have $m_i \le m_{i-1}$ for all i, $m_1 \ge \deg(L \cap Z) + m - 2 \ge d + m - 2$ and $m_i = 0$ if $m_{i-1} \le m-1$. As above there is a minimal integer f such $1 \le f \le d-1$ and $h^1(M_f, \mathcal{I}_{M_f \cap W_{f-1}, M_f}(d-f+1)) > 0$. As above we get a contradiction, unless $f = 2$. Assume $f = 2$. Since $m_2 \le z/2 \le 2(d-1)+1$, there is a line $D \subset M_2$ such that $\deg(D \cap W_1) \ge d+1$. Let E be any connected component of Z. If E is reduced, then $L \cap \mathrm{Res}_{M_1}(E) = \emptyset$, because $M_1 \supset L$. If E is not reduced, then $\deg(M_1 \cap \mathrm{Res}_{M_1}(E)) \le \deg(E \cap M_1)$, because $\mathrm{Res}_{M_1}(E) \subseteq E$. Since $\deg(Z) - \deg(Z_{red}) \le d$, we get $D \ne L$. Assume for the moment that either $D \cap L \ne \emptyset$ or $m \ge 4$, i.e. assume the existence of a hyperplane of $\mathbb{P}^m$ containing $D \cup L$. Hence $w_1 \ge 2d+1$. Hence $\deg(Z_1) \le \deg(W_1) \le d$. Hence $h^1(\mathcal{I}_{Z_1}(d-1)) = 0$. Hence $h^1(H_2, \mathcal{I}_{Z_1 \cap H_2}(d-1)) = 0$, contradicting the assumption $e = 2$. Now assume $m = 3$ and $D \cap L = \emptyset$. We may also assume $\deg(L \cap Z) = \deg(D \cap Z) = d + 1$. Let $N \subset \mathbb{P}^3$ be a general quadric surface containing $D \cup L$. The quadric surface N is smooth. Since $\deg(\mathrm{Res}_N(Z)) \le z - 2d - 2 \le d - 1$, we have $h^1(\mathcal{I}_{\mathrm{Res}_N(Z)}(d-2)) = 0$. Hence the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_N(Z)}(d-2) \to \mathcal{I}_Z(d) \to \mathcal{I}_{Z\cap N,N}(d) \to 0 \tag{3}$$

gives $h^1(N, \mathcal{I}_{Z\cap N,N}(d)) > 0$. Since $D \cap L = \emptyset$, D and L belong to the same ruling of N, say $D, L \in |\mathcal{O}_N(1,0)|$. Since $\deg(Z \cap L) = \deg(Z \cap D) = d+1$, we have $h^i(N, \mathcal{I}_{Z\cap N,N}(d,d)) = h^i(N, \mathcal{I}_{\mathrm{Res}_{D\cup L}(Z\cap N),N}(d-2,d))$, $i = 0, 1$. Since $\deg(\mathrm{Res}_{D\cup L}(Z \cap N)) = \deg(Z \cap N) - 2d - 2 \le d - 1$, we have $h^1(N, \mathcal{I}_{\mathrm{Res}_{D\cup L}(Z\cap N),N}(d-2,d)) = 0$, a contradiction. □

We recall the following result ([3], Lemma 1).

Lemma 2. *Fix $P \in \mathbb{P}^n$. Assume the existence of zero-dimensional schemes $A, B \subset \mathbb{P}^m$ such that $A \ne B$, $P \in \langle \nu_d(A)\rangle \cap \langle \nu_d(B)\rangle$, $P \notin \langle \nu_d(A')\rangle$ for any $A' \subsetneq A$ and $P \notin \langle \nu_d(B')\rangle$ for any $B' \subsetneq B$. Then $h^1(\mathcal{I}_{A\cup B}(d)) > 0$.*

Remark 1. Fix integers $m \ge 1$, $d \ge 2$ and $P \in \mathbb{P}^n$ such that $br(P) \le d + 1$. By [8], Lemma 2.1.5 and Lemma 2.4.4, there is a smoothable zero-dimensional and Gorenstein scheme $A \subset \mathbb{P}^m$ such that $\deg(A) = br(P)$, $P \in \langle \nu_d(A)\rangle$ and $P \notin \langle \nu_d(A')\rangle$ for any $A' \subsetneq A$. We will say that A evinces $br(P)$. In this range the smoothable rank and the border rank coincide. Now assume $br(P) \le (d + 1)/2$. Using Lemma 2 and the inequality $2s \le d + 1$ we get that A is the unique zero-dimensional scheme $E \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(E)\rangle$ and $\deg(E) \le s$. The uniqueness of A implies that A also evinces the cactus rank $cr(P)$ of P. In particular $cr(P) = br(P)$ if $br(P) \le (d+1)/2$.

