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NEW JENSEN'S TYPE INEQUALITIES FOR DIFFERENTIABLE LOG-CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

S.S. DRAGOMIR

Abstract. Some new Jensen's type inequalities for differentiable log-convex functions of selfadjoint operators in Hilbert spaces under suitable assumptions for the involved operators are given. Applications for particular cases of interest are also provided.

1. Introduction

Jensen's inequality for convex functions is one of the most important result in the Theory of Inequalities due to the fact that many other famous inequalities are particular cases of this for appropriate choices of the function involved, see for instance [9, p.].

The following result that provides an operator version for the Jensen inequality for convex functions is due to Mond and Pečarić [10] (see also [5, p. 5]):

Theorem 1 (Mond-Pečarić, 1993, [10])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle \tag{MP}$$

for each $x \in H$ with $\|x\| = 1$.

Taking into account the above result and its applications for various concrete examples of convex functions, it is therefore natural to investigate the corresponding results for the case of *log-convex functions*, namely functions $f : I \to (0, \infty)$ for which $\ln f$ is convex.

We observe that such functions satisfy the elementary inequality

$$f((1-t)a + tb) \leq [f(a)]^{1-t} [f(b)]^{t}$$

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for any $a, b \in I$ and $t \in [0,1]$. Also, due to the fact that the weighted geometric mean is less than the weighted arithmetic mean, it follows that any log-convex function is a convex functions. However, obviously, there are functions that are convex but not log-convex.

As an immediate consequence of the Mond-Pečarić inequality above we can provide the following result, see for instance [4]:

Theorem 2. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $g : [m, M] \to (0, \infty)$ is log-convex, then*

$$g(\langle Ax, x\rangle) \leq \exp \langle \ln g(A) x, x\rangle \leq \langle g(A) x, x\rangle \tag{1.1}$$

for each $x \in H$ with $\|x\| = 1$.

The following reverse for the Mond-Pečarić inequality that generalizes the scalar Lah-Ribarić inequality for convex functions is well known, see for instance [5, p. 57]:

Theorem 3. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$\langle f(A) x, x\rangle \leq \frac{M - \langle Ax, x\rangle}{M - m} \cdot f(m) + \frac{\langle Ax, x\rangle - m}{M - m} \cdot f(M) \tag{1.2}$$

for each $x \in H$ with $\|x\| = 1$.

This result can be improved for log-convex functions as follows [4]:

Theorem 4. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $g : [m, M] \to (0, \infty)$ is log-convex, then*

$$\begin{aligned} \langle g(A) x, x\rangle &\leq \left\langle \left[[g(m)]^{\frac{M1_H - A}{M-m}} [g(M)]^{\frac{A - m1_H}{M-m}} \right] x, x \right\rangle \\ &\leq \frac{M - \langle Ax, x\rangle}{M - m} \cdot g(m) + \frac{\langle Ax, x\rangle - m}{M - m} \cdot g(M) \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} g(\langle Ax, x\rangle) &\leq [g(m)]^{\frac{M - \langle Ax, x\rangle}{M-m}} [g(M)]^{\frac{\langle Ax, x\rangle - m}{M-m}} \\ &\leq \left\langle \left[[g(m)]^{\frac{M1_H - A}{M-m}} [g(M)]^{\frac{A - m1_H}{M-m}} \right] x, x \right\rangle \end{aligned} \tag{1.4}$$

for each $x \in H$ with $\|x\| = 1$, where 1_H is the identity operator on H.

The following result that provides both a refinement and a reverse of the multiplicative version of Jensen's inequality for differentiable log-convex functions can be stated as well, see [4]:

Theorem 5. *Let J be an interval and $g : J \to \mathbb{R}$ be a log-convex differentiable function on $\mathring{J}$ whose derivative g' is continuous on $\mathring{J}$. If A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset \mathring{J}$, then*

$$1 \leq \left\langle \exp\left[\frac{g'(\langle Ax, x\rangle)}{g(\langle Ax, x\rangle)}(A - \langle Ax, x\rangle 1_H)\right] x, x \right\rangle$$
$$\leq \frac{\langle g(A)x, x\rangle}{g(\langle Ax, x\rangle)} \leq \left\langle \exp\left[g'(A)[g(A)]^{-1}(A - \langle Ax, x\rangle 1_H)\right] x, x \right\rangle \quad (1.5)$$

for each $x \in H$ with $\|x\| = 1$.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle ., . \rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [5, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;
(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;
(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, $1_H.e.$ $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A) \quad \text{(P)}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [5] and the references therein. For other results, see [12], [6], [11] and [8]. For recent results, see [1] and [2].

