B. Y. CHEN’S INEQUALITIES FOR BI-SLANT SUBMANIFOLDS IN COSYMPLECTIC SPACE FORMS

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Abstract. In this paper we obtain B. Y. Chen’s inequalities for a bi-slant submanifold $M$ of a cosymplectic space form $M(c)$, when the structure vector field $\xi$ of the ambient space is tangent to $M$.

1. Introduction

In the theory of Riemannian submanifolds it is quite interesting to establish a relationship between the intrinsic and extrinsic invariants. Basically, the Riemannian invariants are intrinsic characteristics of Riemannian manifolds. In 1993, B. Y. Chen [6] has obtained an inequality between sectional curvature $K$, the scalar curvature $\tau$ (intrinsic invariant) and the mean curvature function $||H||$ (extrinsic invariant) of a submanifold $M$ of the real space form of constant curvature $c$. Moreover, Chen [4] also introduced a new type of Riemannian invariants of a Riemannian manifold.

Let $M$ be a Riemannian manifold of dimension $m$ and let $\{e_1, e_2, \ldots, e_m\}$ be any orthonormal basis of the tangent space $T_pM$ at any point $p \in M$. Then the scalar curvature $\tau$ at $p \in M$ is given by

$$\tau = \sum_{1 \leq i < j \leq m} K(e_i \wedge e_j)$$ (1.1)

for any point $p \in M$, we denote

$$(\inf K)(p) = \inf\{K(\pi) : \pi \subset T_pM, \dim \pi = 2\}$$ (1.2)

where $K(\pi)$ denotes the sectional curvature of $M$ associated with a plane section $\pi \subset T_pM$ at $p \in M$.

The Chen invariant $\delta_M$ at any point $p \in M$ is defined as

$$\delta_M(p) = \tau(p) - (\inf K)(p).$$ (1.3)
For a submanifold $M$ of a real space form $\overline{M}(c)$, Chen has given a basic inequality in terms of the intrinsic invariant $\delta_M$ and the squared mean curvature of the immersion, as

$$\delta_M \leq \frac{m^2(m-2)}{2(m-1)} ||H||^2 + \frac{1}{2}(m+1)(m-2)c.$$

(1.4)

The above inequality also holds good in case $M$ is an anti-invariant submanifold of complex space form $\overline{M}(c)[7]$. In case of contact manifold, Defever, Mihai and Verstralen [11] obtained an inequality similar to that of (1.4), for C-totally real submanifold of a Sasakian space form with constant $\varphi$-sectional curvature $c$, given by

$$\delta_M \leq \frac{m^2(m-2)}{2(m-1)} ||H||^2 - \frac{1}{2}(m+1)(m-2)c + \frac{3}{4}.$$

(1.5)

2. Preliminaries

A $(2m+1)$-dimensional Riemannian manifold $\overline{M}$ is said to be an almost contact metric manifold if there exists structure tensors $(\phi, \xi, \eta, g)$, where $\phi$ is a $(1,1)$ tensor field, $\xi$ a vector field, $\eta$ a 1-form and $g$ the Riemannian metric on $\overline{M}$ satisfying [9]

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi^2 \xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0$$

(2.1)

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(\xi) = g(X, \xi)$$

for any $X, Y \in \mathfrak{T}\overline{M}$, where $\mathfrak{T}\overline{M}$ denotes the Lie algebra of vector fields on $\overline{M}$.

An almost contact metric manifold $\overline{M}$ is called a cosymplectic manifold if [13],

$$\nabla_X \phi = 0 \quad \text{and} \quad \nabla_X \xi = 0$$

(2.2)

where $\nabla$ denotes the Levi-Civita connection on $\overline{M}$.

The curvature tensor $\mathcal{R}$ of a cosymplectic space form $\overline{M}(c)$ is given by [14],

$$\mathcal{R}(X, Y)Z = \frac{c}{4} \{ g(Y, Z)X - g(X, Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi - g(\phi X, Z)\phi Y + g(\phi Y, Z)\phi X + 2g(X, \phi Y)\phi Z \}$$

(2.3)

for all $X, Y, Z \in \mathfrak{T}\overline{M}$.

