B.-Y. CHEN INEQUALITIES FOR SLANT SUBMANIFOLDS
IN KENMOTSU SPACE FORMS II

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Abstract. In this article, we investigate sharp inequalities involving
Chen invariants for a slant submanifold $M$ of a Kenmotsu space form
$\tilde{M}(c)$, tangent to the structure vector field of the ambient space.

1. Preliminaries

Let $(\tilde{M}, g)$ be an odd-dimensional Riemannian manifold. Then $\tilde{M}$ is
said to be an almost contact metric manifold if it admits an endomorphism $\varphi$ of its tangent bundle $T\tilde{M}$, a vector field $\xi$ (structure vector field) and a
1-form $\eta$, which satisfy:

$$\varphi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1, \eta \circ \varphi = 0,$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \eta(X) = g(X, \xi),$$

for any vector fields $X, Y$ on $\tilde{M}$.

An almost contact metric manifold is called a Kenmotsu manifold if

$$(\tilde{\nabla}_X \varphi)(Y) = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad \tilde{\nabla}_X \xi = -\varphi^2 X = X - \eta(X)\xi,$$

where $\tilde{\nabla}$ denotes the Riemannian connection with respect to $g$.

A plane section $\pi$ in $T_p\tilde{M}$ is called a $\varphi$-section if it is spanned by $X$ and $\varphi X$, where $X$ is a unit tangent vector field orthogonal to $\xi$. The sectional curvature $\tilde{K}(\pi)$ of a $\varphi$-section $\pi$ is called $\varphi$-sectional curvature. A Kenmotsu manifold with constant $\varphi$-sectional curvature $c$ is called a Kenmotsu space form and it is denoted by $\tilde{M}(c)$. Then its curvature tensor $\tilde{R}$ is expressed by

$$4\tilde{R}(X, Y)Z = (c - 3)[g(Y, Z)X - g(X, Z)Y] + (c + 1)[g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X].$$

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Let $M$ be an $n$-dimensional Riemannian manifold with induced metric $g$ isometrically immersed in $\widetilde{M}$. We denote by $TM$ and $T^\perp M$ the tangent and the normal bundles of $M$ respectively.

For any $X \in TM$, we write $\varphi X = PX + FX$, where $PX$ (respectively $FX$) denotes the tangential (respectively normal) component of $\varphi X$.

The equation of Gauss is given by

$$\widetilde{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(h(X, Z), h(Y, W)) + g(h(X, W), h(Y, Z)),$$

for any vectors $X, Y, Z, W$ tangent to $M$.

We denote by $\tau(p) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)$, where $p \in M$ and $\{e_1, \ldots, e_n\} \subset T_p M$ is an orthonormal basis, the scalar curvature of $M$ at $p \in M$.

The mean curvature vector $H$ is defined by $H = \frac{1}{\dim M} \text{trace} h$.

From now on, let $n$ (respectively $2m + 1$) be the dimension of $M$ (respectively $\widetilde{M}$). We denote by $h^r_{ij} = g(h(e_i, e_j), e_r), i, j \in \{1, \ldots, n\}, r \in \{n + 1, \ldots, 2m + 1\}$;

then we have

$$\|H\|^2 = \frac{1}{n^2} \sum_{r=n+1}^{2m+1} \left( \sum_{i=1}^{n} h^r_{ii} \right)^2, \quad \|h\|^2 = \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^{n} (h^r_{ij})^2.$$ 

Also we put

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(P e_i, e_j),$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $T_p M$ and $\{e_{n+1}, \ldots, e_{2m+1}\}$ is an orthonormal basis of $T^\perp_p M$.

A Chen invariant is defined by

$$\delta_M(p) = \tau(p) - \inf \{ K(\pi) | \pi \subset T_p M \text{ a plane section invariant by } P \}.$$ 

If the structure vector field $\xi$ is tangent to $M$, we denote by $D$ the orthogonal distribution to $\xi$ in $TM$ and we can consider the orthogonal direct decomposition $TM = D \oplus (\xi)$.

