COEFFICIENT ESTIMATES FOR SAKAGUCHI TYPE FUNCTIONS

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Abstract. Let $S_{n,\mu}^\alpha(\alpha, t)$ be the class of normalized analytic functions defined in the open unit disk satisfying

$$\Re \left( \frac{(1-t)z(D_{n,\mu}^\alpha f(z))^\prime}{D_{n,\mu}^\alpha f(z) - D_{n,\mu}^\alpha f(tz)} \right) > \alpha, \quad |t| \leq 1, \ t \neq 1$$

for some $0 \leq \alpha < 1$ and $D_{n,\mu}^\alpha$ is a linear multiplier differential operator defined by the authors in [2]. The object of the present paper is to discuss some properties of functions $f(z)$ belonging to the classes $S_{n,\mu}^\alpha(\alpha, t)$ and $T_{n,\mu}^\alpha(\alpha, t)$ where $f(z) \in T_{n,\mu}^\alpha(\alpha, t)$ if and only if $zf'(z) \in S_{n,\mu}^\alpha(\alpha, t)$.

1. Introduction

Let $\mathcal{A}$ denote the family of functions $f$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1.1)$$

which are analytic in the open unit disk $\mathcal{U} = \{ z : |z| < 1 \}$. For $f(z)$ belongs to $\mathcal{A}$, the multiplier differential operator $D_{n,\mu}^\alpha f$ was defined by the authors in [2] as follows

$$D_{\lambda,\mu}^0 f(z) = f(z)$$

$$D_{\lambda,\mu}^1 f(z) = D_{\lambda,\mu} f(z) = \lambda \mu z^2 (f(z))'' + (\lambda - \mu) z (f(z))^\prime + (1 - \lambda + \mu) f(z)$$

$$D_{\lambda,\mu}^2 f(z) = D_{\lambda,\mu} \left( D_{\lambda,\mu}^1 f(z) \right)$$

$$\vdots$$

$$D_{\lambda,\mu}^n f(z) = D_{\lambda,\mu} \left( D_{\lambda,\mu}^{n-1} f(z) \right)$$

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where $\lambda \geq \mu \geq 0$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

If $f$ is given by (1.1) then from the definition of the operator $D^{n}_{\lambda,\mu}f(z)$ it is easy to see that

$$D^{n}_{\lambda,\mu}f(z) = z + \sum_{k=2}^{\infty} \left[ 1 + (\lambda \mu k + \lambda - \mu) (k-1) \right]^n a_k z^k. \quad (1.2)$$

It should be remarked that the $D^{n}_{\lambda,\mu}$ is a generalization of many other linear operators considered earlier by different authors. In particular, for $f \in \mathcal{A}$ we have the following:

- $D^{n}_{1,0}f(z) \equiv D^{n}f(z)$ the operator investigated by Sălăgean (see [4]).
- $D^{n}_{\lambda,0}f(z) \equiv D^{n}_{\lambda}f(z)$ the operator studied by Al-Oboudi (see [3]).
- $D^{n}_{\lambda,\mu}f(z)$ the operator firstly considered for $0 \leq \mu \leq \lambda \leq 1$, by Răducanu and Orhan (see [1]).

A function $f(z) \in \mathcal{A}$ is said to be in the class $S^{n}_{\lambda,\mu}(\alpha,t)$ if it satisfies

$$\Re \left( \frac{(1-t)z(D^{n}_{\lambda,\mu}f(z))'}{D^{n}_{\lambda,\mu}f(z) - D^{n}_{\lambda,\mu}f(tz)} \right) > \alpha, \quad |t| \leq 1, \ t \neq 1 \quad (1.3)$$

for all $z \in U$ and some $\alpha (0 \leq \alpha < 1)$.

