In this paper, a new kind of covering axiom pre-$\omega$-closedness ($P_\omega$-closedness, for short), stronger than p-closedness due to J. Dontchev et. al. [7] is introduced in terms of pre-$\omega$-open sets [16]. Several characterizations via filter bases and grills [23] along with various properties of this concept are obtained. Grill generalizations of $P_\omega$-closedness and associated concepts have also been investigated.

1. Introduction

The notion of $\omega$-open sets introduced by H. Z. Hdeib [8] has been studied extensively in recent years by a good number of researchers. Some of the recent research works related to $\omega$-open sets are found in the papers of H. Z. Hdeib [8, 9], Noiri, Omari and Noorani [16, 17], Omari and Noorani [18, 19], and Zoubi and Nashef [25].

For a long time, topologists have been interested in investigating properties closely related to compactness using different kinds of open-like sets, some of which can be found in papers [2, 3, 4, 5, 6, 7, 11, 14, 20, 24]. Every new invention neighboring compactness, at some stage or other, yields tremendous applications not only within topology itself but also in other branches of applied sciences. Keeping this in mind, a new kind of covering property, $P_\omega$-closedness, stronger than the celebrated concept of p-closedness due to J. Dontchev et. al. [7] is introduced in terms of $\omega$-open sets and allied concepts. We have obtained several properties and investigated various properties along with its grill generalization.

2. Prerequisites

Throughout this paper spaces $(X, \tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$) represent non-empty topological spaces. The closure and the interior of
a subset $A$ of a space $X$ are denoted by $cl(A)$ and $int(A)$ respectively. Let $A \subset X$. A point $x \in X$ is called a condensation point of $A$ if for each open set $U$ containing $x$, $A \cap U$ is uncountable. A set $A$ is called $\omega$-closed [8] if it contains all of its condensation points and the complement of an $\omega$-closed set is called an $\omega$-open set or equivalently, $A \subset X$ is $\omega$-open if and only if for each $x \in A$ there exists an open set $U$ containing $x$ such that $U - A$ is countable. The set of all $\omega$-open sets of a topological space $(X, \tau)$ is denoted by $\tau_\omega$. It is to be noted that $\tau_\omega$ is a topology on $(X, \tau)$ finer than $\tau$. The interior and the closure of a subset $A$ of a space $X$ with respect to the topology $\tau_\omega$ are denoted by $int_\omega(A)$ (or simply by $int(A)$) and $cl_\omega(A)$ (or simply by $cl(A)$) respectively. A subset $A$ of a space $X$ is called semi-open [12] (resp. regular open, $\alpha$-open [15], preopen [13], $\beta$-open [1], semi-$\omega$-open [16], $\alpha$-$\omega$-open [16], pre-$\omega$-open [16], $\beta$-$\omega$-open [16]) if $A \subset cl(int(A))$ (resp. $A = int(cl(A))$, $A \subset int(cl(A)))$, $A \subset cl(int(A)))$, $A \subset cl(int(cl(A)))$, $A \subset cl(int_\omega(A))$, $A \subset cl(int_\omega(cl(A)))$, $A \subset cl_\omega(cl(A))$ and $A \subset cl(int_\omega(cl(A))))). The family of all semi-open (resp. regular open, $\alpha$-open, preopen, $\beta$-open, semi-$\omega$-open $\alpha$-$\omega$-open, pre-$\omega$-open, $\beta$-$\omega$-open) subsets of $(X, \tau)$ is denoted by $SO(X)$ (resp. $RO(X)$, $\tau^n$, $PO(X)$, $\beta O(X)$, $S\omega O(X)$, $\tau^n_\omega$, $P\omega O(X)$, $\beta \omega O(X)$). It is well known that every preopen set is pre-$\omega$-open. The family of all preopen (resp. preclosed i.e. preclosed as well as preopen) pre-$\omega$-open, regular open) subsets of $X$ containing $x \in X$ is denoted by $PO(X, x)$ (resp. $P CO(X, x)$, $P\omega O(X, x)$, $RO(X, x)$). The complement of a pre-$\omega$-open set is called a pre-$\omega$-closed set. pcl($S$) is the intersection of all preclosed subsets of $X$ containing $S$. $\theta$-preclosure [7] of a subset $S$ of $X$ is the set pcl$_\theta(S) = \{x \in X : pcl(U) \cap S \neq \emptyset$ for all $U \in PO(X, x)\}$. If $S = pcl_\theta(S)$, then $S$ is called a $\theta$-preclosed set [7]. The complement of a $\theta$-preclosed set is called a $\theta$-preopen set or equivalently, $S$ is $\theta$-preopen if for each $x \in S$, there exists $U \in PO(X, x)$ such that pcl($U$) $\subset S$. A subset $S$ of a space $X$ is called a $p$-closed set relative to $X$ [7] if every cover of $S$ by preopen sets of $X$ has a finite subfamily whose pre-closures cover $S$. If $S = X$ and $S$ is p-closed set relative to $X$, then $X$ is called a p-closed space. A topological space $X$ is called strongly irresolvable if $S \in PO(X) \Rightarrow S \in SO(X)$. A space $(X, \tau)$ is called strongly compact [10] if every cover of $X$ by preopen sets has a finite subcover.