The main aim of the present paper is to establish other Jensen's type inequality for differentiable log-convex functions. Some applications for the logarithmic convex function $g(t) = t^{-r}$ with $r > 0$ and $t > 0$ are given as well.

2. More inequalities for differentiable log-convex functions

The following results providing companion inequalities for the Jensen inequality for differentiable log-convex functions obtained in Theorem 5 hold:

Theorem 6. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $g : J \to (0, \infty)$ is a differentiable log-convex function with the derivative continuous on $\mathring{J}$ and $[m, M] \subset \mathring{J}$, then*

$$\exp\left[\frac{\langle g'(A) Ax, x\rangle}{\langle g(A) x, x\rangle} - \frac{\langle g(A) Ax, x\rangle}{\langle g(A) x, x\rangle} \cdot \frac{\langle g'(A) x, x\rangle}{\langle g(A) x, x\rangle}\right] \geq \frac{\exp\left[\frac{\langle g(A) \ln g(A) x, x\rangle}{\langle g(A) x, x\rangle}\right]}{g\left(\frac{\langle g(A) Ax, x\rangle}{\langle g(A) x, x\rangle}\right)} \geq 1 \tag{2.1}$$

for each $x \in H$ with $\|x\| = 1$.

If

$$\frac{\langle g'(A) Ax, x\rangle}{\langle g'(A) x, x\rangle} \in \mathring{J} \text{ for each } x \in H \text{ with } \|x\| = 1, \tag{C}$$

then

$$\exp\left[\frac{g'\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}{g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}\left(\frac{\langle g'(A) Ax, x\rangle}{\langle g'(A) x, x\rangle} - \frac{\langle Ag(A) x, x\rangle}{\langle g(A) x, x\rangle}\right)\right] \geq \frac{g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}{\exp\left(\frac{\langle g(A) \ln g(A) x, x\rangle}{\langle g(A) x, x\rangle}\right)} \geq 1, \tag{2.2}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. By the gradient inequality for the convex function $\ln g$ we have

$$\frac{g'(t)}{g(t)}(t - s) \geq \ln g(t) - \ln g(s) \geq \frac{g'(s)}{g(s)}(t - s) \tag{2.3}$$

for any $t, s \in \mathring{J}$, which by multiplication with $g(t) > 0$ is equivalent with

$$g'(t)(t - s) \geq g(t) \ln g(t) - g(t) \ln g(s) \geq \frac{g'(s)}{g(s)}(tg(t) - sg(t)) \tag{2.4}$$

for any $t, s \in \mathring{J}$.

Fix $s \in \mathring{J}$ and apply the property (P) to get that

$$\langle g'(A) Ax, x\rangle - s\langle g'(A) x, x\rangle \geq \langle g(A) \ln g(A) x, x\rangle - \langle g(A) x, x\rangle \ln g(s)$$

$$\geq \frac{g'(s)}{g(s)} \left(\langle Ag(A)x, x \rangle - s \langle g(A)x, x \rangle \right) \quad (2.5)$$

for any $x \in H$ with $\|x\| = 1$, which is an inequality of interest in itself as well.

Since

$$\frac{\langle g(A)Ax, x \rangle}{\langle g(A)x, x \rangle} \in [m, M] \text{ for any } x \in H \text{ with } \|x\| = 1$$

then on choosing $s := \frac{\langle g(A)Ax,x \rangle}{\langle g(A)x,x \rangle}$ in (2.5) we get

$$\left\langle g'(A)Ax, x \right\rangle - \frac{\langle g(A)Ax, x \rangle}{\langle g(A)x, x \rangle} \left\langle g'(A)x, x \right\rangle$$
$$\geq \langle g(A) \ln g(A)x, x \rangle - \langle g(A)x, x \rangle \ln g\left(\frac{\langle g(A)Ax, x \rangle}{\langle g(A)x, x \rangle} \right) \geq 0,$$