We also denote by $T^{n}_{\lambda,\mu}(\alpha,t)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ such that $zf'(z) \in S^{n}_{\lambda,\mu}(\alpha,t)$. The class $S^{0}_{\lambda,\mu}(0,-1)$ was introduced by Sakaguchi [5]. Therefore, a function $f(z) \in S^{0}_{\lambda,\mu}(\alpha,-1)$ is called Sakaguchi function of order $\alpha$ (see [6] and [8]). Further, the class $S^{0}_{\lambda,\mu}(\alpha,t)$ was introduced and studied by Owa et al. [7]. Various Sakaguchi type functions were investigated and studied by many authors including ([9], [10], [11]). We note that $S^{0}_{\lambda,\mu}(0,-1)$ is the class of starlike functions with respect to symmetric points in $U$. Also $S^{0}_{\lambda,\mu}(\alpha,0) = S^*(\alpha)$ and $T^{0}_{\lambda,\mu}(\alpha,0) = C(\alpha)$ which are, respectively, the familiar classes of starlike functions of order $\alpha (0 \leq \alpha < 1)$ and convex functions of order $\alpha (0 \leq \alpha < 1)$. Incidentally the class of uniformly starlike functions introduced by Goodman [12] as follows

$$UST = \left\{ f(z) \in \mathcal{A} : \Re \left( \frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)} \right) > 0 \right\}, \quad (z,\zeta) \in U \times U.$$

Ronning [13] showed the following important result.

**Remark 1.1.** $f(z) \in UST$ if and only if for every $z \in U$, $|t| = 1$

$$\Re \left( \frac{(1-t)zf'(z)}{f(z)-f(tz)} \right) > 0.$$

Now we will give some results for functions belonging to the classes $S^{m}_{0,\lambda,\mu}(\alpha,t)$ and $T^{m}_{0,\lambda,\mu}(\alpha,t)$. 

Theorem 2.1. If $f(z) \in \mathcal{A}$ satisfies
\[
\sum_{k=2}^{\infty} A^n_k \{ |k - u_k| + (1 - \alpha) |u_k| \} |a_k| \leq 1 - \alpha,
\]
\[
u_k = 1 + t + t^2 + \cdots + t^{k-1}, \quad t(|t| \leq 1, t \neq 1) \quad (2.1)
\]
for some $\alpha (0 \leq \alpha < 1)$ then $f(z) \in S^n_{\lambda, \mu}(\alpha, t)$, where
\[
A^n_k = [1 + (\lambda k + \lambda - \mu)(k - 1)]^n.
\]

Proof. To prove Theorem 2.1, we show that if $f(z)$ satisfies (2.1) then
\[
\left| \frac{(1 - t) z (D^n_{\lambda, \mu} f(z))'}{D^n_{\lambda, \mu} f(z) - D^n_{\lambda, \mu} f(tz)} - 1 \right| < 1 - \alpha.
\]
Evidently, since
\[
\frac{(1 - t) z (D^n_{\lambda, \mu} f(z))'}{D^n_{\lambda, \mu} f(z) - D^n_{\lambda, \mu} f(tz)} - 1 = \frac{z + \sum_{k=2}^{\infty} k A^n_k a_k z^k - 1}{z + \sum_{k=2}^{\infty} k A^n_k u_k a_k z^k - 1} = \frac{\sum_{k=2}^{\infty} (k - u_k) A^n_k a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} A^n_k u_k a_k z^{k-1}}
\]
we see that
\[
\left| \frac{(1 - t) z (D^n_{\lambda, \mu} f(z))'}{D^n_{\lambda, \mu} f(z) - D^n_{\lambda, \mu} f(tz)} - 1 \right| \leq \frac{\sum_{k=2}^{\infty} A^n_k |k - u_k| |a_k|}{1 - \sum_{k=2}^{\infty} A^n_k |u_k| |a_k|}.
\]
Therefore, if $f(z)$ satisfies (2.1), then we have
\[
\left| \frac{(1 - t) z (D^n_{\lambda, \mu} f(z))'}{D^n_{\lambda, \mu} f(z) - D^n_{\lambda, \mu} f(tz)} - 1 \right| < 1 - \alpha.
\]
This completes the proof of Theorem 2.1.