A filter base $\mathcal{F}$ on a topological space $(X, \tau)$ is said to pre-$\theta$-converge [7] to a point $x \in X$ if for each $V \in PO(X, x)$, there exists an $F \in \mathcal{F}$ such that $F \subset pcl(V)$. A filter base $\mathcal{F}$ is said to pre-$\theta$-accumulate [7] (or pre-$\theta$-adhere) at $x \in X$ if $pcl(V) \cap F \neq \emptyset$ for every $V \in PO(X, x)$ and every $F \in \mathcal{F}$. The collection of all points of $X$ at which a filter base $\mathcal{F}$ pre-$\theta$-adheres is denoted by $p-\theta-ad\mathcal{F}$. Thron [23] has defined a grill as a non-empty family $\mathcal{G}$ of non-empty subsets of $X$ satisfying (a) $A \in \mathcal{G}$ and
Let $A \subset B \Rightarrow B \in \mathcal{G}$ and (b) $A \cup B \in \mathcal{G}$ \Rightarrow either $A \in \mathcal{G}$ or $B \in \mathcal{G}$. Thron [23] has also shown that $\mathcal{F}(\mathcal{G}) = \{A \subset X : A \cap F \neq \emptyset, \forall F \in \mathcal{G}\}$ is a filter on $X$ and that there exists an ultrafilter $\mathcal{F}$ such that $\mathcal{F}(\mathcal{G}) \subset \mathcal{F} \subset \mathcal{G}$. Let $\mathcal{G}$ be a grill on a topological space $(X, \tau)$ and $\phi : P(X) \rightarrow P(X)$ be a mapping defined by $\phi(A) = \{x \in X : U \cap A \in \mathcal{G}, \text{ for all } U \in \tau(x)\}$. B. Roy and M. N. Mukherjee [21] proved that $\psi : P(X) \rightarrow P(X)$, where $\psi(A) = A \cup \phi(A)$ for all $A \in P(X)$, is a Kuratowski closure operator and hence induces a topology $\tau_G$ on $X$ finer than $\tau$.

3. pre-$\omega$-$\theta$-open sets

**Definition 3.1.** Let $A$ be a subset of a topological space $X$. Then the pre-$\omega$-interior (resp. pre-$\omega$-closure) of $A$ is denoted by $\text{pint}_\omega(A)$ (resp. $\text{pcl}_\omega(A)$) and is defined as the set $\text{pint}_\omega(A) = \bigcup \{G \subset A : G \in P\omega O(X)\}$ (resp. $\text{pcl}_\omega(A) = \bigcap \{G \supset A : X - G \in P\omega O(X)\}$). If no confusion arises, the pre-$\omega$-interior (resp. pre-$\omega$-closure) of $A$ is denoted by $\text{pint}_\omega(A)$ (resp. $\text{pcl}_\omega(A)$).

Now we state following theorem.

**Theorem 3.2.** For subsets $A, B$ of a topological space $X$, the following properties hold:

(a) $\text{pcl}_\omega(A) \subset \text{pcl}(A)$ and $\text{pcl}_\omega(A) \subset cl_\omega(A)$.

(b) $A \subset B$ implies $\text{pcl}_\omega(A) \subset \text{pcl}_\omega(B)$ and $\text{pint}_\omega(A) \subset \text{pint}_\omega(B)$.

(c) $\text{pcl}_\omega(\text{pcl}_\omega(A)) = \text{pcl}_\omega(A)$ and $\text{pint}_\omega(\text{pint}_\omega(A)) = \text{pint}_\omega(A)$.

(d) $A$ is pre-$\omega$-closed if and only if $\text{pcl}_\omega(A) = A$.

(e) $A$ is pre-$\omega$-open if and only if $\text{pint}_\omega(A) = A$.

(f) $\text{pcl}_\omega(X - A) = X - \text{pint}_\omega(A)$.

(g) $\text{pint}_\omega(X - A) = X - \text{pcl}_\omega(A)$.

**Remark 3.3.** For a subset $A$ of a topological space, $\text{pcl}_\omega(A) \neq \text{pcl}(A)$ in general, which is reflected in the following example.

**Example 3.4.** Consider the space $X = \mathbb{N}$ with the topology generated by the base $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ where $B_n = \{1, n\}$. Then the topology on $X$ is $\tau = \{\emptyset\} \cup \{G \subset \mathbb{N} : G \text{ contains } 1\} = P\omega O(X)$. Since $\mathbb{N}$ is countable, $\tau_\omega = P(X) = P\omega O(X)$, where $P(X)$ is the power set of $X$. Let $A$ be a subset of $X$ containing 1. Then $\text{pcl}(A) = A$ and $\text{pcl}(A) = \mathbb{N}$.

**Definition 3.5.** A point $x \in X$ is said to be a pre-$\omega$-$\theta$-accumulation point of a subset $A$ of a topological space $(X, \tau)$ if $\text{pcl}_\omega(U) \cap A \neq \emptyset$ for every $U \in P\omega O(X, x)$. The set of all pre-$\omega$-$\theta$-accumulation points of $A$ is called the pre-$\omega$-$\theta$-closure of $A$ and is denoted by $\text{pcl}_\theta(A)$. A subset $A$ of a topological space $(X, \tau)$ is said to be pre-$\omega$-$\theta$-closed if $\text{pcl}_\theta(A) = A$. The complement of a pre-$\omega$-$\theta$-closed set is called a pre-$\omega$-$\theta$-open set.
Lemma 3.6. A subset $A$ of a space $X$ is pre-$\omega$-$\theta$-open if and only if for each $x \in A$, there exists $V \in PO(X,x)$ such that $pcl_\omega(V) \subset A$.

Proof. Let $A$ be pre-$\omega$-$\theta$-open and $x \in A$. Since $X - A$ is pre-$\omega$-$\theta$-closed then for $x \in A$, there exists a $V \in PO(X,x)$ such that $pcl_\omega(V) \cap (X - A) = \emptyset$ and thus $pcl_\omega(V) \subset A$.

Conversely, suppose that the condition does not hold. Then there exists an $x \in A$ such that $pcl_\omega(V) \not\subset A$ for all $V \in PO(X,x)$. Thus $pcl_\omega(V) \cap (X - A) \neq \emptyset$ for all $V \in PO(X,x)$ and so $x$ is a pre-$\omega$-$\theta$-accumulation point of $X - A$. Hence $X - A$ is not pre-$\omega$-$\theta$-closed. \hfill \qed

Theorem 3.7. Let $A$ and $B$ be any subsets of a space $X$. The following properties hold:

(a) $\theta$-preclosed sets are pre-$\omega$-$\theta$-closed sets.
(b) $p_\omega cl_\theta(A) \subset pcl_\theta(A)$,
(c) if $A \subset B$, then $p_\omega cl_\theta(A) \subset p_\omega cl_\theta(B)$,
(d) the intersection of an arbitrary family of pre-$\omega$-$\theta$-closed sets is pre-$\omega$-$\theta$-closed in $X$.

