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MORE ON THE BORWEIN-DITOR THEOREM

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Dedicated to Professor Mustafa Kulenović on the occasion of his 60^{th} *birthday*

Abstract. In 1978 D. Borwein and S. Z. Ditor published a paper answering a question of P. Erdos. Since then several authors including N. Bingham, P. Komjath, H. Miller, L. Miller-Van Wieren, A. Ostaszewski have generalized and extended their result. In this paper a significant generalization of all previous results is presented.

1. Introduction

D. Borwein and S.Z. Ditor [1] have proved the following theorem, answering a question of P. Erdos.

Theorem 1.1. (Borwein, Ditor 1978)

(1) *If A is a measurable set in R with $m(A) > 0$, and (d_n) is a sequence of reals converging to 0, then for almost all $x \in A$, $x + d_n \in A$ for infinitely many n.*
(2) *There exists a measurable set A in R with $m(A) > 0$, and a (decreasing) sequence (d_n) converging to 0, such that, for each x, $x + d_n \notin A$ for infinitely many n.*

In [2] H. Miller extended the Borwein-Ditor theorem by using a general function $f : RXR \longrightarrow R$ instead of addition.

In [3] N. Bingham and A. Ostaszewski considered homotopy and its relation to the Borwein-Ditor theorem.

In [5] H. Miller and A. Ostaszewski considered general spaces, group action and shift-compactness and their relations to the Borwein-Ditor theorem.

In [4] H. Miller and L. Miller-Van Wieren extended a result in the last mentioned paper from R to R^2.

In this paper we prove a double extension of the original Borwein-Ditor theorem.

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2. Result

In [4], it is shown that if A is any nowhere dense subset of $[0,1]$, then there exists a sequence (d_n) converging to zero such that, for each x, $x+d_n \notin A$ for infinitely many n. We now present a result analogous to the last mentioned theorem for a general f, $f : RxR \longrightarrow R$, in place of addition.

Theorem 2.1. *Suppose $f : RxR \longrightarrow R$ is continuous and satisfies:*

(1) *there exists $e \in R$ such that $f(x,e) = x$ for all $x \in R$,*
(2) *there exist $(x_1, y_1) \in R^2$, $x_0 \in R$, $x_1 > x_0$, $y_1 > e$, such that the partial derivatives f_x, f_y exist and are continuous on T the closed rectangle with corners (x_0, e), (x_0, y_1), (x_1, e), and(x_1, y_1),*
(3) *there exist $a, b \in R$, with $a, b > 0$, $b > 1$ and $a < f_x, f_y < b$ on T.*

If $E \subset [x_0, x_1]$ is an arbitrary nowhere dense set, then there exists $\{e_n\}_{n=1}^{\infty}$, monotonically converging to e such that $f(x, e_n) \notin E$ infinitely often for each $x \in R$.

Proof. Suppose $n \geq 2$ is arbitrarily fixed. Divide $[x_0, x_1]$ into 2^n adjoining intervals each of length $\frac{x_1-x_0}{2^n}$ and denote them $\{I_{nk}\}$, $k = 1, 2 \ldots 2^n$.

Since E is nowhere dense, each I_{nk} contains an open interval J_{nk} ($k = 1, 2 \ldots, 2^n$) disjoint from E. Let s_n denote the minimum length among the intervals J_{n_k}, $k = 1, 2 \ldots 2^n$.

Now suppose $x \in [x_0, x_1 - \frac{x_1-x_0}{2^n}]$ is arbitrary.

Consider the finite sequence $d_n, 2d_n, 3d_n, \ldots, m(n)d_n$, where $d_n = \frac{s_n}{2b}$ and $m(n)$ is the smallest integer such that $m(n)d_n a > \frac{x_1-x_0}{2^{n-1}}$.

Notice $d_n < \dfrac{s_n}{2}$ and from our choice of $m(n)$ we have that $(m(n)-1)d_n a \leq \frac{x_1-x_0}{2^{n-1}}$ from which $m(n)d_n \leq \frac{x_1-x_0}{a2^{n-1}} + d_n$.

Now examine the sequence

$$f(x,e) = x, f(x, e+d_n), f(x, e+2d_n), \ldots,\ f(x, e+m(n)\,d_n).$$

Since $x \in I_{n_k}$ for some $k \in \{1, 2, \ldots, 2^n - 1\}$, then we claim that the above sequence is strictly increasing and the difference of successive terms is less than $\frac{s_n}{2}$ and $f(x, e+m(n)d_n) - f(x,e) > \frac{2(x_1-x_0)}{2^n}$.

To see this observe:

$$\frac{s_n}{2} = bd_n > f(x, e+(k+1)d_n) - f(x, e+kd_n) > ad_n > 0$$

for $0 \leq k \leq m(n) - 1$ and

$$\begin{aligned} f(x, e+m(n)\,d_n) - f(x,e) > m(n)\,ad_n &> \frac{x_1-x_0}{2^{n-1}} \\ &> \frac{2(x_1-x_0)}{2^n}. \end{aligned}$$

So if $x \in [x_0, x_1 - \frac{x_1 - x_0}{2^n}]$, there exists a $j_x \in \{0, 1, \ldots, m(n)\}$ such that $f(x, e + j_x d_n) \notin E$.

The set $\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{m(n)} \{kd_n\}$ can be arranged as a monotonic non-increasing sequence $\{h_n\}_{n=1}^{\infty}$ converging to zero (which is clear from earlier computations) and let $\{e_n\}_{n=1}^{\infty} = \{e + h_n\}_{n=1}^{\infty}$. Then clearly, for each $x \in [x_0, x_1)$, $f(x, e_n) \notin E$ for infinitely many n. The same is trivially true for x_1, due to the positive partial derivatives at (x_1, e) and for $x \notin [x_0, x_1]$, due to the continuity of f.

This completes the proof. □

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