which, by division with $\langle g(A)x, x \rangle > 0$, produces

$$\frac{\langle g'(A)Ax, x \rangle}{\langle g(A)x, x \rangle} - \frac{\langle g(A)Ax, x \rangle}{\langle g(A)x, x \rangle} \cdot \frac{\langle g'(A)x, x \rangle}{\langle g(A)x, x \rangle}$$
$$\geq \frac{\langle g(A) \ln g(A)x, x \rangle}{\langle g(A)x, x \rangle} - \ln g\left(\frac{\langle g(A)Ax, x \rangle}{\langle g(A)x, x \rangle} \right) \geq 0 \quad (2.6)$$

for any $x \in H$ with $\|x\| = 1$.

Taking the exponential in (2.6) we deduce the desired inequality (2.1).

Now, assuming that the condition (C) holds, then by choosing $s := \frac{\langle g'(A)Ax,x \rangle}{\langle g'(A)x,x \rangle}$ in (2.5) we get

$$0 \geq \langle g(A) \ln g(A)x, x \rangle - \langle g(A)x, x \rangle \ln g\left(\frac{\langle g'(A)Ax, x \rangle}{\langle g'(A)x, x \rangle} \right)$$
$$\geq \frac{g'\left(\frac{\langle g'(A)Ax,x \rangle}{\langle g'(A)x,x \rangle} \right)}{g\left(\frac{\langle g'(A)Ax,x \rangle}{\langle g'(A)x,x \rangle} \right)} \left(\langle Ag(A)x, x \rangle - \frac{\langle g'(A)Ax, x \rangle}{\langle g'(A)x, x \rangle} \langle g(A)x, x \rangle \right)$$

which, by dividing with $\langle g(A)x, x \rangle > 0$ and rearranging, is equivalent with

$$\frac{g'\left(\frac{\langle g'(A)Ax,x \rangle}{\langle g'(A)x,x \rangle} \right)}{g\left(\frac{\langle g'(A)Ax,x \rangle}{\langle g'(A)x,x \rangle} \right)} \left(\frac{\langle g'(A)Ax, x \rangle}{\langle g'(A)x, x \rangle} - \frac{\langle Ag(A)x, x \rangle}{\langle g(A)x, x \rangle} \right)$$
$$\geq \ln g\left(\frac{\langle g'(A)Ax, x \rangle}{\langle g'(A)x, x \rangle} \right) - \frac{\langle g(A) \ln g(A)x, x \rangle}{\langle g(A)x, x \rangle} \geq 0 \quad (2.7)$$

for any $x \in H$ with $\|x\| = 1$.

Finally, on taking the exponential in (2.7) we deduce the desired inequality (2.2). □

Remark 1. We observe that a sufficient condition for (C) to hold is that either $g'(A)$ or $-g'(A)$ is a positive definite operator on H.

Corollary 1. *Assume that A and g are as in Theorem 6. If the condition (C) holds, then we have the double inequality*

$$\ln g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)\geq\frac{\langle g(A)\ln g(A)x,x\rangle}{\langle g(A)x,x\rangle}\geq\ln g\left(\frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\right),\quad(2.8)$$

for any $x\in H$ with $\|x\|=1$.

Remark 2. Assume that A is a positive definite operator on H. Since for $r>0$ the function $g(t)=t^{-r}$ is log-convex on $(0,\infty)$ and

$$\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}=\frac{\langle A^{-r}x,x\rangle}{\langle A^{-r-1}x,x\rangle}>0$$

for any $x\in H$ with $\|x\|=1$, then on applying the inequality (2.8) we deduce the following interesting result

$$\ln\left(\frac{\langle A^{-r}x,x\rangle}{\langle A^{-r-1}x,x\rangle}\right)\leq\frac{\langle A^{-r}\ln Ax,x\rangle}{\langle A^{-r}x,x\rangle}\leq\ln\left(\frac{\langle A^{-r+1}x,x\rangle}{\langle A^{-r}x,x\rangle}\right)\qquad(2.9)$$

for any $x\in H$ with $\|x\|=1$.