Let $\pi \subset D_p$ a plane section at $p \in M$, orthogonal to $\xi_p$. Then, $\Phi^2(\pi) = g^2(P e_1, e_2)$ is a real number which is independent of the choice of the orthonormal basis $\{e_1, e_2\}$ of $\pi$.

Let $L$ be a subspace of $T_p M$ of dimension $r \geq 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of $L$. The scalar curvature $\tau(L)$ of the $r$-plane section $L$
is given by:
\[ \tau(L) = \sum_{1 \leq \alpha < \beta \leq r} K(e_\alpha \wedge e_\beta) \]
and we denote by
\[ \Psi(L) = \sum_{1 \leq i < j \leq r} g^2(P e_i, e_j). \]

For an integer \( k \geq 0 \), we denote by \( S(n, k) \) the finite set consisting of \( k \)-tuples \((n_1, \ldots, n_k)\) of integers \( \geq 2 \) satisfying \( n_1 < n, n_1 + \cdots + n_k \leq n \). Denote by \( S(n) \) the set of \( k \)-tuples with \( k \geq 0 \) for a fixed \( n \).

For each \( k \)-tuples \((n_1, \ldots, n_k) \in S(n)\), Chen introduced a Riemannian invariant defined by:
\[ \delta(n_1, \ldots, n_k)(p) = \tau(p) - S(n_1, \ldots, n_k)(p), \]
where \( S(n_1, \ldots, n_k)(p) = \inf \{ \tau(L_1) + \cdots + \tau(L_k) \} \) and at each point \( p \in M \), \( L_1, \ldots, L_k \) run over all \( k \) mutually orthogonal subspaces of \( T_pM \) such that \( \dim L_j = n_j, j = 1, \ldots, k \).

We will consider the Chen invariant
\[ \delta'(n_1, \ldots, n_k)(p) = \tau(p) - \inf \{ \tau(L_1) + \cdots + \tau(L_k) \}, \]
where \( L_1, \ldots, L_k \) run over all \( k \) mutually orthogonal subspaces of \( T_pM \), invariant by \( P \), such that \( \dim L_j = n_j, j = 1, \ldots, k \).

For each \((n_1, \ldots, n_k) \in S(n)\), let:
\[
d(n_1, \ldots, n_k) = \frac{n^2 \left( n + k - 1 - \sum_{j=1}^{k} n_j \right)}{2 \left( n + k - \sum_{j=1}^{k} n_j \right)}, \]
\[
b(n_1, \ldots, n_k) = \frac{1}{2} \left[ n(n - 1) - \sum_{j=1}^{k} n_j(n_j - 1) \right]. \]

According to Lotta’s definition (see [10]), a submanifold \( M \) immersed into an almost contact metric manifold \( \widetilde{M} \) is called slant if the angle \( \theta(X) \) between \( \varphi X \) and \( T_pM \) is a constant \( \theta \), which is independent of the choice of \( p \in M \) and \( X \in T_pM - \langle \xi_p \rangle \). The angle \( \theta \) of a slant immersion is called the slant angle of the immersion.

Invariant and anti-invariant immersions are slant immersions with slant angle \( \theta \) equal to 0 and \( \frac{\pi}{2} \), respectively. A slant immersions which is neither invariant nor anti-invariant is called a proper slant immersion.

2. Inequalities

We recall the following lemma due to Chen.
Lemma 1. Let $a_1, \ldots, a_n, b$ be $n + 1$ ($n \geq 2$) real numbers such that:

$$\left( \sum_{i=1}^{n} a_i \right)^2 = (n-1) \left( \sum_{i=1}^{n} a_i^2 + b \right).$$

Then, $2a_1a_2 \geq b$, with the equality holding if and only if $a_1 + a_2 = a_3 = \ldots = a_n$.

Using the above lemma, we proved a Chen first inequality (see [7]).