Lemma 3. *Fix a proper linear subspace L of $\mathbb{P}^m$, an integer $d \ge 2$ and a finite set $E \subset \mathbb{P}^m \setminus L$ such that $\sharp(E) \le d$. Then* $\dim(\langle \nu_d(E \cup L)\rangle) = \dim(\langle \nu_d(L)\rangle) + \sharp(E)$. *For any closed subscheme $U \subseteq L$ we have $\langle \nu_d(U \cup E)\rangle \cap \langle \nu_d(L)\rangle = \langle \nu_d(U)\rangle$. For any $O \in \langle \nu_d(L \cup E)\rangle \setminus \langle \nu_d(E)\rangle$, the set $\langle \{O\} \cup \nu_d(E)\rangle \cap \langle \nu_d(L)\rangle$ is a unique point.*

Proof. Since E is a finite set and $E \cap L = \emptyset$, a general hyperplane H containing L contains no point of E. Since $E \cap H = \emptyset$, we have $\mathcal{I}_{E\cup H}(d) \cong \mathcal{I}_E(d-1)$. Since $\sharp(E) \le d$, we have $h^1(\mathcal{I}_E(d-1)) = 0$. Hence $\dim(\langle \nu_d(H \cup E)\rangle) = \dim(\langle \nu_d(H)\rangle) + \sharp(E)$. Since $L \subseteq H$, we get $\dim(\langle \nu_d(E \cup L)\rangle) = \dim(\langle \nu_d(L)\rangle) + \sharp(E)$. Grassmann's formula give $\langle \nu_d(L)\rangle \cap \langle \nu_d(E)\rangle = \emptyset$. Hence $\langle \nu_d(U \cup E)\rangle \cap \langle \nu_d(L)\rangle = \langle \nu_d(U)\rangle$ for any $U \subseteq L$. Fix any $O \in \langle \nu_d(L \cup E)\rangle \setminus \langle \nu_d(E)\rangle$. Since $O \notin \langle \nu_d(E)\rangle$, we have $\dim(\langle \{O\} \cup \nu_d(E)\rangle) = \dim(\langle \nu_d(E)\rangle) + 1$. Since $O \in \langle \nu_d(L \cup E)\rangle$ and $\langle \nu_d(L)\rangle \cap \langle \nu_d(E)\rangle = \emptyset$, Grassmann's formula gives that $\langle \{O\} \cup \nu_d(E)\rangle \cap \langle \nu_d(L)\rangle$ is a unique point. □

In the same way we get the following result.

Lemma 4. *Fix a conic $T \subset \mathbb{P}^m$, an integer $d \geq 5$ and a finite set $E \subset \mathbb{P}^m \setminus T$ such that $\sharp(E) \leq d-1$. Then $\dim(\langle \nu_d(E \cup T) \rangle) = \dim(\langle \nu_d(T) \rangle) + \sharp(E)$. For any closed subscheme $U \subseteq T$ we have $\langle \nu_d(U \cup E) \rangle \cap \langle \nu_d(T) \rangle = \langle \nu_d(U) \rangle$. For any $O \in \langle \nu_d(T \cup E) \rangle \setminus \langle \nu_d(E) \rangle$, the set $\langle \{O\} \cup \nu_d(E) \rangle \cap \langle \nu_d(T) \rangle$ is a unique point.*

The following lemma was proved (with D a hyperplane) in [2], Lemma 8. The same proof works for an arbitrary hypersurface D of $\mathbb{P}^m$ (see also Remark 2 below).

Lemma 5. *Fix $P \in \mathbb{P}^n$. Assume the existence of zero-dimensional schemes $A, B \subset \mathbb{P}^m$ such that $A \neq B$, $P \in \langle \nu_d(A) \rangle \cap \langle \nu_d(B) \rangle$, $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$ and $P \notin \langle \nu_d(B') \rangle$ for any $B' \subsetneq B$. Assume that B is reduced. Assume the existence of a positive integer $t \leq d$ and of a degree t hypersurface $D \subset \mathbb{P}^m$ such that $h^1(\mathcal{I}_{Res_D(A \cup B)}(d-t)) = 0$. Set $E := B \setminus B \cap D$. Then $\nu_d(E)$ is linearly independent, $E = Res_D(A)$ and every unreduced connected component of A is contained in D. The linear space $\langle \nu_d(A) \rangle \cap \langle \nu_d(B) \rangle$ is the linear span of its supplementary subspaces $\langle \nu_d(E) \rangle$ and $\langle \nu_d(A \cap D) \rangle \cap \langle \nu_d(B \cap D) \rangle$.*

Remark 2. Take the set-up of Lemma 5.

Claim : We have $\langle \nu_d(A \cap D) \rangle \cap \langle \nu_d(B \cap D) \rangle \neq \emptyset$ and there is $Q \in \langle \nu_d(A \cap D) \rangle \cap \langle \nu_d(B \cap D) \rangle$ such that $P \in \langle \{Q\} \cup \nu_d(E) \rangle$.

Proof of the Claim : Lemma 2 gives $h^1(\mathcal{I}_{A \cup B}(d)) > 0$. Since $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$ and $P \notin \langle \nu_d(B') \rangle$ for any $B' \subsetneq B$, we get $E \neq A$ and $E \neq B$, i.e. $A \cap D \neq \emptyset$ and $B \cap D \neq \emptyset$. The residual exact sequence (1) gives the following exact sequence:

$$0 \to \mathcal{I}_{\mathrm{Res}_D(A \cup B)}(d-t) \to \mathcal{I}_{A \cup B}(d) \to \mathcal{I}_{(A \cup B) \cap, D, D}(d) \to 0 \qquad (4)$$