The details of the proof are left to the interested reader.

The case of sequences of operators is embodied in the following corollary:

Corollary 2. *Let A_j, $j\in\{1,\dots,n\}$ be selfadjoint operators on the Hilbert space H and assume that $Sp(A_j)\subseteq[m,M]$ for some scalars m,M with $m<M$ and each $j\in\{1,\dots,n\}$. If $g:J\to(0,\infty)$ is a differentiable log-convex function with the derivative continuous on $\mathring{J}$ and $[m,M]\subset\mathring{J}$, then*

$$\exp\left[\frac{\sum_{j=1}^n\langle g'(A_j)A_jx_j,x_j\rangle}{\sum_{j=1}^n\langle g(A_j)x_j,x_j\rangle}-\frac{\sum_{j=1}^n\langle g(A_j)A_jx_j,x_j\rangle}{\sum_{j=1}^n\langle g(A_j)x_j,x_j\rangle}\cdot\frac{\sum_{j=1}^n\langle g'(A_j)x_j,x_j\rangle}{\sum_{j=1}^n\langle g(A_j)x_j,x_j\rangle}\right]$$
$$\geq\frac{\exp\left[\frac{\sum_{j=1}^n\langle g(A_j)\ln g(A_j)x_j,x_j\rangle}{\sum_{j=1}^n\langle g(A_j)x_j,x_j\rangle}\right]}{g\left(\frac{\sum_{j=1}^n\langle g(A_j)A_jx_j,x_j\rangle}{\sum_{j=1}^n\langle g(A_j)x_j,x_j\rangle}\right)}\geq1\qquad(2.10)$$

for each $x_j\in H$, $j\in\{1,\dots,n\}$ with $\sum_{j=1}^n\|x_j\|^2=1$.

If

$$\frac{\sum_{j=1}^{n}\langle g'(A_j)A_jx_j,x_j\rangle}{\sum_{j=1}^{n}\langle g'(A_j)x_j,x_j\rangle}\in\mathring{J} \tag{2.11}$$

for each $x_j\in H, j\in\{1,\dots,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=1$, *then*

$$\exp\left[\frac{g'\left(\frac{\sum_{j=1}^{n}\langle g'(A_j)A_jx_j,x_j\rangle}{\sum_{j=1}^{n}\langle g'(A_j)x_j,x_j\rangle}\right)}{g\left(\frac{\sum_{j=1}^{n}\langle g'(A_j)A_jx_j,x_j\rangle}{\sum_{j=1}^{n}\langle g'(A_j)x_j,x_j\rangle}\right)}\times\left(\frac{\sum_{j=1}^{n}\langle g'(A_j)A_jx_j,x_j\rangle}{\sum_{j=1}^{n}\langle g'(A_j)x_j,x_j\rangle}-\frac{\sum_{j=1}^{n}\langle A_jg(A_j)x_j,x_j\rangle}{\sum_{j=1}^{n}\langle g(A_j)x_j,x_j\rangle}\right)\right]$$
$$\geq\frac{g\left(\frac{\sum_{j=1}^{n}\langle g'(A_j)A_jx_j,x_j\rangle}{\sum_{j=1}^{n}\langle g'(A_j)x_j,x_j\rangle}\right)}{\exp\left(\frac{\sum_{j=1}^{n}\langle g(A_j)\ln g(A_j)x_j,x_j\rangle}{\sum_{j=1}^{n}\langle g(A_j)x_j,x_j\rangle}\right)}\geq 1, \tag{2.12}$$

for each $x_j\in H$, $j\in\{1,\dots,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=1$.

Proof. As in [5, p. 6], if we put

$$\widetilde{A}:=\begin{pmatrix}A_1 & . & . & . & 0\\ & . & & & \\ & & . & & \\ & & & . & \\ 0 & . & . & . & A_n\end{pmatrix}\text{ and }\widetilde{x}=\begin{pmatrix}x_1\\ .\\ .\\ .\\ x_n\end{pmatrix}$$

then we have $Sp\left(\widetilde{A}\right)\subseteq[m,M]$, $\|\widetilde{x}\|=1$,

$$\left\langle g\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle=\sum_{j=1}^{n}\langle g(A_j)x_j,x_j\rangle,\left\langle\widetilde{A}\widetilde{x},\widetilde{x}\right\rangle=\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle$$

and so on.

