This paper is dedicated to G. Ćupona, S. Kurepa, W. Rudin, V. Perić and B. Schweizer who have passed away recently.

Abstract. D. Borwein and S. Z. Ditor have found a measurable subset $A$ of the real line having positive Lebesgue measure and a decreasing sequence $(d_n)$ of reals converging to 0 such that, for each $x$, $x + d_n \notin A$ for infinitely many $n$. The set they constructed is nowhere dense. This result prompted us to further explore the question of subsets of $R$ and $R^2$ that are of "small size" and the existence of null sequences with the described property and hence attain some related results.

1. Introduction

D. Borwein and S.Z. Ditor [1] have proved the following theorem, answering a question of P. Erdos.

Theorem 1.1. (Borwein, Ditor 1978)

1. If $A$ is a measurable set in $R$ with $m(A) > 0$, and $(d_n)$ is a sequence of reals converging to 0, then for almost all $x \in A$, $x + d_n \in A$ for infinitely many $n$.

2. There exists a measurable set $A$ in $R$ with $m(A) > 0$, and a (decreasing) sequence $(d_n)$ converging to 0, such that, for each $x$, $x + d_n \notin A$ for infinitely many $n$.

We recollect that in the proof of (2), the set of natural numbers $N$ is partitioned

$$N = \{1, 2, \ldots N_1\} \cup \{N_1 + 1, N_1 + 2, \ldots N_2\} \cup \ldots \cup \{N_k + 1, N_k + 2, \ldots N_{k+1}\} \cup \ldots$$

with $N_1 < N_2 < N_3 < \cdots < N_k < \cdots$ and $A$ is constructed as follows:

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\[ x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}, \quad a_n \in \{0,1\}, \quad \text{is in } A \]

iff for every \( k \), there exist \( m \in I_k \) so that \( a_m = 0 \), and \( a_n = 1 \) for infinitely many \( n \), where

\[ I_1 = \{1,2,\ldots,N_1\}, I_2 = \{N_1+1,\ldots,N_2\}, \ldots I_k = \{N_{k-1}+1,\ldots,N_k\}, \ldots. \]

A is closed and \( m(A) \) can be made arbitrarily close to 1.

The sequence \((d_n)\) is constructed as follows: For \( k \geq N \),

\[ D_k = \left\{ x = \sum_{m \in I_k} \frac{a_m}{2^m}, \quad a_m \in \{0,1\}, \quad \text{at least one } a_m = 1 \right\} \]

\[ x = 0.000 \ldots 0a_{N_{k-1}+1} \ldots a_{N_k}00 \ldots \text{ and } D = \bigcup_{k=1}^{\infty} D_k \] is countable. The elements of \( D \) converge to 0.

We see that \( m(A) > 0 \), but \( A \) is nowhere dense. In 2010, H. I. Miller [2] proved a generalization of part (2) of the Borwein-Ditor theorem for nowhere dense subsets of \([0,1]\) (and consequently of \( R \)).

**Theorem 1.2.** (H.I. Miller 2010) *Suppose \( A \) is a nowhere dense subset of \([0,1]\). There exists a (decreasing) sequence \((d_n)\) converging to 0, such that, for each \( x \), \( x + d_n \notin A \) for infinitely many \( n \).*

Here we show a two-dimensional version of the above theorem.

**Theorem 1.3.** *Suppose \( A \) is a nowhere dense subset of \([0,1] \times \[0,1]\). There exists a sequence \((d_n) \in \mathbb{R}^2\) converging to 0, such that, for each \( x \), \( x + d_n \notin A \) for infinitely many \( n \).*