From (4) and the definition of E we get the last assertion of Lemma 5. Since $P \in \langle \nu_d(A) \rangle \cap \langle \nu_d(B) \rangle$, $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$ and $P \notin \langle \nu_d(B') \rangle$ for any $B' \subsetneq B$, $\nu_d(A)$ and $\nu_d(B)$ are linearly independent. Since $h^1(\mathcal{I}_{A \cup B}(d)) > 0$ (Lemma 2), the exact sequence (4) gives $h^1(D, \mathcal{I}_{(A \cup B) \cap, D, D}(d)) > 0$. Hence the linear independence of $\nu_d(A)$ and $\nu_d(B)$ implies $\langle \nu_d(A \cap D) \rangle \cap \langle \nu_d(B \cap D) \rangle \neq \emptyset$. Since $\nu_d(A)$ is linearly independent, $P \in \langle \nu_d(A) \rangle \cap \langle \nu_d(B) \rangle$, $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$ and $P \notin \langle \nu_d(B') \rangle$ for any $B' \subsetneq B$, the sets $\langle \{P\} \cup \nu_d(E) \rangle \cap \langle \nu_d(A \cap D) \rangle$ and $\langle \{P\} \cup \nu_d(E) \rangle \cap \langle \nu_d(A \cap D) \rangle$ are given by a unique point. Call it Q_A and Q_B, respectively. Obviously $P \in \langle \nu_d(E) \cup \{Q_A\} \rangle \cap \langle \nu_d(E) \cup \{Q_B\} \rangle$. Since $P \notin \langle \nu_d(E) \rangle$, we get $Q_A = Q_B$. Set $Q := Q_A$.

For any reduced projective set $Y \subset \mathbb{P}^r$ spanning $\mathbb{P}^r$ and any $P \in \mathbb{P}^r$ let $r_Y(P)$ denote the minimal cardinality of a finite set $S \subset Y$ such that $P \in \langle S \rangle$. The positive integer $r_Y(P)$ is often called the Y-rank of P.

Lemma 6. *Assume $m = 2$. Fix integers $w \geq 3$ and $d \geq 4w - 1$. Take lines $L_1, L_2 \subset \mathbb{P}^2$ such that $L_1 \neq L_2$. Set $\{O\} := L_1 \cap L_2$. Let $A_1 \subset L_1$ be the degree w effective divisor of L_1 with O as its support. Fix $O_2 \in L_2 \setminus \{O\}$ and let $A_2 \subset L_2$ the degree w effective divisor of L_2 with O_2 as its reduction. Set $A := A_1 \cup A_2$. Fix $P \in \langle \nu_d(A) \rangle$ such that $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$. Then $br(P) = 2w$ and $sr(P) \geq 2d + 3 - 2w$. There is P as above with $sr(P) = 2d + 3 - 2w$.*

Proof. Since $\deg(A) = 2w \leq (d + 1)/2$ and $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$, A is the only scheme evincing $br(P)$ (Remark 1).

(a) In this step we prove the existence of P as above and with $sr(P) \leq 2d+3-2w$. Fix $B_1 \subset L_1 \setminus \{O\}$ such that $\sharp(B_1) = d-w+2$ and $B_2 \subset L_2 \setminus \{P_2\}$ such that $\sharp(B_2) = d-w+1$. Since $O \notin B_1$, $\deg(A_1 \cup B_1) = d+2$ and $\nu_d(L_1)$ is a degree d rational normal curve in its linear span, the set $\langle \nu_d(A_1) \rangle \cap \langle \nu_d(B_1) \rangle$ is a single point (call it P'). We have $P' \notin \langle W \rangle$ if either $W \subsetneq A_1$ or $W \subsetneq B_1$. Since $\dim(\langle \nu_d(L_1 \cup L_2) \rangle) = 2d$, $\nu_d(A_1 \cup A_2 \cup B_1 \cup B_2)$ spans $\langle \nu_d(L_1 \cup L_2) \rangle$ and $\dim(\langle \nu_d(B_1 \cup B_2) \rangle) = 2d + 2 - 2w$, the set $\langle \nu_d(A_1 \cup A_2) \rangle \cap \langle \nu_d(B_1 \cup B_2) \rangle$ is a line, M. Obviously $P' \in M$. Take as P any of the points of $M \setminus \{P'\}$. Since $P \in \langle \nu_d(B_1 \cup B_2) \rangle$, we have $sr(P) \leq 2d + 3 - 2w$.

(b) To conclude the proof it is sufficient to prove that $sr(P) \geq 2d+3-2w$. Assume $sr(P) \leq 2d + 2 - 2w$ and fix $B \subset \mathbb{P}^2$ evincing $sr(P)$. We have $h^1(\mathcal{I}_{A \cup B}(d)) > 0$ (Lemma 2) and $\deg(A \cup B) \leq 2d + 2$. Hence either there is a line $D \subset \mathbb{P}^2$ such that $\deg(D \cap (A \cup B)) \geq d + 2$ or there is a conic T such that $\deg(T \cap (A \cup B)) \geq 2d + 2$ (Lemma 1).

(b1) Assume the existence of a line D such that $\deg(D \cap (A \cup B)) \geq d + 2$. Since $\deg(\mathrm{Res}_D(A \cup B)) \leq (2d - 2w + 2) + 2w - d - 2 = d$, we have $h^1(\mathcal{I}_{\mathrm{Res}_D(A \cup B)}(d - 1)) = 0$. Hence Lemma 5 gives $A \subset D$, a contradiction.

(b2) Assume the existence of a conic T such that $\deg(T \cap (A \cup B)) \geq 2d+2$. Since $\deg(A)+\deg(B) \leq 2d+2$, we get $A \cap B = \emptyset$, $\sharp(B) = 2d+2-2w$ and $A \cup B \subset T$. We have $\deg(L_2 \cap A) = w + 1$ and $\deg(L_1 \cap A) = w$. Since $w \geq 3$, the Bezout theorem implies that $L_1 \cup L_2$ is the unique conic containing A. Hence $T = L_1 \cup L_2$. Set $B_i := B \cap L_i$. First assume $\sharp(B_1) \geq d - w + 2$. Since $\sharp(B_1) \leq d - w$, the scheme $\mathrm{Res}_{L_1}(A \cup B) = A_2 \cup B_2$ has degree $\leq d$. Hence $h^1(\mathcal{I}_{\mathrm{Res}_{L_1}(A \cup B)}(d - 1)) = 0$. Lemma 5 gives $A \subset L_1$, a contradiction. Now assume $\sharp(B_1) \leq d - w + 1$. Let $G \subset L_1$ be the degree $w - 1$ effective divisor of L_1 with O as its support. Since $\mathrm{Res}_{L_2}(A \cup B) = G \cup B_1$ has degree $\leq d$, Lemma 5 implies $A \subset L_2$, a contradiction. $\square$