Applying Theorem 6 for $\widetilde{A}$ and $\widetilde{x}$ we deduce the desired results. □

The following particular case for sequences of operators also holds:

Corollary 3. *With the assumptions of Corollary* 2 *and if* $p_j\geq 0$, $j\in\{1,\dots,n\}$ *with* $\sum_{j=1}^{n}p_j=1$, *then*

$$\exp\left[\frac{\left\langle\sum_{j=1}^{n}p_jg'(A_j)A_jx,x\right\rangle}{\left\langle\sum_{j=1}^{n}p_jg(A_j)x,x\right\rangle}\right.$$

$$
\left.-\frac{\left\langle\sum_{j=1}^{n} p_j g\left(A_j\right) A_j x, x\right\rangle}{\left\langle\sum_{j=1}^{n} p_j g\left(A_j\right) x, x\right\rangle} \cdot \frac{\left\langle\sum_{j=1}^{n} p_j g'\left(A_j\right) x, x\right\rangle}{\left\langle\sum_{j=1}^{n} p_j g\left(A_j\right) x, x\right\rangle}\right]
$$

$$
\geq \frac{\exp\left[\frac{\left\langle\sum_{j=1}^{n} p_j g(A_j) \ln g(A_j) x, x\right\rangle}{\left\langle\sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle}\right]}{g\left(\frac{\left\langle\sum_{j=1}^{n} p_j g(A_j) A_j x, x\right\rangle}{\left\langle\sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle}\right)} \geq 1 \quad (2.13)
$$

for each $x \in H$, *with* $\|x\| = 1$.

If

$$
\frac{\left\langle\sum_{j=1}^{n} p_j g'\left(A_j\right) A_j x, x\right\rangle}{\left\langle\sum_{j=1}^{n} p_j g'\left(A_j\right) x, x\right\rangle} \in \mathring{J} \quad (2.14)
$$

for each $x \in H$, *with* $\|x\| = 1$, *then*

$$
\exp\left[\frac{g'\left(\frac{\left\langle\sum_{j=1}^{n} p_j g'(A_j) A_j x, x\right\rangle}{\left\langle\sum_{j=1}^{n} p_j g'(A_j) x, x\right\rangle}\right)}{g\left(\frac{\left\langle\sum_{j=1}^{n} p_j g'(A_j) A_j x, x\right\rangle}{\left\langle\sum_{j=1}^{n} p_j g'(A_j) x, x\right\rangle}\right)}\right.
$$

$$
\left.\times\left(\frac{\left\langle\sum_{j=1}^{n} p_j g'\left(A_j\right) A_j x, x\right\rangle}{\left\langle\sum_{j=1}^{n} p_j g'\left(A_j\right) x, x\right\rangle} - \frac{\left\langle\sum_{j=1}^{n} p_j A_j g\left(A_j\right) x, x\right\rangle}{\left\langle\sum_{j=1}^{n} p_j g\left(A_j\right) x, x\right\rangle}\right)\right]
$$

$$
\geq \frac{g\left(\frac{\left\langle\sum_{j=1}^{n} p_j g'(A_j) A_j x, x\right\rangle}{\left\langle\sum_{j=1}^{n} p_j g'(A_j) x, x\right\rangle}\right)}{\exp\left(\frac{\left\langle\sum_{j=1}^{n} p_j g(A_j) \ln g(A_j) x, x\right\rangle}{\left\langle\sum_{j=1}^{n} p_j g(A_j) x, x\right\rangle}\right)} \geq 1, \quad (2.15)
$$

for each $x \in H$, *with* $\|x\| = 1$.