**Proof.** Suppose \( n \in \mathbb{N} \) is arbitrarily fixed. Divide \([0,1] \times [0,1]\) into \( nxn \) squares by partitioning the interval \([0,1]\) into \( n \) subintervals \( 0 < \frac{1}{n} < \frac{2}{n} < \cdots < 1 \). In each square, fix the disk of radius \( \frac{1}{4n} \) centered around the center of the square. Inside that disk, fix a disk that is disjoint with \( A \). There exists \( \delta_n \in \mathbb{N} \) and \( \epsilon_n > 0 \) such that: for each \( x \in [0,1] \times [0,1] \), one of the rays \( x + \lambda e^{\frac{\pi}{n}}, x + \lambda e^{\frac{2\pi}{n}}, x + \lambda e^{\frac{3\pi}{n}}, \ldots, x + \lambda e^{\frac{2(n-1)\pi}{n}} \) \( (\lambda > 0) \) crosses one of the fixed \( nxn \) disks disjoint with \( A \) with a segment of length \( \epsilon_n > 0 \).

Fix the maximal \( k_n \in \mathbb{N} \) with \( k_n \epsilon_n < \frac{2}{n} \). Let

\[ D_n = \left\{ \epsilon_n \cdot e^{\frac{\pi}{nn}}, 2\epsilon_n \cdot e^{\frac{\pi}{nn}}, \ldots, k_n \epsilon_n \cdot e^{\frac{\pi}{nn}}, \epsilon_n \cdot e^{\frac{2\pi}{nn}}, 2\epsilon_n \cdot e^{\frac{2\pi}{nn}}, \ldots, k_n \epsilon_n \cdot e^{\frac{2\pi}{nn}}, \epsilon_n \cdot e^{\frac{3\pi}{nn}}, 2\epsilon_n \cdot e^{\frac{3\pi}{nn}}, \ldots, k_n \epsilon_n \cdot e^{\frac{3\pi}{nn}}, \ldots, 2\epsilon_n \cdot e^{\frac{2(n-1)\pi}{nn}}, \ldots, k_n \epsilon_n \cdot e^{\frac{2(n-1)\pi}{nn}} \right\}. \]
For each \( x \in A \), there exists \( d \in D_n \), \( x + d \notin A \). Let \( D = \bigcup_{n=1}^{\infty} D_n \). \( D \) is countable. The elements of \( D \) converge to 0. If they are arranged as a sequence, the proof is complete.

The question naturally arises whether an analogous statement can be proved for other types of small sets in \( \mathbb{R} \), (for example sets of measure 0). We show that there exists a set of outer measure 0 in \( \mathbb{R} \) for which the opposite of the statement in part (2) of the Borwein-Ditor theorem holds. First we show a more general result.

**Theorem 1.4.** Suppose \( A = [0,1] \setminus X \), where \( X \) is a set of first category. Then for every sequence \((d_n)\) converging to 0, there exists \( x \in A \), \( x + d_n \notin A \) for \( n \) large enough.

**Proof.** Since \( X \) is a set of first category, \( X = \bigcup_{i=1}^{\infty} X_i \) where \( X_i \) is nowhere dense in \([0,1]\) for \( i \in N \). Then \( A = \bigcap_{i=1}^{\infty} A_i \) where \( A_i = [0,1] \setminus X_i \) for \( i \in N \).

Suppose \((d_n)\) is a sequence of reals converging to 0. Let \( n_0 \in N \) be fixed so that \( |d_n| < \frac{1}{4} \) for \( n \geq n_0 \). We need to find \( x \in R \) that satisfies:

- (A) \( x \in A_i \) for \( i \in N \),
- (B) \( x + d_n \in A_i \) for \( i \in N \), \( n \geq n_0 \).

There are countably many conditions in (A) as well as in (B), so we can order them together as a sequence of conditions \( C_1, C_2, \ldots C_k \ldots \).

**Claim:** The set of \( x \in [\frac{1}{4}, \frac{3}{4}] \) that satisfies condition \( C_k \) has a complement that is nowhere dense in \( [\frac{1}{2}, \frac{3}{2}] \), for \( k \in N \).