Proof. The proof is straightforward and is thus omitted. \hfill \qed

Remark 3.8. In a topological space, $p_\omega cl_\theta(A) \neq pcl_\theta(A)$ and a pre-$\omega$-$\theta$-closed set may not be $\theta$-preclosed in general, which is reflected in the following example.

Example 3.9. In Example 3.4, consider $A = N - \{1\}$. Then $1 \not\in p_\omega cl_\theta(A)$ because $\{1\} \in PO(X,1)$ and $pcl_\omega(\{1\}) \cap A = \emptyset$ but $pcl_\theta(A) = N$. It is also clear from this example that $A$ is a pre-$\omega$-$\theta$-closed set but not a $\theta$-preclosed set in $X$.

Definition 3.10. Let $X$ be a topological space and $A \subset X$. Then $A$ is called a $\omega$-regular (resp. mist-$\omega$-regular) open set if $A = int_\omega(cl(A))$ (resp. $A = int(cl_\omega(A))$). The family of all $\omega$-regular open sets of $X$ is denoted by $R_\omega O(X)$.

Lemma 3.11. The family $R_\omega O(X)$ of all $\omega$-regular open sets of $X$ is a base of some topology on $X$.

Proof. Let $x \in X$. Suppose $A$ and $B$ are any two $\omega$-regular open sets of $X$ containing $x$. Consider $C = A \cap B$. Then $C$ is $\omega$-open (so pre-$\omega$-open) containing $x$ and so $int_\omega(cl(C)) \supset C$. On the other hand, $C = int_\omega(cl(A)) \cap int_\omega(cl(B)) = int_\omega(cl(A) \cap cl(B)) \supset int_\omega(cl(A \cap B)) = int_\omega(cl(C))$. Hence the family $R_\omega O(X)$ of all $\omega$-regular open sets of $X$ is a base of some topology on $X$. \hfill \qed

In this paper, we consider $\tau_{R_\omega}$ as the topology generated by the base $R_\omega O(X)$.
4. $P_\omega$-closed spaces

**Definition 4.1.** A topological space $X$ is called $P_\omega$-closed (resp. quasi-$H$-$\omega$-closed) iff every preopen cover of $X$ has a finite subfamily whose pre-$\omega$-closures (resp. $\omega$-closures) cover $X$.

**Theorem 4.2.** Let $(X, \tau)$ be quasi-$H$-$\omega$-closed and strongly irresolvable. Then $(X, \tau)$ is $p$-closed.

**Proof.** Let $\{U_\alpha : \alpha \in \Delta\}$ be a preopen cover of $(X, \tau)$. Since $(X, \tau)$ is quasi-$H$-$\omega$-closed, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta$ such that $X = \bigcup_{i=1}^n cl_\omega(U_{\alpha_i})$. Since $X$ is strongly irresolvable then each $U_\alpha \in SO(X)$ and so $cl_\omega(U_\alpha) \subset cl(U_\alpha) = cl(int(U_\alpha)) = pcl(U_\alpha)$ for each $\alpha \in \Delta$. Thus $(X, \tau)$ is $p$-closed. $\square$

**Definition 4.3.** A filter base $F$ (resp. a grill $G$) on a topological space $(X, \tau)$ is said to $pre$-$\omega$-$\theta$-converge to a point $x \in X$ if for each $V \in PO(X, x)$, there exists $F \in F$ (resp. $F \in G$) such that $F \subset pcl_\omega(V)$. A filter base $F$ is said to $pre$-$\omega$-$\theta$-accumulate (or $pre$-$\omega$-$\theta$-adhere) at $x \in X$ if $pcl_\omega(V) \cap F \neq \emptyset$ for every $V \in PO(X, x)$ and every $F \in F$. The collection of all points of $X$ at which the filter base $F$ $pre$-$\omega$-$\theta$-adheres is denoted by $p_{\omega, \theta}$-ad$F$.

**Theorem 4.4.** For a topological space $(X, \tau)$ the following conditions are equivalent:

(a) $(X, \tau)$ is $P_\omega$-closed,
(b) every ultrafilter base $pre$-$\omega$-$\theta$-converges to some point of $X$,
(c) every filter base $pre$-$\omega$-$\theta$-accumulates at some point of $X$,
(d) for every family $\{V_\alpha : \alpha \in \Delta\}$ of preclosed subsets such that $\cap\{V_\alpha : \alpha \in \Delta\} = \emptyset$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta$ such that $\cap_{i=1}^n pint_\omega(V_{\alpha_i}) = \emptyset$.

**Proof.** (a)$\Rightarrow$(b). Let $(X, \tau)$ be $P_\omega$-closed and $F$ be an ultrafilter base on $X$ which does not $pre$-$\omega$-$\theta$-converge to any point of $X$. Since $F$ is an ultrafilter base on $X$, then it can not $pre$-$\omega$-$\theta$-accumulate at any point of $X$. Thus for each $x \in X$, there is an $F_x \in F$ and a $V_x \in PO(X, x)$ such that $pcl_\omega(V_x) \cap F_x = \emptyset$. Then the family $\{V_x : x \in X\}$ forms a cover of $X$ by preopen subsets. Since $X$ is $P_\omega$-closed, there exists a finite number of points $x_1, x_2, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^n pcl_\omega(V_{x_i})$. Since $F$ is a filter base on $X$, there exists an $F' \in F$ such that $F' \subset \cap_{i=1}^n (F_{x_i})$ and thus $F' = \emptyset$ which is a contradiction.