Theorem 2.2. If $f(z) \in \mathcal{A}$ satisfies
\[
\sum_{k=2}^{\infty} k A^n_k \{ |k - u_k| + (1 - \alpha) |u_k| \} |a_k| \leq 1 - \alpha, \quad u_k = 1 + t + t^2 + \cdots + t^{k-1}
\]
for some $\alpha (0 \leq \alpha < 1)$ then $f(z) \in T^n_{\lambda, \mu}(\alpha, t)$, where
\[
A^n_k = [1 + (\lambda k + \lambda - \mu)(k - 1)]^n.
\]
Proof. Noting that \( f \in T_{\lambda,\mu}^n(\alpha, t) \) if and only if \( zf' \in S_{\lambda,\mu}^n(\alpha, t) \), we can prove Theorem 2.2.

We now define

\[
S_{0,\lambda,\mu}^n(\alpha, t) = \{ f \in \mathcal{A} : f \text{ satisfies (2.1)} \}
\]

and

\[
T_{0,\lambda,\mu}^n(\alpha, t) = \{ f \in \mathcal{A} : f \text{ satisfies (2.2)} \}
\]

In view of the above theorems, we see :

**Example 2.1.** Let us consider a function \( f(z) \) given by

\[
f(z) = z + (1 - \alpha) \left( \frac{\eta \delta_2}{2A_2^2(2 - \alpha)} z^2 + \frac{(1 - \eta)\delta_3}{A_3^2(7 - 3\alpha)} z^3 \right),
\]

\[
0 \leq \eta \leq 1, \quad |\delta_2| = |\delta_3| = 1.
\]

Then for any \( t(|t| \leq 1, t \neq 1) \), \( f(z) \in S_{0,\lambda,\mu}^n(\alpha, t) \subset S_{\lambda,\mu}^n(\alpha, t) \).

**Example 2.2.** Let us consider a function \( f(z) \) given by

\[
f(z) = z + (1 - \alpha) \left( \frac{\eta \delta_2}{4A_2^2(2 - \alpha)} z^2 + \frac{(1 - \eta)\delta_3}{3A_3^2(7 - 3\alpha)} z^3 \right),
\]

\[
0 \leq \eta \leq 1, \quad |\delta_2| = |\delta_3| = 1.
\]

Then for any \( t(|t| \leq 1, t \neq 1) \), \( f(z) \in T_{0,\lambda,\mu}^n(\alpha, t) \subset T_{\lambda,\mu}^n(\alpha, t) \).

**Remark 2.3.** If we take \( n = 0, t = -1 \) in Theorems 2.1 and 2.2 then we get the results given by Cho et al. [6].

3. **Coefficient inequalities**

Applying Carathéodory function \( p(z) \) defined by

\[
p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k
\]

in \( \mathcal{U} \), we discuss the coefficient inequalities for the functions \( f \) in the subclasses \( S_{\lambda,\mu}^n(\alpha, t) \) and \( T_{\lambda,\mu}^n(\alpha, t) \).
Theorem 3.1. If \( f(z) \in S^n_{\lambda,\mu}(\alpha, t) \), then

\[
|a_k| \leq \frac{\beta}{A_k^n} |v_k| \left\{ 1 + \beta \sum_{j=2}^{k-1} \frac{|u_j|}{|v_j|} + \beta^2 \sum_{j_2 > j_1, j_1 = 2}^{k-2} \frac{|u_{j_1} u_{j_2}|}{|v_{j_1} v_{j_2}|} + \beta^3 \sum_{j_3 > j_2, j_2 > j_1, j_1 = 2}^{k-3} \frac{|u_{j_1} u_{j_2} u_{j_3}|}{|v_{j_1} v_{j_2} v_{j_3}|} + \cdots + \beta^{k-2} \prod_{j=2}^{k-1} \frac{|u_j|}{|v_j|} \right\},
\]

where

\[
\beta = 2(1 - \alpha), \quad v_k = k - u_k.
\] (3.2)