Now, let $M$ be an $m$-dimensional isometrically immersed Riemannian submanifold of a cosymplectic manifold $\overline{M}$ with induced metric $g$. Denoting by
the tangent bundle of $M$ and by $T^\perp M$ the set of all vector fields normal to $M$, we write
\[ \phi X = PX + FX \] (2.4)
for any $X \in TM$, where $PX$ (resp. $FX$) denotes the tangential (resp. normal) component of $\phi X$.

From now on we assume that the structure vector field $\xi$ is tangent to $M$. We make the direct orthogonal decomposition $TM = D \oplus \xi$.

A submanifold $M$ is said to be slant if for any non zero vector $X$ tangent to $M$ at $p$ such that $X$ is not proportional to $\xi_p$, the angle $\theta(X)$ between $\phi X$ and $T_pM$ is constant i.e., is independent of the choice of $p \in M$ and $X \in T_pM - \{\xi_p\}$. Sometimes the angle $\theta(X)$ is termed as the Wirtinger angle of the slant immersion.

Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a proper slant immersion.

A submanifold $M$ tangent to structure vector field $\xi$ is said to be a bi-slant submanifold of a cosymplectic manifold $\mathcal{M}$, if there exist two orthogonal differentiable distributions $D_1$ and $D_2$ on $M$, such that

(i) $TM$ possesses an orthogonal direct decomposition of $D_1$ and $D_2$ i.e. $TM = D_1 \oplus D_2 \oplus \xi$.

(ii) $D_i$ is slant distribution with slant angle $\theta_i$ for any $i = 1, 2$.

If we take the $\text{dim} D_1 = 2n_1$ and $\text{dim} D_2 = 2n_2$, then it is obvious that in case either $n_1$ vanishes or $n_2$, the bi-slant submanifold reduces to a slant submanifold. Hence, the bi-slant submanifolds are generalized cases of slant submanifolds. Moreover, slant submanifolds, invariant submanifolds and anti-invariant submanifolds are particular cases of bi-slant submanifolds.

Let $R$ and $\mathcal{R}$ denote the curvature tensors of the submanifold $M$ and cosymplectic space form $\mathcal{M}(c)$, respectively. Then the equation of Gauss is given by
\[ \mathcal{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(h(X, W), h(Y, Z)) + g(h(X, Z), h(Y, W)) \] (2.5)

for all $X, Y, Z, W \in TM$.

We denote by $h$ the second fundamental form of $M$ and by $A_N$ the Weingarten map associated with $N \in T^\perp M$. We put
\[ h_{i,j}^r = g(h(e_i, e_j), e_r) \quad \text{and} \quad \|h\|^2 = \sum_{i,j=1}^{m} g(h(e_i, e_j), h(e_i, e_j)) \] (2.6)
for any $e_i, e_j \in TM$ and $e_r \in T^\perp M$.

The mean curvature vector $H$ is defined as $H = \frac{1}{m}(\text{trace } h)$. We say that the submanifold $M$ is minimal, if the mean curvature vector $H$ vanishes.
identically. It is well known that for a cosymplectic manifold
\[ h(X, \xi) = 0. \] (2.7)

For a given orthonormal frame \( \{e_1, e_2, \ldots, e_m\} \) of a differentiable distribution \( D \), we denote the squared norms of \( P \) and \( F \) respectively, by
\[
\|P\|^2 = \sum_{i,j=1}^{m} g^2(e_i, Pe_j) \quad \text{and} \quad \|F\|^2 = \sum_{i=1}^{m} \|Fe_i\|^2. \] (2.8)

It can be readily seen that \( \|P\|^2 \) and \( \|F\|^2 \) are independent of the choice of the above orthonormal frame.

For any \( i = 1, 2, \ldots, m \) where \( \{e_1, e_2, \ldots, e_m, \xi\} \) is a local orthonormal frame, we have
\[
\sum_{j=1}^{m} g^2(e_i, \phi e_j) = \cos^2 \theta. \] (2.9)

A plane section \( \pi \) in a cosymplectic manifold \( M \) is said to be a \( \phi \)-section, if it is spanned by a unit tangent vector \( X \) orthonormal to \( \xi \) and \( \phi X \), i.e.
\[ K(\pi) = K(X, \phi X) = g(R(X, \phi X)\phi X, X). \] (2.10)

The sectional curvature of a \( \phi \)-section is called \( \phi \)-sectional curvature. A cosymplectic manifold \( M \) with constant \( \phi \)-sectional curvature \( c \) is said to be a cosymplectic space form and is usually denoted by \( M(c) \).