(b)$\Rightarrow$(c). Let $F$ be any filter base on $X$. Then there is an ultrafilter base $F'$ containing $F$. By the hypothesis, $F'$ $pre$-$\omega$-$\theta$-converges to some point $x \in X$. Now consider $V \in PO(X, x)$ and every $F \in F$. Then there exists an $F' \in F'$ such that $F' \subset pcl_\omega(V)$ and $F \cap F' \neq \emptyset$. Hence $\emptyset \neq F \cap F' \subset pcl_\omega(V) \cap F$. So the filter base $F$ $pre$-$\omega$-$\theta$-accumulates at $x \in X$. 

(c)⇒(d). Let \( \{V_{\alpha} : \alpha \in \Delta \} \) be a family of preclosed subsets of \( X \) such that \( \cap \{V_{\alpha} : \alpha \in \Delta \} = \emptyset \). Let \( \mathcal{G} \) be the family of all finite subsets of \( \Delta \). Suppose \( \cap \{\text{pint}_\omega(V_{\beta}) , \beta \in \delta \} \neq \emptyset \) for each \( \delta \in \mathcal{G} \). Then \( \mathcal{F} = \{\cap \{\text{pint}_\omega(V_{\beta}) , \beta \in \delta \} : \delta \in \mathcal{G} \} \) is a filter base on \( X \). For, if \( F_1, F_2 \in \mathcal{F} \), then \( F_1 = \cap \{\text{pint}_\omega(V_{\beta}) , \beta \in \delta \} \) and \( F_2 = \cap \{\text{pint}_\omega(V_{\gamma}) , \gamma \in \delta' \} \) for some \( \delta, \delta' \in \mathcal{G} \) and so \( F_3 = F_1 \cap F_2 = \cap \{\text{pint}_\omega(V_{\lambda}) , \lambda \in \delta \cup \delta' \} \in \mathcal{F} \). Then by (c), \( \mathcal{F} \) pre-\( \omega \)-\( \theta \)-accumulates at some point \( x \) of \( X \). Since \( \{X - V_{\alpha} : \alpha \in \Delta \} \) is a preopen cover of \( X, x \in X - V_{\alpha_0} \) for some \( \alpha_0 \in \Delta \). Let \( G = X - V_{\alpha_0} \). Then \( G \in PO(X,x) \) and \( \text{pint}_\omega(V_{\alpha_0}) \in \mathcal{F} \) such that \( \text{pcl}_\omega(G) \cap \text{pint}_\omega(V_{\alpha_0}) = \emptyset \) which is a contradiction.

(d)⇒(a). Let \( \{U_{\alpha} : \alpha \in \Delta \} \) be a family of preopen subsets of \( X \) covering \( X \). Then \( \{X - U_{\alpha} : \alpha \in \Delta \} \) is a family of preclosed subsets of \( X \) having empty intersection. Thus by (d), there exist \( \alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta \) such that
\[
\cap_{i=1}^n \text{pint}_\omega(X - U_{\alpha_i}) = \emptyset \quad \text{i.e.} \quad \cup_{i=1}^n \text{pcl}_\omega(U_{\alpha_i}) = X.
\]
So \( (X, \tau) \) is \( P_\omega \)-closed. 

**Theorem 4.5.** If the topological space \( X \) is \( P_\omega \)-closed, then every pre-\( \omega \)-\( \theta \)-open cover of \( X \) has a finite subcover.

**Proof.** Let \( X \) be \( P_\omega \)-closed. Let \( \Sigma = \{U_{\alpha} : \alpha \in \Delta \} \) be a cover of \( X \) by pre-\( \omega \)-\( \theta \)-open sets of \( X \). Let \( x \in X \) and \( x \in U_{\alpha_x} \) for some \( \alpha_x \in \Delta \). Then by the Lemma 3.6, there exists a \( V_{\alpha_x} \in PO(X,x) \) such that \( \text{pcl}_\omega(V_{\alpha_x}) \subset U_{\alpha_x} \). Therefore \( \Sigma = \{V_{\alpha_x} : x \in X \} \) is a preopen cover of \( X \) and hence there exist \( x(1), x(2), \ldots, x(n) \in X \) such that \( X = \cup_{i=1}^n \text{pcl}_\omega(V_{\alpha_{x(i)}}) \). So \( X = \cup_{i=1}^n U_{\alpha_{x(i)}} \). Hence \( \{U_{\alpha_x(i)} : x(i) \in X, i = 1, 2, \ldots, n \} \) is the required finite subcover of \( \Sigma \).

It is clear that every \( P_\omega \)-closed space is p-closed but the converse need not be true. This fact has been established with the following example.

**Example 4.6.** Consider the space \( (X, \tau) \) from Example 3.4. Then clearly, \( X \) is p-closed because for any \( A \in PO(X), \text{pcl}(A) = N \). Now observe the cover \( \{A_n = \{1, n\} : n \in \mathbb{N} \} \) of \( X \) by preopen sets of \( X \). Again it is noted that \( \text{pcl}_\omega(A_n) = \{1, n\} \) and so \( \{A_n : n \in \mathbb{N} \} \) is a cover of \( X \) by pre-\( \omega \)-\( \theta \)-open sets of \( X \). But it has no finite subcover. Hence by theorem 4.5, \( X \) is not \( P_\omega \)-closed.

**Definition 4.7.** A topological space \( (X, \tau) \) is said to be strongly \( P_\omega \)-regular if for each point \( x \in X \) and each preclosed set \( F \) such that \( x \notin F \), there exist \( V \in PO(X,x) \) and \( W \in P(\omega\mathcal{O}(X)) \) such that \( F \subset W \) and \( V \cap W = \emptyset \).

**Theorem 4.8.** A topological space \( X \) is strongly \( P_\omega \)-regular if and only if for each \( x \in X \) and for each preopen set \( U \) containing \( x \), there exists \( V \in PO(X,x) \) such that \( x \in V \subset \text{pcl}_\omega(V) \subset U \).