Proof. We define the function \( p(z) \) by

\[
p(z) = \frac{1}{1 - \alpha} \left( (1 - t)z \left(D_{\lambda,\mu}^n f(z)\right)' - \alpha \right) = 1 + \sum_{k=1}^{\infty} p_k z^k \quad (3.3)
\]

for \( f(z) \in S^n_{\lambda,\mu}(\alpha, t) \). Then \( p(z) \) is a Caratheodory function and satisfies

\[
|p_k| \leq 2 \quad (k \geq 1). \quad (3.4)
\]

Since

\[
(1 - t)z \left(D_{\lambda,\mu}^n f(z)\right)' = [D_{\lambda,\mu}^n f(z) - D_{\lambda,\mu}^n f(tz)] \left[ \alpha + (1 - \alpha)p(z) \right],
\]

we have

\[
z + \sum_{k=2}^{\infty} kA_k^n a_k z^k = \left( z + \sum_{k=2}^{\infty} kA_k^n u_k a_k z^k \right) \left( 1 + (1 - \alpha) \sum_{k=1}^{\infty} p_k z^k \right)
\]

where

\[
u_k = 1 + t + t^2 + \cdots + t^{k-1}.
\]

So we get

\[
a_k = \frac{1 - \alpha}{A_k^n (k - u_k)} \left( p_1 A_{k-1}^n u_{k-1} a_{k-1} + p_2 A_{k-2}^n u_{k-2} a_{k-2} + \cdots + p_{k-2} A_2^n u_2 a_2 + p_{k-1} \right). \quad (3.5)
\]

From Eq. (3.5), we easily have that

\[
|a_2| = \left| \frac{(1 - \alpha)}{A_2^n (2 - u_2)} p_1 \right| \leq \frac{2 (1 - \alpha)}{A_2^n |2 - u_2|},
\]

\[
|a_3| \leq \frac{2 (1 - \alpha)}{A_3^n |3 - u_3|} (A_2^n |u_2 a_2| + 1) \leq \frac{2 (1 - \alpha)}{A_3^n |3 - u_3|} \left( 1 + 2 (1 - \alpha) \frac{|u_2|}{|2 - u_2|} \right)
\]
and
\[ |a_4| \leq \frac{2(1-\alpha)}{A_n^4 |4-u_4|} \left\{ 1 + 2(1-\alpha) \left( \frac{|u_2|}{|2-u_2|} + \frac{|u_3|}{|3-u_3|} \right) + 2^2 (1-\alpha)^2 \left( \frac{|u_2 u_3|}{|2-u_2| |3-u_3|} \right) \right\}. \]

Thus, using the mathematical induction, we obtain the inequality (3.2).

**Remark 3.2.** If we write \( \alpha = t = n = 0 \) in Theorem 3.1 then we have well known the result
\[ f \in S^* \implies |a_k| \leq k, \]
where \( S^* \) is usual the class of starlike functions.

**Remark 3.3.** If we take \( \alpha = \frac{1}{2}, t = 0, n = 1, \lambda = 1, \mu = 0 \) in Theorem 3.1 then we obtain
\[ |a_k| \leq \frac{1}{k}. \]

**Remark 3.4.** If we take \( \alpha = 0, t = -1, n = 1, \) in Theorem 3.1 then we have
\[ |a_k| \leq \frac{1}{A_k^k}, \]
where \( A_k = 1 + (\lambda \mu k + \lambda - \mu)(k-1) \) and \( \lambda \geq \mu \geq 0 \).