Proof. Follows from Corollary 2 by choosing $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, \ldots, n\}$ where $x \in H$ with $\|x\| = 1$. □

The following result providing different inequalities also holds:

Theorem 7. *Let* A *be a selfadjoint operator on the Hilbert space* H *and assume that* $Sp(A) \subseteq [m, M]$ *for some scalars* m, M *with* $m < M$. *If* $g : J \to (0, \infty)$ *is a differentiable log-convex function with the derivative continuous on* $\mathring{J}$ *and* $[m, M] \subset \mathring{J}$, *then*

$$
\left\langle \exp\left[g'(A)\left(A - \frac{\langle g(A) A x, x\rangle}{\langle g(A) x, x\rangle} 1_H\right)\right] x, x\right\rangle
$$

$$\geq \left\langle \left(\frac{g(A)}{g\left(\frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\right)} \right)^{g(A)} x, x \right\rangle$$

$$\geq \left\langle \exp\left[\frac{g'\left(\frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\right)}{g\left(\frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle}\right)} \left(Ag(A) - \frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle} g(A) \right) \right] x, x \right\rangle \geq 1 \tag{2.16}$$

for each $x \in H$ *with* $\|x\| = 1$.

If the condition (C) *from Theorem* 6 *holds, then*

$$\left\langle \exp\left[\frac{g'\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)}{g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}\right)} \left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle} g(A) - Ag(A) \right) \right] x, x \right\rangle$$

$$\geq \left\langle \left(g\left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle} \right) [g(A)]^{-1} \right)^{g(A)} x, x \right\rangle$$

$$\geq \left\langle \exp\left[g'(A) \left(\frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle} 1_H - A \right) \right] x, x \right\rangle \geq 1 \tag{2.17}$$

for each $x \in H$ *with* $\|x\| = 1$.

Proof. By taking the exponential in (2.4) we have the following inequality

$$\exp\left[g'(t)(t-s)\right] \geq \left(\frac{g(t)}{g(s)} \right)^{g(t)} \geq \exp\left[\frac{g'(s)}{g(s)} (tg(t) - sg(t)) \right] \tag{2.18}$$

for any $t, s \in \mathring{J}$.

If we fix $s \in \mathring{J}$ and apply the property (P) to the inequality (2.18), we deduce

$$\langle \exp\left[g'(A)(A - s1_H)\right] x, x \rangle \geq \left\langle \left(\frac{g(A)}{g(s)} \right)^{g(A)} x, x \right\rangle$$

$$\geq \left\langle \exp\left[\frac{g'(s)}{g(s)} (Ag(A) - sg(A)) \right] x, x \right\rangle \tag{2.19}$$

for each $x \in H$ with $\|x\| = 1$, where 1_H is the identity operator on H.

By Mond-Pečarić's inequality applied for the convex function exp we also have

$$\left\langle \exp\left[\frac{g'(s)}{g(s)} (Ag(A) - sg(A)) \right] x, x \right\rangle$$

$$\geq \exp\left(\frac{g'(s)}{g(s)} (\langle Ag(A)x, x\rangle - s\langle g(A)x, x\rangle) \right) \tag{2.20}$$

for each $s \in \mathring{J}$ and $x \in H$ with $\|x\| = 1$.

Now, if we choose $s := \frac{\langle g(A)Ax,x\rangle}{\langle g(A)x,x\rangle} \in [m, M]$ in (2.19) and (2.20) we deduce the desired result (2.16).

Observe that, the inequality (2.18) is equivalent with

$$\exp\left[\frac{g'(s)}{g(s)}(sg(t) - tg(t))\right] \geq \left(\frac{g(s)}{g(t)}\right)^{g(t)} \geq \exp\left[g'(t)(s-t)\right] \tag{2.21}$$

for any $t, s \in \mathring{J}$.

If we fix $s \in \mathring{J}$ and apply the property (P) to the inequality (2.21) we deduce

$$\left\langle \exp\left[\frac{g'(s)}{g(s)}(sg(A) - Ag(A))\right] x, x\right\rangle \geq \left\langle \left(g(s)[g(A)]^{-1}\right)^{g(A)} x, x\right\rangle \geq \left\langle \exp\left[g'(A)(s1_H - A)\right] x, x\right\rangle \tag{2.22}$$

for each $x \in H$ with $\|x\| = 1$.

By Mond-Pečarić's inequality we also have

$$\left\langle \exp\left[g'(A)(s1_H - A)\right] x, x\right\rangle \geq \exp\left[s\left\langle g'(A)x, x\right\rangle - \left\langle g'(A)Ax, x\right\rangle\right] \tag{2.23}$$

for each $s \in \mathring{J}$ and $x \in H$ with $\|x\| = 1$.