Lemma 7. *Assume $m = 2$. Fix integers $w \geq 2$ and $d \geq 4w + 1$. Take lines $L_1, L_2 \subset \mathbb{P}^2$ such that $L_1 \neq L_2$. Set $\{O\} := L_1 \cap L_2$. Let $A_1 \subset L_1$ be the degree $w + 1$ effective divisor of L_1 with O as its support. Fix $O_2 \in L_2 \setminus \{O\}$ and let $A_2 \subset L_2$ the degree w effective divisor of L_2 with O_2 as its reduction.*

Set $A := A_1 \cup A_2$. *Fix* $P \in \langle \nu_d(A) \rangle$ *such that* $P \notin \langle \nu_d(A') \rangle$ *for any* $A' \subsetneq A$. *Then* $br(P) = 2w + 1$ *and* $sr(P) \geq 2d + 2 - 2w$. *There is a* P *as above with* $sr(P) = 2d + 2 - 2w$.

Proof. Copy the proof of Lemma 6. In step (b2) we have $T = L_1 \cup L_2$, because $\deg(L_1 \cap A) = \deg(L_2 \cap A) = w + 1 \geq 3$). □

Lemma 8. *Fix integers* $w \geq 3$, $s \geq 2w$ *and assume* $d \geq 2s - 1$. *Fix lines* $D, R \subset \mathbb{P}^m$ *such that* $D \neq R$ *and* $D \cap R \neq \emptyset$. *Set* $\{O\} := D \cap R$. *Let* U *be the plane spanned by* $D \cup R$. *Let* $E \subset U$ *be a general subset with cardinality* $s - 2w$. *Let* $A_1 \subset D$ *be the zero-dimensional degree* w *subscheme of* D *with* O *as its support. Fix* $O' \in R \setminus \{O\}$ *and call* A_2 *the zero-dimensional subscheme of* R *with* O' *as its support and degree* w. *Set* $A := A_1 \cup A_2$. *There is* $P \in \langle \nu_d(A_1 \cup A_2 \cup E) \rangle$ *such that* $br(P) = s$ *and* $sr(P) = 2d + 3 + s - 4w$.

Proof. We will always compute the residual schemes with respect to divisors of U. Notice that $A := A_1 \cup A_2 \cup E$ is curvilinear and hence it only has finitely many subschemes. Hence there is $P \in \langle \nu_d(A_1 \cup A_2 \cup E) \rangle$ such that $P \notin \langle \nu_d(F) \rangle$ for any $F \subsetneq A_1 \cup A_2 \cup E$. Since $\deg(A) = s \leq (d+1)/2$, we get $sr(P) = s$ and that A is the only subscheme of $\mathbb{P}^m$ evincing $sb(P)$ (Remark 1).

(a) Fix $P \in \langle \nu_d(A_1 \cup A_2 \cup E) \rangle$ such that $P \notin \langle \nu_d(F) \rangle$ for any $F \subsetneq A_1 \cup A_2 \cup E$. In this step we prove that $sr(P) \geq 2d + 3 + s - 2w$. Assume $sr(P) \leq 2d + 2 + s - 2w$. By [9], Proposition 3.1, or [15], subsection 3.2, there is $B \subset U$ evincing $sr(P)$. Since A is not reduced, we have $A \neq B$. Hence $h^1(U, \mathcal{I}_{A \cup B}(d)) > 0$. Set $W := A \cup B$. Since $\deg(W) \leq 2d + 2s - 6 < 3d$, either there is a line $L \subset U$ such that $\deg(L \cap W) \geq d + 2$ or there is a conic $T \subset U$ such that $\deg(T \cap W) \geq 2d + 2$ (Lemma 1).

(a1) Assume the existence of a line $L \subset U$ such that $\deg(L \cap W) \geq d + 2$. If $h^1(U, \mathcal{I}_{\mathrm{Res}_{L}(W)}(d-1)) = 0$, then Lemma 5 gives $A_1 \cup A_2 \subset L$, absurd. Hence $h^1(U, \mathcal{I}_{\mathrm{Res}_{L}(W)}(d-1)) > 0$. Since $\deg(\mathrm{Res}_L(W)) \leq 2(d-1) + 1$, there is a line $L' \subset U$ such that $\deg(L' \cap \mathrm{Res}_L(W)) \geq d + 1$. Since $\deg(\mathrm{Res}_{L \cup L'}(W)) \leq d - 1$, Lemma 5 gives $A_1 \cup A_2 \subset L \cup L'$ and $\mathrm{Res}_{L \cup L'}(A) = B \setminus B \cap (L \cup L')$. Since $w \geq 3$, the Bezout theorem gives $L \cup L' = D \cup R$. Hence $E \subset B$ and $B \setminus E \subset D \cup R$. Lemma 5 and Remark 2 applied to $D \cup R$ give the existence of $Q \in \langle \nu_d(A \cap (D \cup R)) \rangle$ such that $A_1 \cup A_2$ evinces $br(Q)$, while $B \setminus E$ evinces $sr(Q)$. Since $\sharp(B \setminus E) \leq 2d + 2 - 2w$, either $\sharp((B \setminus E) \cap D) \leq d + 1 - w$ or $\sharp((B \setminus E) \cap R) \leq d - w$. First assume $\sharp((B \setminus E) \cap D) \leq d + 1 - w$. Since $\deg(\mathrm{Res}_R(A_1 \cup A_2)) = w - 1$, we have $h^1(\mathcal{I}_{\mathrm{Res}_R((A \cup B) \cap (D \cup R))}(d-1)) = 0$. Hence Lemma 5 applied to $A \cap (D \cap R)$ and $B \cap (D \cup R)$ gives $\mathrm{Res}_R(A_1 \cup A_2) = \mathrm{Res}_R(B \cap (D \cup R))$. Since B is reduced, we get $w \leq 2$, a contradiction.