Theorem 2. Let $\tilde{M}(c)$ be a $(2m+1)$-dimensional Kenmotsu space form and $M$ an $(n = 2k+1)$-dimensional non anti-invariant $\theta$-slant submanifold, tangent to $\xi$. Then, for any point $p \in M$ and any plane section $\pi \subset D_p$, we have:

$$\tau - K(\pi) \leq \frac{n-2}{2} \left\{ \frac{n}{n-1} \|H\|^2 + \frac{(c-3)(n+1)}{4} \right\} + \frac{(c+1)(n-1)}{8} (3\cos^2 \theta - 2) - \frac{c+1}{4} \Phi^2(\pi).$$

The equality case of the inequality holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_pM$ and an orthonormal basis $\{e_{n+1}, \ldots, e_{2m+1}\}$ of $T_p^\perp M$ such that $e_n = \xi$, $\pi$ is spanned by $e_1, e_2$ and the shape operators of $M$ in $\tilde{M}(c)$ at $p$ take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 \\ 0 & b & 0 & \ldots & 0 \\ 0 & 0 & \mu & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \mu \end{pmatrix}, \ a + b = \mu,$$

$$A_r = \begin{pmatrix} h_{11} & h_{12} & 0 & \ldots & 0 \\ h_{12} & -h_{11} & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \ r \in \{n+2, \ldots, 2m+1\}.$$

From Theorem 2 we derive the following.

Corollary 3. Let $\tilde{M}(c)$ be a $(2m+1)$-dimensional Kenmotsu space form and $M$ an $(n = 2k+1)$-dimensional invariant submanifold, tangent to $\xi$. Then, for any point $p \in M$ and any plane section $\pi \subset D_p$, we have:

$$\tau - K(\pi) \leq \frac{(c-3)(n-2)(n+1)}{8} + \frac{c+1}{4} \left\{ \frac{n-1}{2} \right\} - 3\Phi^2(\pi).$$
Theorem 4. Let $\tilde{M}(c)$ be a $(2m + 1)$-dimensional Kenmotsu space form and $M$ an $(n = 2k + 1)$-dimensional non anti-invariant $\theta$-slant submanifold, tangent to $\xi$. Then:

$$\delta_M' \leq \frac{n - 2}{2} \left\{ \frac{n^2}{n - 1} \|H\|^2 + \frac{(c - 3)(n + 1)}{4} \right\} + \frac{c + 1}{8} \left[ 3(n - 3) \cos^2 \theta - 2(n - 1) \right].$$

The equality case of the inequality holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_pM$ and an orthonormal basis $\{e_{n+1}, \ldots, e_{2m+1}\}$ of $T_p^\perp M$ such that the shape operators of $M$ in $\tilde{M}(c)$ at $p$ have the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a + b = \mu,$$

$$A_r = \begin{pmatrix} h^r_{11} & h^r_{12} & 0 & \cdots & 0 \\ h^r_{12} & -h^r_{11} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad r \in \{n + 2, \ldots, 2m + 1\}.$$ 

Proof. The proof of Theorem 4 is similar with the proof of Theorem 2 considering $\pi = Sp\{e_1, e_2\}$, with $e_2 = \frac{1}{\cos \theta} Pe_1$, because $\pi$ is invariant by $P\gamma$ and so $\Phi^2(\pi) = \cos^2 \theta$. \hfill $\Box$

Lemma 5. Let $M$ be an $(n = 2k + 1)$-dimensional submanifold, tangent to $\xi$ of a $(2m + 1)$-dimensional Kenmotsu space form $\tilde{M}(c)$. Let $n_1, \ldots, n_k$ be integers $\geq 2$ satisfying $n_1 < n, n_1 + \cdots + n_k \leq n$. For $p \in M$, let $L_j \subset T_pM$ be subspaces of $T_pM$, orthogonal to $\xi$ such that $\dim L_j = n_j, \forall j \in \{1, \ldots, k\}$. Then, we have:

$$\tau(p) - \sum_{j=1}^{k} \tau(L_j) \leq d(n_1, \ldots, n_k) \|H\|^2 + b(n_1, \ldots, n_k) \frac{c - 3}{4}$$

$$+ \frac{c + 1}{8} \left[ 3 \|P\|^2 - 2n + 2 - \sum_{j=1}^{k} 6\Psi(L_j) \right].$$

Proof. Let \( p \in M \) and \( \{e_1, \ldots, e_n = \xi\} \) be an orthonormal basis of \( T_p M \).