Taking into account that the condition (C) is valid, then we can choose in (2.22) and (2.23) $s := \frac{\langle g'(A)Ax,x\rangle}{\langle g'(A)x,x\rangle}$ to get the desired result (2.17). □

Remark 3. If we apply, for instance, the inequality (2.16) for the log-convex function $g(t) = t^{-1}, t > 0$, then, after simple calculations, we get the inequality

$$\left\langle \exp\left(\frac{A^{-2} - \left\langle A^{-1}x, x\right\rangle A^{-1}}{A^{-2} - \langle A^{-1}x, x\rangle}\right) x, x\right\rangle \geq \left\langle \left(\left\langle A^{-1}x, x\right\rangle A^{-1}\right)^{A^{-1}} x, x\right\rangle \geq \left\langle \exp\left(\frac{A^{-1} - \left\langle A^{-1}x, x\right\rangle 1_H}{\langle A^{-1}x, x\rangle^2}\right) x, x\right\rangle \geq 1 \tag{2.24}$$

for each $x \in H$ with $\|x\| = 1$.

Other similar results can be obtained from the inequality (2.17), however the details are left to the interested reader.

3. A reverse inequality

The following reverse inequality that provides a companion for the results in Theorem 4 is also of interest:

Theorem 8. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $g : J \to (0, \infty)$ is a differentiable log-convex function with the derivative continuous on $\mathring{J}$ and $[m, M] \subset \mathring{J}$, then*

$$\begin{aligned}(1 \leq)\ & \frac{[g(m)]^{\frac{M-\langle Ax,x\rangle}{M-m}}[g(M)]^{\frac{\langle Ax,x\rangle-m}{M-m}}}{\exp\langle \ln g(A)x, x\rangle} \\ & \leq \exp\left[\frac{\langle (M1_H - A)(A - m1_H)x, x\rangle}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \\ & \leq \exp\left[\frac{(M - \langle Ax, x\rangle)(\langle Ax, x\rangle - m)}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \\ & \leq \exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \end{aligned} \tag{3.1}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Utilizing the inequality (2.3) we have successively

$$\ln g((1-\lambda)t + \lambda s) - \ln g(s) \geq (1-\lambda)\frac{g'(s)}{g(s)}(t-s) \tag{3.2}$$

and

$$\ln g((1-\lambda)t + \lambda s) - \ln g(t) \geq -\lambda\frac{g'(t)}{g(t)}(t-s) \tag{3.3}$$

for any $t, s \in \mathring{J}$ and any $\lambda \in [0, 1]$.

Now, if we multiply (3.2) by λ and (3.3) by $1-\lambda$ and sum the obtained inequalities, we deduce

$$\begin{aligned}(1-\lambda)\ln g(t) + \lambda \ln g(s) - \ln g((1-\lambda)t + \lambda s) \\ \leq (1-\lambda)\lambda\left[\left(\frac{g'(t)}{g(t)} - \frac{g'(s)}{g(s)}\right)(t-s)\right]\end{aligned} \tag{3.4}$$

for any $t, s \in \mathring{J}$ and any $\lambda \in [0, 1]$.

Now, if we choose $\lambda := \frac{M-u}{M-m}$, $s := m$ and $t := M$ in (3.4) then we get the inequality

$$\begin{aligned}\frac{u-m}{M-m}\ln g(M) + \frac{M-u}{M-m}\ln g(m) - \ln g(u) \\ \leq \left[\frac{(M-u)(u-m)}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right]\end{aligned} \tag{3.5}$$

for any $u \in [m, M]$.

If we use the property (P) for the operator A we get

$$\frac{\langle Ax,x\rangle - m}{M-m}\ln g(M) + \frac{M-\langle Ax,x\rangle}{M-m}\ln g(m) - \langle \ln g(A)x,x\rangle$$
$$\leq \left[\frac{\langle (M1_H - A)(A-m1_H)x,x\rangle}{M-m}\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \quad (3.6)$$

for each $x\in H$ with $\|x\|=1$.

Taking the exponential in (3.6) we deduce the first inequality in (3.1).