For an orthonormal basis \( \{e_1, e_2, \ldots, e_m, e_{m+1} = \xi\} \) of the tangent space \( T_pM \) at \( p \in M \), from (1.1), the scalar curvature \( \tau \) at \( p \) of \( M \) assumes the form
\[
2\tau = \sum_{i \neq j}^{m} K(e_i \wedge e_j) + 2 \sum_{i=1}^{m} K(e_i \wedge \xi). \] (2.11)

Now, we mention the following results for our subsequent use.

**Corollary 2.1.** [12] Let \( M \) be a slant submanifold of an almost contact metric manifold \( \overline{M} \) with slant angle \( \theta \). Then for any \( X, Y \in TM \), we have
\[
g(PX, PY) = \cos^2 \theta \{g(X, Y) - \eta(X)\eta(Y)\} \] (2.12)
\[
g(FX, FY) = \sin^2 \theta \{g(X, Y) - \eta(X)\eta(Y)\}. \] (2.13)

**Lemma 2.1.** [6] Let \( a_1, a_2, \ldots, a_k, c \) be \( k + 1 \) \((k \geq 2)\) real numbers such that
\[
\left( \sum_{i=1}^{k} a_i \right)^2 = (k-1) \left( \sum_{i=1}^{k} a_i^2 + c \right). \]

Then \( 2a_1a_2 \geq c \) and the equality holds if and only if \( a_1 + a_2 = a_3 = \cdots = a_k \).
3. Chen’s inequality for bi-slant submanifolds in cosymplectic space forms

**Theorem 3.1.** Let \( \psi : M \to \overline{M} \) be an isometric immersion from a Riemannian \((m + 1 = 2n_1 + 2n_2 + 1)\)-dimensional bi-slant submanifold \( M \) into a cosymplectic space form \( \overline{M}(c) \) of dimension \( 2m + 1 \). Then, we have

\[
\tau - K(\pi) \leq \frac{(m + 1)^2(m - 1)}{2m} \|H\|^2 + \frac{c}{8}(m + 1)(m - 2) + \frac{3c}{4}((n_1 - 1)\cos^2\theta_1 + n_2\cos^2\theta_2) \quad (3.1)
\]

on \( D_1 \), and

\[
\tau - K(\pi) \leq \frac{(m + 1)^2(m - 1)}{2m} \|H\|^2 + \frac{c}{8}(m + 1)(m - 2) + \frac{3c}{4}(n_1\cos^2\theta_1 + (n_2 - 1)\cos^2\theta_2) \quad (3.2)
\]

on \( D_2 \).

The equality cases in (3.1) and (3.2) hold at a point \( p \in M \) if and only if there exist an orthonormal basis \( \{e_1, e_2, \ldots, e_m, e_{m+1} = \xi\} \) of \( T_pM \) and an orthonormal basis \( \{e_{m+2}, e_{m+3}, \ldots, e_{2m+1}\} \) of \( T^\perp_pM \) such that the shape operators of \( M \) in \( \overline{M}(c) \), at a point \( p \) take the following forms

\[
A_{m+2} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \lambda I_{m-1} \\ \end{pmatrix}, \quad a + b = \lambda \quad (3.3)
\]

\[
A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\ h_{21}^r & h_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \end{pmatrix}, \quad r \in \{m + 3, \ldots, 2m + 1\} \quad (3.4)
\]

**Proof.** Using Gauss equation in the expression of the curvature tensor \( \overline{R} \) of cosymplectic space form \( \overline{M}(c) \) given by (2.3), we obtain

\[
\begin{align*}
R(X, Y, Z, W) &= g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)) \\
&+ \frac{c}{4}\left\{ g(Y, Z)g(X, W) - g(X, Z)g(Y, W) + \eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) - \eta(Y)\eta(W)g(X, Z) - \eta(X)\eta(W)g(Y, Z) - g(\phi X, Z)g(\phi Y, W) + g(\phi Y, Z)g(\phi X, W) + 2g(X, \phi Y)g(\phi Z, W) \right\} 
\end{align*}
\]