**Proof.** Let \( X \) be a strongly \( P_\omega \)-regular space. Suppose \( x \in X \) and \( U \in PO(X,x) \). Then \( F = X - U \) is a preclosed set not containing \( x \). Then
there exist a \( V \in PO(X, x) \) and a \( W \in \mathcal{P} \omega O(X) \) such that \( F \subset W \) and \( V \cap W = \emptyset \). So \( x \in V \subset \mathcal{P} \omega(V) \subset X - W \subset X - F = U \).

Conversely, let \( x \in X \) and \( F \) be preclosed with \( x \notin F \). Then there exists \( V \in PO(X, x) \) such that \( x \in V \subset \mathcal{P} \omega(V) \subset X - F \). Consider \( W = X - \mathcal{P} \omega(V) \). Then \( F \subset W \) and \( V \cap W = \emptyset \). So \( X \) is strongly \( \mathcal{P} \omega \)-regular.

**Theorem 4.9.** If a topological space \( X \) is \( \mathcal{P} \omega \)-closed and strongly \( \mathcal{P} \omega \)-regular, then \( X \) is strongly compact.

**Proof.** Let \( X \) be \( \mathcal{P} \omega \)-closed and strongly \( \mathcal{P} \omega \)-regular. Suppose \( \{U_\alpha : \alpha \in \Delta\} \) is a preopen cover of \( X \). For each \( x \in X \) there exists \( \alpha(x) \in \Delta \) such that \( U_{\alpha(x)} \in PO(X, x) \) and since \( X \) is strongly \( \mathcal{P} \omega \)-regular by Theorem 4.8, there exists \( V_x \in PO(X, x) \) such that \( x \in \mathcal{P} \omega(V_x) \subset U_{\alpha(x)} \). Then \( \{V_x : x \in X\} \) is a family of preopen subsets of \( X \) covering \( X \) and as \( X \) is \( \mathcal{P} \omega \)-closed, there exist \( x_1, x_2, \ldots, x_n \in X \) such that \( X = \bigcup_{i=1}^n \mathcal{P} \omega(V_{x_i}) \subset \bigcup_{i=1}^n U_{\alpha(x_i)} \). Hence \( X \) is strongly compact. \( \square \)

**Definition 4.10.** A subset \( S \) of a topological space \( (X, \tau) \) is said to be \( \mathcal{P} \omega \)-closed relative to \( X \) if every cover \( \{V_\alpha : \alpha \in \Delta\} \) of \( S \) by preopen subsets of \( (X, \tau) \) has a finite subfamily whose pre-\( \omega \)-closures cover \( S \).

**Theorem 4.11.** For a topological space \( (X, \tau) \) and for a subset \( S \) of \( X \), the following conditions are equivalent:

(a) \( S \) is \( \mathcal{P} \omega \)-closed relative to \( X \),
(b) every ultrafilter base on \( X \) which meets \( S \) pre-\( \omega \)-converges to some point of \( S \),
(c) every filter base on \( X \) which meets \( S \) pre-\( \omega \)-accumulates at some point of \( S \),
(d) for every family \( \{V_\alpha : \alpha \in \Delta\} \) of pre-closed subsets of \( (X, \tau) \) such that \( \cap\{V_\alpha : \alpha \in \Delta\} \cap S = \emptyset \), there exists a finite number of indices \( \alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta \) such that \( \cap_{i=1}^n \mathcal{P} \omega(V_{\alpha_i}) \cap S = \emptyset \).

**Proof.** The proof is quite similar to the proof of the Theorem 4.4 and is thus omitted. \( \square \)

**Theorem 4.12.** For a topological space \( (X, \tau) \) and for a subset \( S \) of \( X \) the following two conditions are equivalent:

(a) \( S \) is \( \mathcal{P} \omega \)-closed relative to \( X \),
(b) every grill \( \mathcal{G} \) on \( X \) containing \( S \) pre-\( \omega \)-converges to some point of \( S \).

**Proof.** (a)\( \Rightarrow \)(b). Let \( \mathcal{G} \) be a grill on \( X \) containing \( S \) which does not pre-\( \omega \)-converge to any point of \( S \). Then for each \( x \in S \) and for each \( U_x \in PO(X, x) \), \( F \not\subset \mathcal{P} \omega(U_x) \) for all \( F \in \mathcal{G} \). Thus \( \mathcal{P} \omega(U_x) \notin \mathcal{G} \). Now consider
the cover \( \{ U_x : x \in S \} \) of \( S \). Since \( S \) is \( \omega \)-closed relative to \( X \), there exist \( x_1, x_2, \ldots, x_n \in S \) such that \( S \subset \bigcup_{i=1}^n \text{pcl}_\omega(U_{x_i}) \). Then \( \bigcup_{i=1}^n \text{pcl}_\omega(U_{x_i}) \in \mathcal{G} \) is a contradiction.

(b) \( \Rightarrow \) (a). Let \( S \) not be \( \omega \)-closed relative to \( X \). Then there exists a cover \( \{ U_\alpha \in PO(X) : \alpha \in \Delta \} \) of \( S \) such that \( F = \{ S - \bigcup_{i=1}^n \text{pcl}_\omega(U_{\alpha_i}) : n < \aleph_0 \} \) is a filterbase on \( X \). Now consider \( \mathcal{G} \), an ultrafilter base containing \( F \). Then \( \mathcal{G} \) is a grill containing \( S \) and hence by (b), \( \mathcal{G} \) converges at some point \( x \in S \). Now let \( x \in U_\alpha \) for some \( \alpha \in \Delta \), there exists \( F \in \mathcal{G} \) such that \( F \subset \text{pcl}_\omega(U_\alpha) \). So \( \text{pcl}_\omega(U_\alpha) \in \mathcal{G} \). But \( S - \text{pcl}_\omega(U_\alpha) \in \mathcal{G} \) which is a contradiction. \( \square \)

**Corollary 4.14.** A topological space \( X \) is \( \omega \)-closed if and only if every grill on \( X \) \( \omega \)-converges to some point of \( X \).