(a2) Now we assume the existence of a conic $T \subset U$ such that $\deg(T \cap W) \geq 2d+2$. Since $\deg(\mathrm{Res}_T(W)) \leq d-1$, we have $h^1(\mathcal{I}_{\mathrm{Res}_{T(W)}}(d-2)) = 0$. Hence the case $t = 2$ of Lemma 5 gives $A_1 \cup A_2 \subset T$ and $B \setminus B \cap T = \mathrm{Res}_T(A)$. Since $w \geq 3$, and $A_1 \cup A_2 \subset T$, the Bezout theorem gives $T = D \cup R$. We work as in step (b1) (notice that in the case $L \cup L' = D \cup R$ we only used that $\deg((L \cup L') \cap (A \cup B)) \geq 2d + 2$).

(b) In this step we check the existence of $P \in \langle \nu_d(A_1 \cup A_2 \cup E) \rangle$ such that $P \notin \langle \nu_d(F) \rangle$ for any $F \subsetneq A_1 \cup A_2 \cup E$ and $sr(P) = 2d + 3 + s - 2w$ and $br(P) = s$. Lemma 6 gives the existence of $O \in \langle \nu_d(A_1 \cup A_2) \rangle$ such that $sr(O) \leq 2d + 3 - 2w$ and $O \notin \langle \nu_d(G) \rangle$ for any $G \subsetneq A_1 \cup A_2$. Take a general $P \in \langle \{O\} \cup \nu_d(E) \rangle$. Obviously $sr(P) \leq sr(O) + \sharp(E) \leq 2d+3+s-2w$. Step (b) gives $sr(P) = 2d + 3 + s - 2w$. Assume $br(P) < s$. Since $br(P) \leq d + 1$, there is a zero-dimensional scheme $W \subset \mathbb{P}^m$ such that $\deg(W) = br(P)$, $P \in \langle \nu_d(W) \rangle$ and $P \notin \langle \nu_d(W') \rangle$ for any $W' \subsetneq W$. First assume $W \nsubseteq A_1 \cup A_2 \cup F$. Lemma 2 gives $h^1(\mathcal{I}_{A_1 \cup A_2 \cup E \cup W}(d)) > 0$. Since $bs(P) + s \leq 2d + 1$, there is a line $L \subset \mathbb{P}^m$ such that $\deg(L \cap (A_1 \cup A_2 \cup E \cup W)) \geq d + 2$. As in step (b1) we get a contradiction. Now assume $W \subsetneq A_1 \cup A_2 \cup E$. Set $A' := (A_1 \cup A_2) \cap W$. Lemma 3 gives $\langle \{P\} \cup \nu_d(E) \rangle \cap \langle \nu_d(A_1 \cup A_2) \rangle = \{O\}$ and $\langle \{P\} \cup \nu_d(E') \rangle \cap \langle \nu_d(A_1 \cup A_2) \rangle$. Since $P \in \langle \nu_d(W) \rangle$ and $P \notin \langle \nu_d(W') \rangle$ for any $W' \subsetneq W$, the set $\langle \{P\} \cup \langle \nu_d(W \cap E) \rangle \cap \langle \nu_d(A') \rangle$, is a single point. Hence $E \subseteq W$ and $\{O\} \in \langle \nu_d(A') \rangle$, contradicting the choice of O. □

Quoting Lemma 7 instead of Lemma 6 we get the following result.

Lemma 9. *Fix integers $w \geq 2$ and $s \geq 2w+1$, Assume $d \geq 2s-1$. Fix lines $D, R \subset \mathbb{P}^m$ such that $D \neq R$ and $D \cap R \neq \emptyset$. Set $\{O\} := D \cap R$. Let U be the plane spanned by $D \cup R$. Let $E \subset U$ be a general subset with cardinality $s - 2w - 1$. Let $A_1 \subset D$ be the zero-dimensional degree $w + 1$ subscheme of D with O as its support. Fix $O' \in R \setminus \{O\}$ and call A_2 the zero-dimensional subscheme of R with O' as its support and degree w. Set $A := A_1 \cup A_2$. There is $P \in \langle \nu_d(A_1 \cup A_2 \cup E) \rangle$ such that $br(P) = s$ and $sr(P) = 2d + 1 + s - 4w$.*

Proof of Theorem 1. Since the cases $s = 2, 3$ are true by [4], Theorems 32 and 37, we may assume $s \geq 4$ (the case $(s, r) = (3, 2d - 1)$ does not occur in the statement of Theorem 1, because we assumed $r \leq 2d + s - 7$; this inequality is used in steps (d) and (e) below).

