MORE ON THE BORWEIN-DITOR THEOREM

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Dedicated to Professor Mustafa Kulenović on the occasion of his 60th birthday

Abstract. In 1978 D. Borwein and S. Z. Ditor published a paper answering a question of P. Erdos. Since then several authors including N. Bingham, P. Komjath, H. Miller, L. Miller-Van Wieren, A. Ostaszewski have generalized and extended their result. In this paper a significant generalization of all previous results is presented.

1. Introduction

D. Borwein and S.Z. Ditor [1] have proved the following theorem, answering a question of P. Erdos.

Theorem 1.1. (Borwein, Ditor 1978)

1. If $A$ is a measurable set in $\mathbb{R}$ with $m(A) > 0$, and $(d_n)$ is a sequence of reals converging to 0, then for almost all $x \in A$, $x + d_n \in A$ for infinitely many $n$.

2. There exists a measurable set $A$ in $\mathbb{R}$ with $m(A) > 0$, and a (decreasing) sequence $(d_n)$ converging to 0, such that, for each $x$, $x + d_n \notin A$ for infinitely many $n$.

In [2] H. Miller extended the Borwein-Ditor theorem by using a general function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ instead of addition.


In [4] H. Miller and L. Miller-Van Wieren extended a result in the last mentioned paper from $\mathbb{R}$ to $\mathbb{R}^2$.

In this paper we prove a double extension of the original Borwein-Ditor theorem.

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2. Result

In [4], it is shown that if \( A \) is any nowhere dense subset of \([0, 1]\), then there exists a sequence \((d_n)\) converging to zero such that, for each \( x, x + d_n \notin A \) for infinitely many \( n \). We now present a result analogous to the last mentioned theorem for a general \( f, f : RxR \rightarrow R \), in place of addition.

**Theorem 2.1.** Suppose \( f : RxR \rightarrow R \) is continuous and satisfies:

1. there exists \( e \in R \) such that \( f(x, e) = x \) for all \( x \in R \),
2. there exist \((x_1, y_1) \in R^2, x_0 \in R, x_1 > x_0, y_1 > e, \) such that the partial derivatives \( f_x, f_y \) exist and are continuous on \( T \) the closed rectangle with corners \((x_0, e), (x_0, y_1), (x_1, e), \) and \((x_1, y_1), \)
3. there exist \( a, b \in R, \) with \( a, b > 0, b > 1 \) and \( a < f_x, f_y < b \) on \( T \).

If \( E \subset [x_0, x_1] \) is an arbitrary nowhere dense set, then there exists \( \{e_n\}_{n=1}^\infty, \) monotonically converging to \( e \) such that \( f(x, e_n) \notin E \) infinitely often for each \( x \in R \).

**Proof.** Suppose \( n \geq 2 \) is arbitrarily fixed. Divide \([x_0, x_1]\) into \( 2^n \) adjoining intervals each of length \( \frac{x_1-x_0}{2^n} \) and denote them \( \{I_{nk}\}, k = 1, 2 \ldots 2^n \).

Since \( E \) is nowhere dense, each \( I_{nk} \) contains an open interval \( J_{nk} (k = 1, 2 \ldots, 2^n) \) disjoint from \( E \). Let \( s_n \) denote the minimum length among the intervals \( J_{nk}, k = 1, 2 \ldots 2^n \).

Now suppose \( x \in [x_0, x_1 - \frac{x_1-x_0}{2^n}] \) is arbitrary.

Consider the finite sequence \( d_n, 2d_n, 3d_n, \ldots, m(n)d_n, \) where \( d_n = \frac{s_n}{2^n} \) and \( m(n) \) is the smallest integer such that \( m(n)d_n > \frac{x_1-x_0}{2^n} \).

Notice \( d_n < \frac{s_n}{2^n} \) and from our choice of \( m(n) \) we have that \( (m(n)-1)d_n a \leq \frac{x_1-x_0}{2^n} \) from which \( m(n)d_n \leq \frac{x_1-x_0}{2^n} + d_n \).

Now examine the sequence

\[
f(x, e) = x, f(x, e + d_n), f(x, e + 2d_n), \ldots, f(x, e + m(n)d_n).
\]

Since \( x \in I_{nk} \) for some \( k \in \{1, 2, \ldots, 2^n - 1\} \), then we claim that the above sequence is strictly increasing and the difference of successive terms is less than \( \frac{s_n}{2^n} \) and \( f(x, e + m(n)d_n) - f(x, e) > \frac{2(x_1-x_0)}{2^n} \).

To see this observe:

\[
\frac{s_n}{2} = bd_n > f(x, e + (k + 1)d_n) - f(x, e + kd_n) > ad_n > 0
\]

for \( 0 \leq k \leq m(n) - 1 \) and

\[
f(x, e + m(n)d_n) - f(x, e) > m(n) ad_n > \frac{x_1-x_0}{2^n-1} > \frac{2(x_1-x_0)}{2^n}.
\]
So if \( x \in [x_0, x_1] \), there exists a \( j_x \in \{0, 1, \ldots, m(n)\} \) such that \( f(x, e + j_x d_n) \notin E \).

The set \( \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{m(n)} \{kd_n\} \) can be arranged as a monotonic non-increasing sequence \( \{h_n\}_{n=1}^{\infty} \) converging to zero (which is clear from earlier computations) and let \( \{e_n\}_{n=1}^{\infty} = \{e + h_n\}_{n=1}^{\infty} \). Then clearly, for each \( x \in [x_0, x_1] \), \( f(x, e_n) \notin E \) for infinitely many \( n \). The same is trivially true for \( x_1 \), due to the positive partial derivatives at \((x_1, e)\) and for \( x \notin [x_0, x_1] \), due to the continuity of \( f \).

This completes the proof. \( \square \)

**References**


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