for any \( X, Y, Z, W \in TM \).
For an orthonormal basis \( \{e_1, e_2, \ldots, e_m, e_{m+1} = \xi \} \) of \( T_p M \) at \( p \in M \), putting \( X = W = e_i \) and \( Y = Z = e_j, \forall i, j \in \{1, \ldots, m+1\} \), in (3.5), we get
\[
\sum_{i,j=1}^{m+1} R(e_i, e_j, e_j, e_i) = g(h(e_i, e_i), h(e_j, e_j)) - g(h(e_i, e_j), h(e_j, e_i)) \\
+ \frac{c}{4} \left\{ g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i) \right\} + \frac{c}{4} \left\{ \eta(e_i)\eta(e_j)g(e_j, e_j) - \eta(e_j)\eta(e_i)g(e_j, e_j) - g(\phi e_i, e_j)g(\phi e_j, e_i) + 2g(e_i, \phi e_j)g(\phi e_j, e_i) \right\}
\]
or,
\[
\sum_{i,j=1}^{m+1} R(e_i, e_j, e_j, e_i) = (m+1)^2\|H\|^2 - \|h\|^2 + \frac{c}{4} \left\{ (m+1)^2 - (m+1) \right\}
\]
or,
\[
\sum_{i \neq j}^m R(e_i, e_j, e_j, e_i) + 2 \sum_{i=1}^m R(e_i, \xi, \xi, e_i) = (m+1)^2\|H\|^2 - \|h\|^2 \\
+ \frac{c}{4} \left\{ (m+1)^2 - (m+1) \right\} + \frac{c}{4} \left\{ -2m + 3 \sum_{i,j=1}^{m+1} g^2(e_i, \phi e_j) \right\}
\]
Now using (2.11) in the above equation, we get
\[
2\tau = (m+1)^2\|H\|^2 - \|h\|^2 + \frac{c}{4} m(m+1) + \frac{c}{4} \left\{ -2m + 3 \sum_{i,j=1}^{m+1} g^2(e_i, \phi e_j) \right\}
\]
or,
\[
2\tau = (m+1)^2\|H\|^2 - \|h\|^2 + \frac{c}{4} m(m-1) + 3\frac{c}{4} \sum_{i,j=1}^{m+1} g^2(e_i, \phi e_j).
\]
Since \( M^{m+1} \) is bi-slant submanifold of a cosymplectic space form \( M^{2m+1} \), where \( (m+1) = 2n_1 + 2n_2 + 1 \), we may consider an adapted bi-slant orthonormal frame as follows:
\[
e_1, e_2 = \sec \theta_1 Pe_1, \ldots, e_{2n_1-1}, e_{2n_1} = \sec \theta_1 Pe_{2n_1-1} \\
e_{2n_1+1}, e_{2n_1+2} = \sec \theta_2 Pe_{2n_1+1}, \ldots, e_{2n_1+2n_2-1}, e_{2n_1+2n_2} = \sec \theta_2 Pe_{2n_1+2n_2-1} \quad \text{and} \quad e_{2n_1+2n_2+1} = \xi.
\]
Then, we have
\[ g(e_1, \phi e_2) = -g(\phi e_1, e_2) = -g(\phi_1, \sec \theta_1 Pe_1) \]
or,
\[ g(e_1, \phi e_2) = -\sec \theta_1 g(Pe_1, Pe_1). \]
Now, using (2.12), we get
\[ g(e_1, \phi e_2) = -\cos \theta_1 \]
or,
\[ g^2(e_1, \phi e_2) = \cos^2 \theta_1. \]
Similarly,
\[ g^2(e_i, \phi e_{i+1}) = \begin{cases} \cos^2 \theta_1, & \text{for } i = 1, \ldots, 2n_1 - 1 \\ \cos^2 \theta_2, & \text{for } i = 2n_1 + 1, \ldots, 2n_1 + 2n_2 - 1. \end{cases} \]
Hence, we have
\[ m+1 \sum_{i, j=1}^{m+1} g^2(e_i, \phi e_j) = 2 \{ n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2 \}. \]
Using this relation in (3.6), we obtain
\[ 2\tau = (m+1)^2 \|H\|^2 - \|h\|^2 + \frac{c}{4} m(m-1) + \frac{3c}{4} [2(n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2)]. \quad (3.7) \]
Putting
\[ \epsilon = 2\tau - \frac{(m+1)^2(m-1)}{m} \|H\|^2 - \|h\|^2 + \frac{c}{4} m(m+1)(m-2) - \frac{3c}{2} [n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2] \]
in (3.7), we get
\[ \epsilon = \frac{(m+1)^2}{m} \|H\|^2 - \|h\|^2 + \frac{c}{2} \]
or,
\[ (m+1)^2 \|H\|^2 = m\|h\|^2 + m \left\{ \epsilon - \frac{c}{2} \right\}. \quad (3.9) \]
Let \( p \in M, \pi \subset T_p M, \dim \pi = 2 \) and \( \pi \) is orthogonal to \( \xi \).
Now, we consider the following two cases:
**Case (i).** Let \( \pi \) be tangent to the differentiable distribution \( D_1 \) and let it be spanned by orthonormal basis vectors \( e_1 \) and \( e_2 \). If we take \( e_{m+2} \) in the direction of mean curvature vector \( H \) i. e. \( \epsilon_{m+2} = \frac{H}{\|H\|} \), then from (3.9), we get
\[ \left( \sum_{i=1}^{m+1} h_{i,i}^{m+2} \right)^2 = m \left\{ \sum_{i=1}^{m+1} \left( h_{i,i}^{m+2} \right)^2 + \sum_{i \neq j} \left( h_{i,j}^{m+2} \right)^2 + \sum_{r=m+3}^{2m+1} \sum_{i,j} \left( h_{i,j}^r \right)^2 + \epsilon - \frac{c}{2} \right\}. \quad (3.10) \]
Now using lemma (2.1) in (3.10), we get