From the Gauss equation we get
\[
2\tau = n^2 \|H\|^2 - \|h\|^2 + \frac{c - 3}{4} n(n - 1) + \frac{c + 1}{4} \left[ 3 \|P\|^2 - 2(n - 1) \right].
\]

Denoting by
\[
\eta = 2\tau - 2d(n_1, \ldots, n_k) \|H\|^2 - \frac{c - 3}{4} n(n - 1) - \frac{c + 1}{4} \left[ 3 \|P\|^2 - 2(n - 1) \right],
\]

it follows that
\[
n^2 \|H\|^2 = \left( \eta + \|h\|^2 \right) \gamma,
\]

where \( \gamma = n + k - \sum_{j=1}^{k} n_j \).

Let \( e_{n+1} \) be a unit normal vector at \( p \) parallel to \( H(p) \) and \( \{e_{n+1}, \ldots, e_{2m+1}\} \) an orthonormal basis of \( T_p^\perp M \).

We denote by \( a_i = h_{ii}^{n+1} = g(h(e_i, e_i), e_{n+1}) \).

The relation (1) becomes
\[
\left( \sum_{i=1}^{n} a_i \right)^2 = \gamma \left[ \eta + \sum_{i \neq j} (h_{i+1}^{n+1})^2 + \sum_{i=1}^{n} (h_{ii}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2 \right].
\]

(2)

Let \( L_1, \ldots, L_k \) be \( k \) mutually orthogonal subspaces of \( T_p M \), \( \dim L_j = n_j \), defined by:
\[
L_1 = Sp \{e_1, \ldots, e_{n_1}\},
\]
\[
L_2 = Sp \{e_{n_1+1}, \ldots, e_{n_1+n_2}\},
\]
\[
\vdots
\]
\[
L_k = Sp \{e_{n_1+\cdots+n_{k-1}+1}, \ldots, e_{n_1+\cdots+n_k}\}.
\]

We denote by \( D_j, j = 1, \ldots, k \) the sets:
\[
D_1 = \{1, \ldots, n_1\},
\]
\[
D_2 = \{n_1 + 1, \ldots, n_1 + n_2\},
\]
\[
\vdots
\]
\[
D_k = \{n_1 + \cdots + n_{k-1} + 1, \ldots, n_1 + \cdots + n_k\}.
\]

Also we put:
\[
b_1 = a_1,
\]
\[
b_2 = a_2 + \cdots + a_{n_1},
\]
\[
b_3 = a_{n_1+1} + \cdots + a_{n_1+n_2},
\]
\[
\vdots
\]
\[
b_{k+1} = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+\cdots+n_k},
\]
\[
b_{k+2} = a_{n_1+\cdots+n_{k+1}},
\]
\[
\vdots
\]
\[
b_{\gamma+1} = a_n.
\]
Then the relation (2) is equivalent to

\[
\left( \sum_{i=1}^{\gamma+1} b_i \right)^2 = \gamma \left[ \eta + \sum_{i=1}^{\gamma+1} b_i^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{i=1}^{n} (h_{ij}^r)^2 \right. \\
\left. - 2 \sum_{2 \leq \alpha_1 < \beta_1 \leq n_1} a_{\alpha_1} a_{\beta_1} - 2 \sum_{\alpha_2 < \beta_2 \alpha_2, \beta_2 \in D_2} a_{\alpha_2} a_{\beta_2} - \cdots - 2 \sum_{\alpha_k < \beta_k \alpha_k, \beta_k \in D_k} a_{\alpha_k} a_{\beta_k} \right].
\]