**Theorem 4.15.** Let \( A, B \) be subsets of a space \( X \). If \( A \) is \( \omega \)-\( \theta \)-closed and \( B \) is \( \omega \)-closed relative to \( X \), then \( A \cap B \) is \( \omega \)-closed relative to \( X \).

**Proof.** Let \( \{ U_\alpha : \alpha \in \Delta \} \) be a cover of \( A \cap B \) by preopen subsets of \( X \). Since \( A \) is \( \omega \)-\( \theta \)-closed, then for each \( x \in B - A \) there exists \( V_x \in PO(X, x) \) such that \( \text{pcl}_\omega(V_x) \cap A = \emptyset \). Then the family \( \{ U_\alpha : \alpha \in \Delta \} \cup \{ V_x : x \in B - A \} \) is a cover of \( B \) by preopen subsets of \( X \). Since \( B \) is \( \omega \)-closed relative to \( X \), then there exists a finite number of points \( x_1, x_2, \ldots, x_n \in B - A \) and a finite number of indices \( \alpha_1, \alpha_2, \ldots, \alpha_m \in \Delta \) such that \( B \subset (\bigcup_{i=1}^n \text{pcl}_\omega(V_{x_i})) \cup (\bigcup_{j=1}^m \text{pcl}_\omega(U_{\alpha_j})) \) and so \( A \cap B \subset \bigcup_{i=1}^n \text{pcl}_\omega(U_{\alpha_i}) \). So \( A \cap B \) is \( \omega \)-closed relative to \( X \). \( \square \)

**Corollary 4.16.** If \( X \) is a \( \omega \)-closed space, then every \( \omega \)-\( \theta \)-closed subset of \( X \) is \( \omega \)-closed relative to \( X \).

**Definition 4.17.** Let \( X \) be a topological space and \( A \subset X \). Then \( \text{mist}\omega \)-boundary of \( A \) is the set \( \omega\text{-Fr}(A) = \text{cl}_\omega(A) - \text{int}(A) \).

**Definition 4.18.** A topological space \( X \) is called \( \omega \)-\( \kappa \)-nearly compact if every cover of \( X \) by \( \omega \)-\( \kappa \)-regular open sets has a finite subcover.

**Definition 4.19.** For an infinite cardinal number \( \kappa \), a topological space \( X \) is called \( \omega \)-\( \kappa \)-extremely disconnected if the cardinality of the \( \omega \)-boundary of every \( \omega \)-\( \kappa \)-regular open set is less than \( \kappa \).

**Theorem 4.20.** If a topological space \( X \) is \( \omega \)-closed and \( \omega \)-\( \aleph_0 \)-extremely disconnected, then \( X \) is \( \omega \)-\( \kappa \)-nearly compact.

**Proof.** Let \( \Sigma = \{ U_\alpha : \alpha \in \Delta \} \) be a cover of \( X \) by \( \omega \)-\( \kappa \)-regular open sets of \( X \). Since \( U_\alpha = \text{int}(\text{cl}_\omega(U_\alpha)) \subset \text{int}(\text{cl}(U_\alpha)) \) for each \( \alpha \in \Delta \), \( \Sigma \) is a preopen cover of \( X \). Since \( X \) is \( \omega \)-closed, there exists a finite set \( \Delta_0 \subset \Delta \) such that \( X = \bigcup_{\alpha \in \Delta_0} \text{pcl}_\omega(U_\alpha) \subset \bigcup_{\alpha \in \Delta_0} \text{cl}_\omega(U_\alpha) \). Therefore \( X = \bigcup_{\alpha \in \Delta_0} \text{cl}_\omega(U_\alpha) \). Now for each \( \alpha \in \Delta_0 \), \( \text{cl}_\omega(U_\alpha) = U_\alpha \cup F_\alpha \) where \( F_\alpha = \text{cl}_\omega(U_\alpha) - U_\alpha = \)]
\[ \text{Let (It is obvious that if } \text{Let } \text{Let } \text{Let} \text{∪ } \text{cl } G \subset \text{∪ } \text{quasi-} \in \text{and so } \text{pcl } U \text{α } \text{α } \text{X } \text{X } \text{relative to } \text{cl } \text{the grill } G \text{X of any topological space } H \text{closed (resp. quasi-} \text{X } \text{H } \text{Proof. } \text{in the topological space } \text{Proof. } \text{Proof. } G \text{is pre-} \text{G } \text{is cover of } \text{g } 0 \text{− } \Delta \in \text{∈ } \text{ω } \text{G } \text{− closed set relative to any grill } \text{− closed) with respect to any grill } \text{− closed) space } X \text{ is closed in the topological space } (X, \tau). \text{Proof. } \text{Let } x \in X − G. \text{ Then for each } g \in G, \text{ there exist two open sets } U_g \text{ and } V_g \text{ such that } x \in U_g \text{ and } g \in V_g \text{ and } U_g \cap V_g = \emptyset. \text{ Then } \{V_g : g \in G \} \text{ is a cover of } G \text{ by open (and so preopen) sets of } X. \text{ Since } G \text{ is a } p_\omega \text{-closed set relative to } X, \text{ there exist } g_1, g_2, \ldots, g_n \in G \text{ such that } G \subset \cup_{i=1}^n \text{pcl}_\omega(V_{g_i}) \subset \text{pcl}_\omega(\cup_{i=1}^n (V_{g_i})). \text{ Now consider } U = \cap U_{g_i} \text{ and } V = \cup V_{g_i}. \text{ Then } U \cap V = \emptyset \text{ and so } \text{pcl}_\omega(V) \subset \text{cl}(V) \subset X − U. \text{ Therefore } \text{pcl}_\omega(V) \cap U = \emptyset \text{ and so } \text{int}(\text{pcl}_\omega(V)) \cap U = \emptyset. \text{ Hence } \text{int}(\text{pcl}_\omega(V)) \cap \text{cl}(U) = \emptyset \text{ and since } G \text{ is pre-} \omega \text{-open, } G \cap \text{pcl}_\omega(U) = \emptyset. \text{ Thus } G \text{ is pre-} \omega \text{-closed.} \]
Theorem 4.25. Let $\mathcal{G}$ be a grill on a topological space $(X, \tau)$ containing all nonempty $\omega$-open sets and $X$ be quasi-$H$-$\omega$-closed with respect to the grill $\mathcal{G}$. Then $X$ is quasi-$H$-$\omega$-closed.