Notice that $\sigma_s(X_{m,d}) \setminus \sigma_{s-1}(X_{m,d}) \neq \emptyset$ (e.g., because $s(m + 1) < \binom{m+d}{m}$). Fix $P \in \sigma_s(X_{m,d}) \setminus \sigma_{s-1}(X_{m,d})$ and write $r := sr(P)$. Since $P \notin \sigma_{s-1}(X_{m,d})$ we have $r \geq s$. Since $\sigma_s(X_{m,d}) \neq \sigma_{s-1}(X_{m,d})$ a non-empty open subset of $\sigma_s(X_{m,d})$ is formed by points with rank s. Hence to prove Theorem 1 we may assume $r > s$. By Remark 1 there is a unique degree s zero-dimensional scheme $A \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(A) \rangle$ and this scheme is smoothable. By Remark 1 there is no zero-dimensional scheme $A_1 \subset \mathbb{P}^m$

such that $\deg(A_1) < s$ and $P \in \langle \nu_d(A_1) \rangle$. Hence P has cactus rank s and $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$. Since $sr(P) = r$ there is a finite set $B \subset \mathbb{P}^m$ such that $\sharp(B) = r$, $P \in \langle \nu_d(B) \rangle$ and $P \notin \langle \nu_d(B') \rangle$ for any $B' \subsetneq B$. Set $W := A \cup B$. We have $\deg(W) \le \deg(A) + \deg(B) = r + s$ and equality holds if and only if $A \cap B = \emptyset$. Lemma 2 gives $h^1(\mathcal{I}_W(d)) > 0$. Hence $\deg(W) \ge d + 2$ (e.g., by [4], Lemma 34). Therefore $\sigma_{s,x}(X_{m,d}) = \emptyset$ if $s + 1 \le x \le d - s + 1$.

We have $\sigma_{s,d-s+2}(X_{m,d}) \neq \emptyset$, because $\sigma_{s,d-s+2}(X_{1,d}) \neq \emptyset$ by a theorem of Sylvester ([9], [15], Theorem 4.1, or [4]) and for any line $L \subset \mathbb{P}^m$ and any $P \in \langle \nu_d(L) \rangle$ the symmetric rank and the border rank of P are the same with respect to $X_{m,d}$ or with respect to $\nu_d(L) \cong X_{1,d}$ ([15], subsection 3.2).

(a) In this step we prove that $\sigma_{s,r}(X_{m,d}) \neq \emptyset$ for every $r \in \{d - s + 3, \dots, d + s - 2\}$ such that $r + s \equiv d \pmod 2$. Fix $r \in \{d - s + 3, \cdots, d + s - 2\}$ such that $r + s \equiv d \pmod 2$. Set $b := (d + 2 + s - r)/2$. Since $r + s \equiv d \pmod 2$, we have $b \in \mathbb{Z}$. Since $r \le d + s - 2$, we have $b \ge 2$. Since $r \ge d - s + 3$, we have $b < s$. Since $d \ge 2s - 2$, we have $r > s$ and hence $2b < d + 2$. Fix a line $L \subset \mathbb{P}^m$ and a connected zero-dimensional scheme $Z' \subset L$ such that $\deg(Z') = b$. Take any $Q \in \langle \nu_d(Z') \rangle$ such that $Q \notin \langle \nu_d(Z'') \rangle$ for any $Z'' \subsetneq Z'$ (Q exists and the set of all such points Q is a non-empty open subset of a projective space of dimension $\deg(Z') - 1$, because Z' is a divisor of the smooth curve L). Fix any set $E \subset \mathbb{P}^m \setminus L$ such that $\sharp(E) = s - b = (s + r - d - 2)/2$. Since $E \cap L = \emptyset$, we have $\deg(Z' \cup E) = s$. Since $d \ge s - 1$, we have $\dim(\langle \nu_d(Z' \cup E) \rangle) = s - 1$. Since L is contained in a smooth curve, Z' is curvilinear. Since E is a finite set, the scheme $Z' \cup E$ is curvilinear. We claim that any zero-dimensional curvilinear subscheme $W \subset \mathbb{P}^m$ has only finitely many subschemes. Indeed, W is contained in a smooth curve C and hence we may write $W = \sum_{i=1}^{x} a_i Q_i$ for some $x \in \mathbb{N} \setminus \{0\}$, $a_i \in \mathbb{N} \setminus \{0\}$ and $Q_i \in C$. The subschemes of W are the effective divisors of C of the form $\sum_{i=1}^{x} b_i Q_i$ for some $b_i \in \{0, \dots, a_i\}$. Hence W has exactly $\prod_{i=1}^{x} (a_i + 1)$ subschemes. Hence $Z' \cup E$ has only finitely many closed subschemes. Fix any $O \in \langle \nu_d(Z' \cup E) \rangle$ such that $O \notin \langle \nu_d(F) \rangle$ for any $F \subsetneq Z' \cup E$ (O exists and the set of all such points O is a non-empty open subset of the $(s - 1)$-dimensional projective space $\langle \nu_d(Z' \cup E) \rangle$, because $Z' \cup E$ has only finitely many subschemes and $O \notin \langle \nu_d(L \cup E') \rangle$ for any $E' \subsetneq E$). Let $A' \subset L$ be a set evincing $sr(Q)$ with respect to the rational normal curve $\nu_d(L)$. A theorem of Sylvester gives $\sharp(A') = d - \deg(Z') + 2$ ([9], [15], Theorem 4.1, or [4]). Set $G := A' \cup E$. Since $E \cap L = \emptyset$, we have $\sharp(G) = r$.

Claim 1: $br(O) = s$.

Proof of Claim 1: We have $O \in \langle \nu_d(Z' \cup E)\rangle$ and $O \notin \langle \nu_d(F)\rangle$ for any $F \subsetneq Z' \cup E$. Apply Remark 1.