Now, consider the function $h:[m,M]\to\mathbb{R}$, $h(t)=(M-t)(t-m)$. This function is concave in $[m,M]$ and by Mond-Pečarić's inequality (MP) we have

$$\langle (M1_H - A)(A-m1_H)x,x\rangle \leq (M-\langle Ax,x\rangle)(\langle Ax,x\rangle - m)$$

for each $x\in H$ with $\|x\|=1$, which proves the second inequality in (3.1).

For the last inequality, we observe that

$$(M-\langle Ax,x\rangle)(\langle Ax,x\rangle - m) \leq \frac{1}{4}(M-m)^2,$$

and the proof is complete. □

Corollary 4. *Assume that g is as in Theorem 8 and A_j are selfadjoint operators with $Sp(A_j)\subseteq [m,M]\subset \mathring{J}$, $j\in\{1,\ldots,n\}$.*

If and $x_j\in H, j\in\{1,\ldots,n\}$ with $\sum_{j=1}^n \|x_j\|^2=1$, then

$$(1\leq)\ \frac{[g(m)]^{\frac{M-\sum_{j=1}^n\langle A_jx_j,x_j\rangle}{M-m}}[g(M)]^{\frac{\sum_{j=1}^n\langle A_jx_j,x_j\rangle - m}{M-m}}}{\exp\left(\sum_j^n\langle \ln g(A_j)x_j,x_j\rangle\right)}$$
$$\leq \exp\left[\frac{\sum_{j=1}^n\langle (M1_H-A_j)(A_j-m1_H)x_j,x_j\rangle}{M-m}\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right]$$
$$\leq \exp\left[\frac{\left(M-\sum_{j=1}^n\langle A_jx_j,x_j\rangle\right)\left(\sum_{j=1}^n\langle A_jx_j,x_j\rangle - m\right)}{M-m}\right.$$
$$\left.\times\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right] \leq \exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right]. \quad (3.7)$$

If $p_j\geq 0$, $j\in\{1,\ldots,n\}$ with $\sum_{j=1}^n p_j=1$, then

$$(1\leq)\ \frac{[g(m)]^{\frac{M-\left\langle\sum_{j=1}^n p_jA_jx,x\right\rangle}{M-m}}[g(M)]^{\frac{\left\langle\sum_{j=1}^n p_jA_jx,x\right\rangle - m}{M-m}}}{\left\langle\prod_{j=1}^n [g(A_j)]^{p_j}x,x\right\rangle}$$
$$\leq \exp\left[\frac{\sum_{j=1}^n p_j\langle (M1_H-A_j)(A_j-m1_H)x_j,x_j\rangle}{M-m}\left(\frac{g'(M)}{g(M)}-\frac{g'(m)}{g(m)}\right)\right]$$

$$\leq \exp\left[\frac{\left(M - \left\langle \sum_{j=1}^{n} p_j A_j x, x\right\rangle\right)\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x\right\rangle - m\right)}{M-m}\right.$$

$$\left.\times\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \leq \exp\left[\frac{1}{4}(M-m)\left(\frac{g'(M)}{g(M)} - \frac{g'(m)}{g(m)}\right)\right] \quad (3.8)$$

for each $x \in H$ *with* $\|x\| = 1$.

Remark 4. Let A be a selfadjoint positive operator on a Hilbert space H. If A is invertible, then

$$(1 \leq) \frac{m^{\frac{\langle Ax,x\rangle - M}{M-m}} M^{\frac{m - \langle Ax,x\rangle}{M-m}}}{\exp\langle \ln A^{-1}x, x\rangle} \leq \exp\left[\frac{\langle (M1_H - A)(A - m1_H)x, x\rangle}{Mm}\right]$$

$$\leq \exp\left[\frac{(M - \langle Ax,x\rangle)(\langle Ax,x\rangle - m)}{Mm}\right] \leq \exp\left[\frac{1}{4}\frac{(M-m)^2}{mM}\right] \quad (3.9)$$

for all $x \in H$ with $\|x\| = 1$.

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Mathematics, School of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia
E–mail: sever.dragomir@vu.edu.au

URL: http://www.staff.vu.edu.au/rgmia/dragomir/