Proof. Let $X$ be quasi-$H$-$\omega$-closed with respect to the grill $\mathcal{G}$ and $\{U_\alpha : \alpha \in \Delta\}$ be a cover of $X$ by the preopen sets of $X$. Then there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta$ such that $X - \bigcup_{i=1}^n \text{cl}_\omega(U_{\alpha_i}) \notin \mathcal{G}$. If $\text{int}_\omega(X - \bigcup_{i=1}^n \text{cl}_\omega(U_{\alpha_i})) \neq \emptyset$, then $\text{int}_\omega(X - \bigcup_{i=1}^n \text{cl}_\omega(U_{\alpha_i})) \in \mathcal{G}$. But $\text{int}_\omega(X - \bigcup_{i=1}^n \text{cl}_\omega(U_{\alpha_i})) = X - \text{cl}_\omega(\bigcup_{i=1}^n U_{\alpha_i}) = X - \bigcup_{i=1}^n \text{cl}_\omega(U_{\alpha_i})$. So $X - \bigcup_{i=1}^n \text{cl}_\omega(U_{\alpha_i}) \in \mathcal{G}$, which is a contradiction. Hence $\emptyset = \text{int}_\omega(X - \bigcup_{i=1}^n \text{cl}_\omega(U_{\alpha_i})) = X - \bigcup_{i=1}^n \text{cl}_\omega(U_{\alpha_i})$. Thus $X = \bigcup_{i=1}^n \text{cl}_\omega(U_{\alpha_i})$. Therefore $X$ is quasi-$H$-$\omega$-closed.

Definition 4.26. A topological space $X$ is called weakly $P_\omega$-closed (resp. strongly $P_\omega$-closed, strongly compact) with respect to a grill $\mathcal{G}$ if every preopen (resp. open, preopen) cover $\{V_{\alpha} : \alpha \in \Delta\}$ of $X$ has a finite subfamily $\{V_{\alpha_i} : \alpha_i \in \Delta, i = 1, 2, \ldots, n\}$ such that $X - \bigcup_{i=1}^n \text{int}_\omega(V_{\alpha_i}) \notin \mathcal{G}$ (resp. $X - \bigcup_{i=1}^n \text{pcl}_\omega(V_{\alpha_i}) \notin \mathcal{G}$, $X - \bigcup_{i=1}^n V_{\alpha_i} \notin \mathcal{G}$).

Definition 4.27. A topological space $X$ is called strongly pre-$\omega$-regular with respect to a grill $\mathcal{G}$ if for each $x \in X$ and preclosed set $F$ not containing $x$. Then there exist disjoint sets $U \in \text{PO}(X, x)$ and $V \in \text{P}_\omega\text{O}(X)$ such that $F - V \notin \mathcal{G}$.

Theorem 4.28. A $P_\omega$-closed strongly pre-$\omega$-regular space with respect to a grill $\mathcal{G}$ is strongly compact with respect to the grill $\mathcal{G}$.

Proof. Let $\{U_{\alpha} : \alpha \in \Delta\}$ be a cover of $X$ by preclosed sets of $X$. Then for each $x \in X$, there exists $\alpha(x) \in \Delta$ such that $x \in U_{\alpha(x)}$. Since $X$ is strongly pre-$\omega$-regular with respect to the grill $\mathcal{G}$, there exist disjoint sets $P_{\alpha(x)} \in \text{PO}(X, x)$ and $Q_{\alpha(x)} \in \text{P}_\omega\text{O}(X)$ such that $(X - U_{\alpha(x)}) - Q_{\alpha(x)} \notin \mathcal{G}$. Here $\{P_{\alpha(x)} : x \in X\}$ is a cover of $X$ by preopen sets of $X$. Since $X$ is $P_\omega$-closed, there exist $x(1), x(2), \ldots, x(n) \in X$ such that $X = \bigcup_{i=1}^n \text{pcl}_\omega(P_{\alpha(x(i))})$. Consider $S_{\alpha(x)} = (X - U_{\alpha(x)}) - Q_{\alpha(x)}$. Here $P_{\alpha(x)} \cap Q_{\alpha(x)} = \emptyset$ implies that $\text{pcl}_\omega(P_{\alpha(x)}) \cap Q_{\alpha(x)} = \emptyset$. Now we claim that $\text{pcl}_\omega(P_{\alpha(x)}) \subset S_{\alpha(x)} \cup U_{\alpha(x)}$. Since $x \notin \text{pcl}_\omega(P_{\alpha(x)})$, but $x \notin U_{\alpha(x)}$ implies that $q \in X - Q_{\alpha(x)}$ and so $q \in ((X - U_{\alpha(x)}) - Q_{\alpha(x)}) = S_{\alpha(x)}$. Thus $X = \bigcup_{i=1}^n \text{pcl}_\omega(P_{\alpha(x(i))}) \subset \bigcup_{i=1}^n (S_{\alpha(x(i)}) \cup U_{\alpha(x(i))})$, and so $X - \bigcup_{i=1}^n U_{\alpha(x(i))} \subset \bigcup_{i=1}^n (S_{\alpha(x(i)}) \notin \mathcal{G}$. But for each $i = 1, 2, \ldots, n$, $S_{\alpha(x(i))} \notin \mathcal{G}$ and so $X - \bigcup_{i=1}^n U_{\alpha(x(i))} \notin \mathcal{G}$. Hence $X$ is strongly compact with respect to the grill $\mathcal{G}$.