\[ 2h_{11}^{m+2}h_{22}^{m+2} \geq \sum_{i \neq j} (h_{ij}^{m+2})^2 + \sum_{r=m+3}^{2m+1} \sum_{i,j} (h_{ij}^r)^2 + \epsilon - \frac{c}{2}. \]  

(3.11)

On the other hand, we have

\[ K(\pi) = R(e_1, e_2, e_2, e_1) = g(h(e_1, e_1), h(e_2, e_2)) - g(h(e_1, e_2), h(e_1, e_2)) + \frac{c}{4} + \frac{3c}{4} \cos^2 \theta_1 \]

or,

\[ K(\pi) = \sum_{r=m+2}^{2m+1} \left\{ g(h(e_1, e_1), e_r) g(h(e_2, e_2), e_r) - g(h(e_1, e_2), e_r) g(h(e_1, e_2), e_r) + \frac{c}{4} + \frac{3c}{4} \cos^2 \theta_1 \right\} \]

or,

\[ K(\pi) = \sum_{r=m+2}^{2m+1} \left\{ h_{11}^r h_{22}^r - (h_{12}^r)^2 \right\} + \frac{c}{4} + \frac{3c}{4} \cos^2 \theta_1 \]

(3.12)

or,

\[ K(\pi) = h_{11}^{m+2}h_{22}^{m+2} + \sum_{r=m+3}^{2m+1} h_{11}^r h_{22}^r - \sum_{r=m+2}^{2m+1} (h_{12}^r)^2 + \frac{c}{4} + \frac{3c}{4} \cos^2 \theta_1. \]

Using (3.11) in the above equation, we obtain

\[ k(\pi) \geq \frac{1}{2} \sum_{i \neq j} (h_{ij}^{m+2})^2 + \frac{1}{2} \sum_{r=m+3}^{2m+1} \sum_{i,j=1}^{2m+1} (h_{ij}^r)^2 \]

\[ + \frac{\epsilon}{2} - \frac{c}{4} + \sum_{r=m+3}^{2m+1} h_{11}^r h_{22}^r - \sum_{r=m+2}^{2m+1} (h_{12}^r)^2 + \frac{c}{4} + \frac{3c}{4} \cos^2 \theta_1 \]

or,

\[ K(\pi) \geq \frac{\epsilon}{2} + \frac{3c}{4} \cos^2 \theta_1. \]

(3.13)