**Remark 3.5.** If we put \( \lambda = \mu = 1 \) in Remark 3.4 then we have the following
\[ |a_k| \leq \frac{1}{k^2 - k + 1}. \]

**Remark 3.6.** Equalities in Theorem 3.1 are attended for \( f(z) \) given by
\[ \frac{(1-t)z \left( D_{\lambda,\mu}^n f(z) \right)'}{D_{\lambda,\mu}^n f(z) - D_{\lambda,\mu}^n f(tz)} = 1 + (1-2\alpha)z \]
\[ \frac{1}{1-z}. \]

**Theorem 3.7.** If \( f(z) \in T_{\lambda,\mu}^n (\alpha, t) \), then
\[ |a_k| \leq \frac{\beta}{kA_k^n|v_k|} \left\{ 1 + \beta \sum_{j=2}^{k-1} \frac{|u_j|}{|v_j|} + \beta^2 \sum_{j_2>j_1}^{k-1} \sum_{j_1=2}^{k-2} \frac{|u_{j_1} u_{j_2}|}{|v_{j_1} v_{j_2}|} + \beta^3 \sum_{j_3>j_2>j_1}^{k-3} \prod_{j_1=2}^{k-1} \frac{|u_{j_1} u_{j_2} u_{j_3}|}{|v_{j_1} v_{j_2} v_{j_3}|} + \cdots + \beta^{k-2} \prod_{j=2}^{k-1} \frac{|u_j|}{|v_j|} \right\}, \]
where
\[ \beta = 2(1-\alpha), \ v_k = k - u_k. \]
4. DISTORTION INEQUALITIES

For the functions \( f(z) \) in the classes \( S_{0,\lambda,\mu}(\alpha, t) \) and \( T_{0,\lambda,\mu}^{n}(\alpha, t) \), we derive

**Theorem 4.1.** If \( f(z) \in S_{0,\lambda,\mu}^{n}(\alpha, t) \), then

\[
|z| - \sum_{k=2}^{j} |a_k| |z|^k - B_j |z|^{j+1} \leq |f(z)| \leq |z| + \sum_{k=2}^{j} |a_k| |z|^k + B_j |z|^{j+1}
\]

(4.1)

where

\[
B_j = \frac{1 - \alpha - \sum_{k=2}^{j} A_{k}^{n} \{|k - u_k| + (1 - \alpha) |u_k|\} |a_k|}{(j + 1 - \alpha |u_{j+1}|) A_{j+1}^{n}} \quad (j \geq 2).
\]

(4.2)

**Proof.** From the inequality (2.1) we know that

\[
\sum_{k=j+1}^{\infty} A_{k}^{n} \{|k - u_k| + (1 - \alpha) |u_k|\} |a_k| \leq 1 - \alpha
\]

\[
- \sum_{k=2}^{j} A_{k}^{n} \{|k - u_k| + (1 - \alpha) |u_k|\} |a_k|.
\]

On the other hand

\[
\{|k - u_k| + (1 - \alpha) |u_k|\} \geq k - \alpha |u_k|,
\]

and \( k - \alpha |u_k| \) is monotonically increasing with respect to \( k \). Thus we deduce

\[
(j + 1 - \alpha |u_{j+1}|) A_{j+1}^{n} \sum_{k=j+1}^{\infty} |a_k| \leq 1 - \alpha - \sum_{k=2}^{j} A_{k}^{n} \{|k - u_k| + (1 - \alpha) |u_k|\} |a_k|,
\]

which implies that

\[
\sum_{k=j+1}^{\infty} |a_k| \leq B_j.
\]

(4.3)

Therefore we have the following

\[
|f(z)| \leq |z| + \sum_{k=2}^{j} |a_k| |z|^k + B_j |z|^{j+1}
\]

and

\[
|f(z)| \geq |z| - \sum_{k=2}^{j} |a_k| |z|^k - B_j |z|^{j+1}.
\]

This completes the proof of theorem. \( \square \)