Applying the algebraic lemma we have:

\[
2b_1 b_2 \geq \eta + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{i=1}^{n} (h_{ij}^r)^2 \\
- \left( \sum_{2 \leq \alpha_1 < \beta_1 \leq n_1} a_{\alpha_1} a_{\beta_1} + \sum_{\alpha_2 < \beta_2 \alpha_2, \beta_2 \in D_2} a_{\alpha_2} a_{\beta_2} + \cdots + \sum_{\alpha_k < \beta_k \alpha_k, \beta_k \in D_k} a_{\alpha_k} a_{\beta_k} \right),
\]

which is equivalent to:

\[
\sum_{\alpha_1 < \beta_1} a_{\alpha_1} a_{\beta_1} + \cdots + \sum_{\alpha_k < \beta_k} a_{\alpha_k} a_{\beta_k} \geq \frac{1}{2} \left[ \eta + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1} \sum_{i=1}^{n} (h_{ij}^r)^2 \right],
\]

with \( \alpha_i, \beta_i \in D_i, \forall i = 1, \ldots, k. \)

From the Gauss equation we obtain:

\[
\tau(L_j) = \frac{n_j(n_j - 1)(c - 3)}{8} + \frac{3(c + 1)}{4} \Psi(L_j) \\
+ \sum_{r=n+1}^{2m+1} \sum_{\alpha_j < \beta_j} \left[ h_{\alpha_j \alpha_j} h_{\beta_j \beta_j} - (h_{\alpha_j \beta_j})^2 \right].
\]

It follows that

\[
\sum_{j=1}^{k} \sum_{r=n+1}^{2m+1} \sum_{\alpha_j < \beta_j} \left[ h_{\alpha_j \alpha_j} h_{\beta_j \beta_j} - (h_{\alpha_j \beta_j})^2 \right] \geq \frac{\eta}{2} + \frac{1}{2} \sum_{r=n+1}^{2m+1} \sum_{(\alpha, \beta) \in D^2} (h_{\alpha \beta})^2 \\
+ \frac{1}{2} \sum_{r=n+1}^{2m+1} \sum_{j=1}^{k} \left( \sum_{\alpha_j \in D_j} h_{\alpha_j \alpha_j} \right)^2 \geq \frac{\eta}{2},
\]

where \( D^2 = (D_1 \times D_1) \cup \cdots \cup (D_k \times D_k). \)
Thus
\[ k \sum_{j=1}^{\infty} \tau(L_j) \geq \frac{\eta}{2} + \sum_{j=1}^{\infty} \left[ \frac{n_j(n_j - 1)(c - 3)}{8} + \frac{3(c + 1)}{4} \Psi(L_j) \right] \]
\[ = \tau - d(n_1, \ldots, n_k) \|H\|^2 - \frac{c - 3}{8} n(n-1) - \frac{c + 1}{8} \left[ 3\|P\|^2 - 2(n - 1) \right] \]
\[ + \sum_{j=1}^{k} \left[ \frac{n_j(n_j - 1)(c - 3)}{8} + \frac{3(c + 1)}{4} \Psi(L_j) \right], \]
which is equivalent with the relation that we want to prove. \( \square \)

In particular, for slant submanifolds we derive:

**Theorem 6.** Let \( M \) be an \( (n = 2k + 1) \)-dimensional non anti-invariant \( \theta \)-slant submanifold, tangent to \( \xi \) of a \( (2m + 1) \)-dimensional Kenmotsu space form \( \widetilde{M}(c) \). Let \( n_1, \ldots, n_k \) be integers \( \geq 2 \) satisfying \( n_1 < n, n_1 + \cdots + n_k \leq n \). For \( p \in M \), let \( L_j \subset T_pM \) be subspaces of \( T_pM \), orthogonal to \( \xi \) such that \( \dim L_j = n_j, \forall j \in \{1, \ldots, k\} \). Then, we have:

\[ \tau(p) - \sum_{j=1}^{k} \tau(L_j) \leq d(n_1, \ldots, n_k) \|H\|^2 + b(n_1, \ldots, n_k) \frac{c - 3}{4} \]
\[ + \frac{c + 1}{8} \left( (n - 1)(3\cos^2 \theta - 2) - \sum_{j=1}^{k} 6\Psi(L_j) \right). \]

The equality case of the inequality holds at \( p \in M \) if and only if there exists an orthonormal basis \( \{e_1, \ldots, e_n\} \) of \( T_pM \) and an orthonormal basis \( \{e_{n+1}, \ldots, e_{2m+1}\} \) of \( T_p^\perp M \) such that the shape operators of \( M \) in \( \widetilde{M}(c) \) at \( p \) have the following forms:

\[ A_{n+1} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}, \quad a_1 + \cdots + a_{n_1} = a_{n_1+1} + \cdots + a_{n_1+n_2} = \cdots \]
\[ = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+\cdots+n_k} = a_{n_1+\cdots+n_{k+1}} = \cdots = a_n. \]
For a θ-slant submanifold of a Kenmotsu space form we have \( \|P\|^2 = (n - 1) \cos^2 \theta \).