**Proof of claim:** If \( C_k \) is the condition that \( x \in A_i \) for some \( i \), then since \( X_i \) is nowhere dense in \([0,1]\), the claim is true. Suppose \( C_k \) is the condition that \( x + d_n \in A_i \) for some \( n \geq n_0 \), and some \( i \in N \). Since \( A_i \) has a complement that is nowhere dense in \([0,1], -d_n + A_i \) has a complement that is nowhere dense in \([-d_n, 1 - d_n] \) and consequently nowhere dense in \([\frac{1}{4}, \frac{3}{4}] \) (since \( |d_n| < \frac{1}{4} \)).

So the set of \( x \) for which \( x + d_n \in A_i \) has a complement that is nowhere dense in \([\frac{1}{4}, \frac{3}{4}] \).

The claim is proved.

From the above claim, we see that the set of \( x \) in \([\frac{1}{4}, \frac{3}{4}] \) satisfying conditions \( C_1, C_2, \ldots C_k \ldots \), is an intersection of countably many sets that have nowhere dense complements. This set has a complement of first category and is therefore nonempty. This completes the proof.

**Remark 1.5.** In the proof of the above theorem, it has been shown that the set of \( x \) satisfying the conclusion is actually large (has a complement of first category).

The next theorem is a corollary of Theorem 1.4.
**Theorem 1.6.** There exists a set $A \subset [0, 1]$ with outer measure 0, such that for every sequence $(d_n)$ converging to 0, there exists $x \in A$, $x + d_n \in A$ for $n$ large enough.

**Proof.** By the proof of Borwein-Ditor (2) (see the first page), for each $n$ we can construct $X_n \subset [0, 1]$, $X_n$ nowhere dense, $m(X_n) = 1 - \frac{1}{n}$ (by choosing $N_1, N_2, \ldots, N_k$ appropriately). Let

$$A = [0, 1] \setminus \bigcup_{n=1}^{\infty} X_n = \bigcap_{n=1}^{\infty} X_n^c.$$  

Then $A$ has outer measure 0 and the conclusion follows from Theorem 1.4. \qed

We add a natural generalization of Theorem 1.4 in two dimensions.

**Theorem 1.7.** Suppose $A = [0, 1] \times [0, 1] \setminus X$, where $X$ is a set of first category. Then for every sequence $(d_n) \in \mathbb{R}^2$ converging to 0, there exists $x \in A$, $x + d_n \in A$ for $n$ large enough.

**Proof.** Since $X$ is a set of first category, $X = \bigcup_{i=1}^{\infty} X_i$ where $X_i$ is nowhere dense in $[0, 1] \times [0, 1]$ for $i \in N$. Then $A = \bigcap_{i=1}^{\infty} A_i$ where $A_i = [0, 1] \times [0, 1] \setminus X_i$ for $i \in N$.

Suppose $(d_n)$ is a sequence of vectors in $\mathbb{R}^2$ converging to 0. Let $n_0 \in N$ be fixed so that $|d_n| < \frac{1}{4}$ for $n \geq n_0$. We need to find $x \in \mathbb{R}^2$ that satisfies:

(A) $x \in A_i$ for $i \in N$,

(B) $x + d_n \in A_i$ for $i \in N$, $n \geq n_0$.

There are countably many conditions in (A) as well as in (B), so we can order them together as a sequence of conditions $C_1, C_2, \ldots, C_k$, etc.

Let $B$ denote the closed disk of radius $\frac{1}{4}$ centered around $(\frac{1}{2}, \frac{1}{2})$. The following claim can be verified by the same reasoning that was used in the proof of Theorem 1.4:

The set of $x \in B$ that satisfies condition $C_k$ has a complement that is nowhere dense in $B$, for $k \in N$.

From the above claim, we see that the set of $x$ satisfying conditions $C_1, C_2, \ldots, C_k$, etc., is nonempty. This completes the proof. \qed

**References**


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