Similarly we have
Theorem 4.2. If \( f(z) \in T_{0,\lambda,\mu}^{n}(\alpha, t) \), then
\[
|z| - \sum_{k=2}^{j} |a_k| |z|^k - C_j |z|^j+1 \leq |f(z)| \leq |z| + \sum_{k=2}^{j} |a_k| |z|^k + C_j |z|^j+1 \quad (4.4)
\]
and
\[
1 - \sum_{k=2}^{j} k |a_k| |z|^{k-1} - D_j |z|^j \leq |f'(z)| \leq 1 + \sum_{k=2}^{j} k |a_k| |z|^{k-1} + D_j |z|^j \quad (4.5)
\]
where
\[
C_j = \frac{1 - \alpha - \sum_{k=2}^{j} k A_n^p \{|k - u_k| + (1 - \alpha) |u_k|\} |a_k|}{(j + 1) \{j + 1 - \alpha |u_{j+1}|\} A_{j+1}^n} \quad (j \geq 2) \quad (4.6)
\]
and
\[
D_j = \frac{1 - \alpha - \sum_{k=2}^{j} k A_n^p \{|k - u_k| + (1 - \alpha) |u_k|\} |a_k|}{\{j + 1 - \alpha |u_{j+1}|\} A_{j+1}^n} \quad (j \geq 2). \quad (4.7)
\]

Remark 4.3. If we choose \( n = 0 \), \( t = -1 \), \( j = 2 \) in Theorems 4.1 and 4.2, then we get the results given by Cho et al. [6].

5. Relation between the classes

By the definitions for the classes \( S_{0,\lambda,\mu}^{n}(\alpha, t) \) and \( T_{0,\lambda,\mu}^{n}(\alpha, t) \), evidently we have
\[
S_{0,\lambda,\mu}^{n}(\alpha, t) \subset S_{0,\lambda,\mu}^{n}(\beta, t) \quad (0 \leq \beta \leq \alpha < 1)
\]
and
\[
T_{0,\lambda,\mu}^{n}(\alpha, t) \subset T_{0,\lambda,\mu}^{n}(\beta, t) \quad (0 \leq \beta \leq \alpha < 1).
\]

Let us consider a relation between \( S_{0,\lambda,\mu}^{n}(\alpha, t) \) and \( T_{0,\lambda,\mu}^{n}(\alpha, t) \).

Theorem 5.1. If \( f(z) \in T_{0,\lambda,\mu}^{n}(\alpha, t) \), then \( f(z) \in S_{0,\lambda,\mu}^{n}(\frac{1+\alpha}{2}, t) \).

Proof. Let \( f(z) \in T_{0,\lambda,\mu}^{n}(\alpha, t) \). Then if \( f(z) \) satisfies
\[
\frac{|k - u_k| + (1 - \beta) |u_k|}{1 - \beta} \leq \frac{k |k - u_k| + (1 - \alpha) |u_k|}{1 - \alpha} \quad (5.1)
\]
for all \( k \geq 2 \), then we have that \( f(z) \in S_{0,\lambda,\mu}^{n}(\beta, t) \). From (5.1), we have
\[
\beta \leq 1 - \frac{(1 - \alpha) |k - u_k|}{k |k - u_k| + (1 - \alpha) (k - 1) |u_k|}, \quad (5.2)
\]
Furthermore, since for all $k \geq 2$

$$\frac{|k - u_k|}{k|k - u_k| + (1 - \alpha)(k - 1)|u_k|} \leq \frac{1}{k} \leq \frac{1}{2},$$

we obtain

$$f(z) \in S^n_{0,\lambda,\mu}(\frac{1 + \alpha}{2}, t).$$

\[\square\]

**Remark 5.2.** Taking $n = 0$ in Theorems 2.1- 5.1, we immediately obtain the results due to Owa et al. [7].

**Remark 5.3.** If we put $n = 0$, $t = -1$ in Theorems 3.1- 5.1, then we get the results given by Owa et al. [8].

**REFERENCES**


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