Now using (3.8) in (3.13), we obtain

\[ \tau - K(\pi) \leq \frac{(m+1)^2(m-1)}{2m} ||H||^2 + \frac{c}{8} (m+1)(m-2) + \frac{3c}{4} [n_1 \cos^2 \theta_1 + n_2 \cos^2 \theta_2]. \]
Case (ii). If \( \pi \) is tangent to \( D_2 \), we obtain, as in Case (i)
\[
\tau - K(\pi) \leq \frac{(m + 1)^2(m - 1)}{2m} \|H\|^2 + \frac{c}{8}(m + 1)(m - 2) \bigg( \frac{3c}{4} n_1 \cos^2 \theta_1 + (n_2 - 1) \cos^2 \theta_2 \bigg).
\]
These are the desired inequalities.
If at any point \( p \in M \), equality in (3.1) and (3.2) hold, then the inequalities in (3.11) and (3.13) become equalities. Hence, we have
\[
\begin{align*}
&h^m_{1j} = h^m_{2j} = h^m_{ij} = 0, \quad \forall i \neq j > 2 \\
&h^r_{ij} = 0, \quad \forall i \neq j, \quad i, j = 3, \ldots, 2m + 1, r = m + 3, \ldots, 2m + 1 \\
&h^r_{11} + h^r_{22} = 0, \quad \forall r = m + 3, \ldots, 2m + 1 \\
&h^m_{11} + h^m_{22} = h^m_{33} = \cdots = h^m_{m+1,m+1}.
\end{align*}
\]
Now, if we take \( e_1, e_2 \) such that \( h^m_{12} = 0 \) and letting \( a = h^r_{11} \), \( b = h^r_{22} \), \( \lambda = h^m_{33} = \cdots = h^m_{m+1,m+1} \), it follows that the shape operators assume the desired form.

**Corollary 3.1.** Let \( M \) be an \( m + 1 \)-dimensional contact CR-submanifold with in a \( 2m + 1 \)-dimensional cosymplectic space form \( \overline{M}(c) \). Then, we have
\[
\tau - K(\pi) \leq \frac{(m + 1)^2(m - 1)}{2m} \|H\|^2 + \frac{c}{8}(m + 1)(m - 2) + \frac{3c}{4}(n_1 - 1)
\]
on \( D_1 \), and
\[
\tau - K(\pi) \leq \frac{(m + 1)^2(m - 1)}{2m} \|H\|^2 + \frac{c}{8}(m + 1)(m - 2) + \frac{3c}{4} n_1
\]
on \( D_2 \).

Now, we have the following result.

**Theorem 3.2.** Let \( M \) be an \( (m + 1) \)-dimensional \( \theta \)-slant submanifold with \( \theta_1 = \theta_2 = \theta \) in a \( (2m + 1) \)-dimensional cosymplectic space form \( \overline{M}(c) \). Then, we have
\[
\delta_M \leq \frac{(m + 1)^2(m - 1)}{2m} \|H\|^2 + \frac{c}{8}(m + 1)(m - 2) + \frac{3c}{8}(m - 2) \cos^2 \theta.
\]
The equality holds at a point \( p \in M \) if and only if there exists an orthonormal basis \( \{e_1, e_2, \ldots, e_m, e_{m+1} = \xi\} \) of \( T_pM \) and an orthonormal basis \( \{e_{m+2}, e_{m+3}, \ldots, e_{2m+1}\} \) of \( T_p^\perp M \) such that the shape operators of \( M \)
in cosymplectic space form $\overline{M}(c)$ take the following forms

\[
A_{m+2} = \begin{pmatrix}
a & 0 & 0 & \cdots & 0 \\
0 & b & 0 & \cdots & 0 \\
0 & 0 & \cdots & \cdots & 0 \\
\end{pmatrix}, \quad a + b = \lambda
\]

\[
A_{e_r} = \begin{pmatrix}
h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\
h_{12}^r & -h_{11}^r & 0 & \cdots & 0 \\
0 & 0 & \cdots & \cdots & 0 \\
\end{pmatrix}, \quad r = m + 3, \ldots, 2m + 1.
\]

**Corollary 3.2.** Let $M$ be an $(m+1)$-dimensional invariant submanifold of a $(2m+1)$-dimensional cosymplectic space form $\overline{M}(c)$. Then, we have

\[
\delta_M \leq \frac{c(m^2 + 2m - 8)}{8}.
\]

**Corollary 3.3.** Let $M$ be an $(m+1)$-dimensional anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic space form $\overline{M}(c)$. Then, we have

\[
\delta_M \leq \frac{(m+1)^2(m-1)}{2m} \|H\|^2 + \frac{c}{8}(m+1)(m-2).
\]