Theorem 4.29. A $T_2$ weakly $P_\omega$-closed space with respect to a grill $\mathcal{G}$ is strongly pre-$\omega$-regular with respect to the grill $\mathcal{G}$.

Proof. Consider $x \in X$ and a preclosed set $F$ not containing $x$. Then for each $y \in F$, there exist disjoint open sets $U_y$ and $V_y$ containing $x$ and $y$. Then $X - \bigcup_{y \in F} (U_y \cup V_y) \notin \mathcal{G}$, which contradicts the assumption that $X$ is strongly pre-$\omega$-regular with respect to the grill $\mathcal{G}$. Thus $X$ is strongly pre-$\omega$-regular with respect to the grill $\mathcal{G}$.
respectively. Therefore \( \{ V_y : y \in F \} \cup \{ X - F \} \) is a preopen cover of \( X \).
Since \( X \) is a weakly \( P_\omega \)-closed space with respect to the grill \( G \), there exist \( y_1, y_2, \ldots, y_n \in F \) such that \( X - \left( \bigcup_{i=1}^n \text{int}_\omega(V_{y_i}) \cup \text{int}_\omega(X - F) \right) \notin G \). Now consider \( U = X - \text{pcl}(\bigcup_{i=1}^n V_{y_i}) \) and \( V = \bigcup_{i=1}^n V_{y_i} \). Then \( U \cap V = \emptyset \), \( U \in \text{PO}(X, x) \), \( V \in \text{PO}(X) \subset P_\omega \text{O}(X) \) and \( F - V = F \cap (X - V) = X - \left( \bigcup_{i=1}^n (V_{y_i}) \cup (X - F) \right) \subset X - \left[ \bigcup_{i=1}^n \text{int}_\omega(V_{y_i}) \cup \text{int}_\omega(X - F) \right] \) and so \( F - V \notin G \). Hence \( X \) is strongly pre-\( \omega \)-regular with respect to the grill \( G \). \( \square \)

**Theorem 4.30.** Let \( G \) be a grill on a topological space \( (X, \tau) \) and \( (X, \tau) \) be strongly compact with respect to the grill \( G \). Then \( (X, \tau_G) \) is strongly \( P_\omega \)-closed with respect to the grill \( G \).

**Proof.** Let \( (X, \tau) \) be strongly compact with respect to the grill \( G \) and consider \( \Sigma \) to be a cover of \( X \) by open sets of \( (X, \tau_G) \). Then for each \( x \in X \), there exists \( U_x \in \Sigma \) such that \( x \in U_x \). Then there exist a \( B_x \in \tau \) and a \( V_x \notin G \) such that \( x \in B_x - V_x \subset U_x \). Then \( \{ B_x : x \in X \} \) is a cover of \( X \) by open (and so preopen) sets of the space \( (X, \tau) \). Since \( (X, \tau) \) is strongly compact with respect to the grill \( G \), there exist \( x(1), x(2), \ldots, x(n) \in X \) such that \( X - \bigcup_{i=1}^n B_{x(i)} \notin G \). Now \( X - \bigcup_{i=1}^n \text{pcl}_\omega(U_{x(i)}) \subset X - \bigcup_{i=1}^n U_{x(i)} \subset X - \bigcup_{i=1}^n (B_{x(i)} - U_{x(i)}) \subset (X - \bigcup_{i=1}^n (B_{x(i)} \cup U_{x(i)})) \notin G \). Hence \( (X, \tau_G) \) is strongly \( P_\omega \)-closed with respect to the grill \( G \). \( \square \)

**Theorem 4.31.** Let \( G \) be a grill on \( X \). A topological space \( (X, \tau) \) is \( P_\omega \)-closed with respect to the grill \( G \) if and only if every \( \text{pre-}\omega-\theta \)-closed subset of \( X \) is \( P_\omega \)-closed with respect to the grill \( G \) and the space \( X \).

**Proof.** Let \( (X, \tau) \) be \( P_\omega \)-closed with respect to the grill \( G \) and \( A \) be a \( \text{pre-}\omega-\theta \)-closed subset of \( X \) and let \( \Sigma = \{ V_\alpha : \alpha \in \Delta \} \) be a cover of \( A \) by preopen sets of \( X \). Since \( X - A \) is a \( \text{pre-}\omega-\theta \)-open set, for each \( x \in X - A \), by the Lemma 3.6, there exists \( U_x \in \text{PO}(X, x) \) such that \( \text{pcl}_\omega(U_x) \subset X - A \). Hence \( \Sigma \cup \{ U_x : x \in X - A \} \) is a preopen cover of \( X \) and so there exist \( \alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta \) and \( x_1, x_2, \ldots, x_m \in X - A \) such that \( X - \left( \bigcup_{i=1}^n \text{pcl}_\omega(V_{\alpha_i}) \cup \bigcup_{i=1}^m \text{pcl}_\omega(U_{x_i}) \right) \notin G \). So \( X - \bigcup_{i=1}^n \text{pcl}_\omega(V_{\alpha_i}) = A - \left( \bigcup_{i=1}^n \text{pcl}_\omega(V_{\alpha_i}) \right) \subset A - \left( \bigcup_{i=1}^n \text{pcl}_\omega(V_{\alpha_i}) \right) \cup \left( \bigcup_{i=1}^m \text{pcl}_\omega(U_{x_i}) \right) \notin G \). Therefore \( A - \bigcup_{i=1}^n \text{pcl}_\omega(V_{\alpha_i}) \notin G \) and hence \( A \) is \( P_\omega \)-closed with respect to the grill \( G \) and \( X \). Again since \( X \) is a \( \text{pre-}\theta-\omega \)-closed subset of \( X \), the converse part of the theorem is obvious. \( \square \)

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