Claim 2: $sr(O) = r$ and G evinces $sr(O)$.

Proof of Claim 2: Since $P \in \langle \nu_d(G)\rangle$, we have $sr(P) \le r$. Since $s \le (d+1)/2$, $Z' \cup E$ is the only scheme evincing $br(O)$. Since Z' is not reduced, we get $sr(P) > s$. Fix any $U \subset \mathbb{P}^m$ evincing $sr(P)$. Since $sr(P) + br(P) \le 2d+1$, $sr(P) > br(P)$ and $Z' \cup E$ evinces $br(O)$, [3], Theorem 1, gives the existence of a line $D \subset \mathbb{P}^m$ such that $(Z' \cup E) \setminus D \cap (Z' \cup E) = U \setminus U \cap D$ and every unreduced connected component of $Z' \cup E$ is contained in D. Since Z' is not reduced, we get $D = L$. Hence $U \setminus U \cap L = E$. Since $O \in \langle \nu_d(Z' \cup E)\rangle \subseteq \langle \nu_d(L \cup E)\rangle$, Lemma 3 gives that $\langle \{O\} \cup \nu_d(E)\rangle \cap \langle \nu_d(L)\rangle$ is a single point (call it Q'). Since $O \in \langle \{Q\} \cup \nu_d(E)\rangle$ and $Q \in \langle \nu_d(L)\rangle$, we have $Q' = Q$. Lemma 3 gives $Q \in \langle \nu_d(L \cap U)\rangle$. Since A' evinces $sr(Q)$, we get $sr(P) = \sharp(U) \ge \sharp(E) + \sharp(A')$, concluding the proof of Claim 2.

The point O shows that $\sigma_{s,r}(X_{m,d}) \ne \emptyset$ for any $r \in \{d+3-s, \dots, d+s-2\}$ such that $r + s \equiv d \pmod 2$.

(b) Fix any integer r such that $r \in \{d-s+3, \dots, d+s-2\}$ and $r+s \equiv d+1 \pmod 2$. In this step we prove that $\sigma_{s,r}(X_{m,d}) = \emptyset$. Assume the existence of $P \in \sigma_{s,r}(X_{m,d})$. Fix $A \subset \mathbb{P}^m$ evincing $br(P)$ and $B \subset \mathbb{P}^m$ evincing $sr(P)$. Since $r > s$ we have $A \ne B$. As in step (a) we get the existence of a line $D \subset \mathbb{P}^m$ such that $\deg((A \cup B) \cap D) \ge d+2$, every unreduced connected component of A is contained in D and $\mathrm{Res}_D(A) = B \setminus B \cap D$. Set $E := B \setminus B \cap D$. By Lemma 3 the set $\langle \nu_d(D)\rangle \cap \langle \nu_d(E) \cup \{P\}\rangle$ is a unique point, O, and $O \in \langle \nu_d(A \cap D)\rangle \cap \langle \nu_d(B \cap D)\rangle$. Since $\deg(A) = \deg(A \cap D) + \sharp(E)$ (resp. $\sharp(B) = \sharp(B \cap D) + \sharp(E)$), $A \cap D$ evinces $br(P)$ (resp. $B \cap D$ evinces $sr(O)$). The quoted theorem of Sylvester gives $sr(O) + br(O) = d+2$. Hence $s + r = 2 \cdot \sharp(E) + sr(O) + br(O) \equiv d \pmod 2$, a contradiction.

(c) In this step we fix an integer r such that $d + s - 1 \le r \le 2d + 1 - s$. In order to obtain a contradiction we assume $\sigma_{s,r}(X_{m,d}) \ne \emptyset$ and fix $P \in \sigma_{s,r}(X_{m,d})$. Take $A \subset \mathbb{P}^m$ evincing $br(P)$ and $B \subset \mathbb{P}^m$ evincing $sr(P)$. Let A_1 be the union of the connected components of A which are not reduced. Since $r > s$, we have $A_1 \ne \emptyset$. Hence $\deg(A_1) \ge 2$. Lemma 2 gives $h^1(\mathcal{I}_{A \cup B}(d)) > 0$. Since $\deg(A \cup B) \le s + r \le 2d+1$, there is a line $D \subset \mathbb{P}^m$ such that $\deg((A \cup B) \cap D) \ge d+2$ (Lemma 1 or [4], Lemma 34). Set $E := B \setminus B \cap D$. Since $e := \deg(\mathrm{Res}_D(A \cup B)) \le r + s - d - 2 \le d$, we have $h^1(\mathcal{I}_{\mathrm{Res}_D(A \cup B)}(d-1)) = 0$. Hence Lemma 5 gives $A_1 \subset D$ and $A \setminus A \cap D = E$, $e = \sharp(E)$ and $e = \deg(A \setminus A \cap D) \le s - \deg(A_1)$. Since A evinces $br(P)$, the set $\langle \nu_d(E) \cup \{P\}\rangle \cap \langle \nu_d(A_1)\rangle$ is a unique point, O.

Sylvester's theorem gives $sr(O) \le d$. Since $P \in \langle \nu_d(E) \cup \{O\}\rangle$, we get $sr(P) \le e + d \le d + s - 2$, a contradiction.