4. **Examples of bi-slant submanifolds of cosymplectic manifolds**

**Example 4.1.** For any $\theta_1, \theta_2 \in [0, \pi/2]$

\[
x(u, v, w, s, z) = (u, 0, w, 0, v \cos \theta_1, v \sin \theta_1, s \cos \theta_2, s \sin \theta_2, z)
\]

defines a 5-dimensional bi-slant submanifold $M$, with slant angles $\theta_1$ and $\theta_2$ in $\mathbb{R}^9$ with its usual cosymplectic structure $(\phi_0, \xi, \eta, g)$, given by:

\[
\eta = dz, \quad \xi = \frac{\partial}{\partial z}
\]

\[
g = \eta \otimes \eta + \left\{ \sum_{i=1}^{4} (dx^i \otimes dx^i + dy^i \otimes dy^i) \right\}
\]

and

\[
\phi_0 \left\{ \sum_{i=1}^{4} \left( X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i} \right) + Z \frac{\partial}{\partial z} \right\} = \sum_{i=1}^{4} \left( -Y_i \frac{\partial}{\partial x^i} + X_i \frac{\partial}{\partial y^i} \right).
\]

Furthermore it is easy to see that:

\[
e_1 = \frac{\partial}{\partial x^1}, \quad e_2 = \cos \theta_1 \frac{\partial}{\partial y^1} + \sin \theta_1 \frac{\partial}{\partial y^2}, \quad e_3 = \frac{\partial}{\partial x^3}
\]

\[
e_4 = \cos \theta_2 \frac{\partial}{\partial y^3} + \sin \theta_1 \frac{\partial}{\partial y^4} \quad \text{and} \quad e_5 = \frac{\partial}{\partial z} = \xi
\]
form a local orthonormal frame of $TM$. If, we define $D_1 = \{e_1, e_2\}$ and $D_2 = \{e_3, e_4\}$, then a simple computation yields, $g(\phi_0 e_1, e_2) = \cos \theta_1$ and $g(\phi_0 e_3, e_4) = \cos \theta_2$ proving that the distribution $D_1$ is $\theta_1$-slant and the distribution $D_2$ is $\theta_2$-slant.

Example 4.2. For any $\theta_1, \theta_2 \in [0, \pi/2]$

$$x(u, v, w, s, z) = (\cos \alpha_1 \cos \alpha_2 u - \sin \alpha_1 s, \sin \alpha_1 \cos \alpha_2 u$$

$$+ \cos \alpha_1 \sin \alpha_2 u, \sin \alpha_1 \sin \alpha_2 u, w, - \sin \alpha_2 v, 0, \cos \alpha_2 v, z)$$

defines a 5-dimensional bi-slant submanifold $M$, with slant angles $\theta_1 = \pi/2$ and $\cos^2 \theta_2 = \sin^2 \alpha_1$ in $R^9$ with its usual cosymplectic structure.

We can choose orthonormal frame on $TM$, given by

$$e_1 = (\cos \alpha_1 \cos \alpha_2, \sin \alpha_1 \cos \alpha_2, \cos \alpha_1 \sin \alpha_2, \sin \alpha_1 \sin \alpha_2, 0, 0, 0, 0, 0)$$

$$e_2 = - \sin \alpha_2 \frac{\partial}{\partial y^2} + \cos \alpha_2 \frac{\partial}{\partial y^3}, \quad e_3 = \frac{\partial}{\partial y^4}$$

$$e_4 = - \sin \alpha_1 \frac{\partial}{\partial x^1} + \cos \alpha_1 \frac{\partial}{\partial x^2} \quad \text{and} \quad e_5 = \frac{\partial}{\partial z} = \xi$$

where, distributions are defined by $D_1 = \{e_1, e_2\}$ and $D_2 = \{e_3, e_4\}$. Then it can be easily seen that $g(e_1, \phi_0 e_2) = 0$ and $g(e_3, \phi_0 e_4) = \sin \alpha_1$, that is, distribution $D_1$ is $\theta_1$-slant with $\theta_1 = \pi/2$ and the distribution $D_2$ is $\theta_2$-slant with $\cos^2 \theta_2 = \sin^2 \alpha_1$.

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References


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