(d) In this step we prove that $\sigma_{s,r}(X_{m,d}) \neq \emptyset$ for every integer r such that $2d + 2 - s \le r \le 2d + s - 7$ and $r + s \equiv 0 \pmod 2$. Set $b := (2d + 2 + s - r)/2$. Since $r + s \equiv 0 \pmod 2$, we have $b \in \mathbb{Z}$. Since $r \le 2d + s - 7$, we have $b \ge 9/2$ and hence $b \ge 5$. Since $r \ge 2d + 2 - s$, we have $b \le s$. We may assume $m = 2$ ([15], subsection 3.2). Fix a smooth conic $C \subset \mathbb{P}^2$, a connected zero-dimensional scheme $A_1 \subset C$ such that $\deg(A_1) = b$ and a general set $E \subset \mathbb{P}^2 \setminus C$ such that $\sharp(E) = s - b$. We have $\sharp(E) \le d - 1$. Set $A := A_1 \cup E$ and $\{O'\} := (A_1)_{red}$. Fix $P \in \langle \nu_d(A)\rangle$ such that $P \notin \langle \nu_d(F)\rangle$ for any scheme $F \subsetneq A$ (P exists, because A is curvilinear and $\dim(\langle \nu_d(A)\rangle) = \deg(A) - 1$). Since $s = \deg(A) \le (d + 1)/2$, then $br(P) = s$ and A is the only scheme evincing $br(P)$ (Remark 1). Lemma 4 gives that the set $\langle \{P\} \cup E\rangle \cap \langle \nu_d(C)\rangle$ is a unique point, O, that $O \in \langle \nu_d(A_1)\rangle$ and that $sr(O) = b$. Since $b \le d+1$, the quoted theorem of Sylvester gives $r_{\nu_d(C)}(O) = 2d+2-b$. Fix $B_1 \subset C$ such that $\nu_d(B_1)$ evinces $r_{\nu_d(C)}(O)$, i.e. take $B_1 \subset C$ such that $\sharp(B_1) = 2d+2-b$, $O \in \langle \nu_d(B_1)\rangle$ and $O \notin \langle \nu_d(F)\rangle$ for any $F \subsetneq B_1$. We have $P \in \langle \nu_d(B_1 \cup E)\rangle$. Lemma 4 also gives $P \notin \langle \nu_d(G)\rangle$ for any $G \subsetneq B_1 \cup E$. Since $br(P) = s$, $P \in \langle \nu_d(B_1 \cup E)\rangle$ and $\sharp(B_1 \cup E) = r$, to prove that $\sigma_{s,r}(X_{m,d}) \neq \emptyset$ it is sufficient to prove that $sr(P) \ge r$. Assume $sr(P) < r$ and take B evincing $sr(P)$. We have $h^1(\mathcal{I}_{A\cup B}(d)) > 0$ (Lemma 2). Since $\deg(A \cup B) \le s + r - 1 < 3d$, either there is a line $D \subset \mathbb{P}^2$ such that $\deg(D \cap (A \cup B)) \ge d + 2$ or there is a conic T such that $\deg(T \cap (A \cup B)) \ge 2d + 2$ (Lemma 1).

(d1) Here we assume the existence of a line $D \subset \mathbb{P}^2$ such that $\deg(D \cap (A\cup B)) \ge d+2$. If $h^1(\mathcal{I}_{\mathrm{Res}_D(A\cup B)}(d-1)) = 0$, then Lemma 5 gives $A_1 \subset D$, a contradiction. Hence $h^1(\mathcal{I}_{\mathrm{Res}_D(A\cup B)}(d - 1)) > 0$. Since $\deg(\mathrm{Res}_D(A \cup B)) \le r + s - 1 - d - 2 \le 2(d - 1) + 1$, Lemma 1 or [4], Lemma 34, give the existence of a line $D' \subset \mathbb{P}^2$ such that $\deg(D' \cap \mathrm{Res}_D(A \cup B)) \ge d + 1$. Hence $\deg((A \cup B) \cap (D \cup D')) \ge 2d + 3$. Hence $\deg(\mathrm{Res}_{D\cup D'}(A \cup B)) \le d - 1$. Hence $h^1(\mathcal{I}_{\mathrm{Res}_{D\cup D'}(A\cup B)}(d - 2)) = 0$. Lemma 5 gives $A_1 \subset D \cup D'$. Since C is an irreducible conic containing A_1, the Bezout theorem gives $b \le 4$, a contradiction.

(d2) Now assume the existence of a conic $T \subset \mathbb{P}^2$ such that $\deg(T\cap(A\cup B)) \ge 2d + 2$. Since $\deg(\mathrm{Res}_T(A \cup B)) \le d - 1$, we have $h^1(\mathcal{I}_{\mathrm{Res}_T(A\cup B)}(d-2)) = 0$. Hence the case $t = 2$ of Lemma 5 gives $A_1 \subset T$ and $\mathrm{Res}_T(A) = B \setminus B \cap T$. Since $b \ge 5$, C is irreducible and $A_1 \subset C$, the Bezout theorem gives $T = C$. We get $B \setminus B \cap C = E$. Hence $\sharp(B \cap C) \le 2d + 1 - b$. Hence $\deg(C \cap (A \cup B)) \le 2d + 1$, a contradiction.

(e) In this step we prove that $\sigma_{s,r}(X_{m,d}) \neq \emptyset$ for every integer r such that $2d + 3 - s \le r \le 2d + s - 7$ and $r + s \equiv 1 \pmod 2$ and hence conclude the proof of Theorem 1. Set $c := (2d + 3 + s - r)/2$ and $w := \lfloor c/2\rfloor$. Since

$r + s \equiv 1 \pmod 2$, we have $c \in \mathbb{Z}$. Since $r \geq 2d + 3 - s$, we have $c \leq s$. Since $r \leq 2d + s - 7$, we have $c \geq 5$. If c is odd, then we apply Lemma 9. If c is even, then we apply Lemma 8. □

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