Equality at a point \( p \in M \) holds if and only if the equality holds in all the previous inequalities and we have the equality in the algebraic lemma:

\[
\sum_{\alpha, \beta \in D} h_{\alpha \beta} = 0, \forall r = n + 2, 2m + 1, \forall j = 1, k,
\]

\[
b_1 + b_2 = b_3 = \cdots = b_{r+1}.
\]

For invariant submanifolds we have the following.

**Corollary 7.** Let \( M \) be an \( (n = 2k+1) \)-dimensional invariant submanifold, tangent to \( \xi \) of a \( (2m+1) \)-dimensional Kenmotsu space form \( \tilde{M}(c) \). Let \( n_1, \ldots, n_k \) be integers \( \geq 2 \) satisfying \( n_1 < n, n_1 + \cdots + n_k \leq n \). For \( p \in M \), let \( L_j \subset T_pM \) be subspaces of \( T_pM \), orthogonal to \( \xi \) such that \( \dim L_j = n_j, \forall j \in \{1, \ldots, k\} \). Then, we have:

\[
\tau(p) - \sum_{j=1}^{k} \tau(L_j) \leq b(n_1, \ldots, n_k) \frac{c - 3}{4} + \frac{c + 1}{8} \left[ (n - 1) - \sum_{j=1}^{k} 6\Psi(L_j) \right].
\]

**Proof.** It is known that every invariant submanifold of a Kenmotsu space form is minimal. \( \square \)

We obtain the following Chen inequality.

**Theorem 8.** Let \( M \) be an \( (n = 2k+1) \)-dimensional non anti-invariant θ-slant submanifold, tangent to \( \xi \) of a \( (2m+1) \)-dimensional Kenmotsu space form \( \tilde{M}(c) \). Let \( n_1, \ldots, n_k \) be integers \( \geq 2 \) satisfying \( n_1 < n, n_1 + \cdots + n_k \leq n \). Then, we have:

\[
\delta'(n_1, \ldots, n_k) \leq d(n_1, \ldots, n_k) \|H\|^2 + b(n_1, \ldots, n_k) \frac{c - 3}{4} + \frac{c + 1}{8} \left[ 3(n - 1) - \sum_{j=1}^{k} n_j \cos^2 \theta - 2(n - 1) \right].
\]
The equality case of the inequality holds at $p \in M$ if and only if there exists an orthonormal basis \( \{ e_1, \ldots, e_n \} \) of \( T_p M \) and an orthonormal basis \( \{ e_{n+1}, \ldots, e_{2m+1} \} \) of \( T_p^\bot M \) such that the shape operators of \( M \) in \( \tilde{M}(c) \) at \( p \) have the following forms:

\[
A_{n+1} = \begin{pmatrix}
a_1 & 0 & \cdots & 0 \\
0 & a_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_n
\end{pmatrix}, \quad a_1 + \cdots + a_{n_1} = a_{n+1} + \cdots + a_{n+n_2} = \cdots = a_n,
\]

\[
A_r = \begin{pmatrix}
a_1^r & 0 & 0 & \cdots & 0 \\
0 & a_2^r & 0 & \cdots & 0 \\
0 & 0 & a_k^r & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad A_r^r \in M_n(\mathbb{R}), \quad A_r^r = A_j^r, \quad \text{Tr} A_j^r = 0,
\]

\[\forall j = 1, k, \forall r \in \{ n+2, \ldots, 2m+1 \}.\]

\[
\Psi(L_j) = \frac{n_j}{2} \cos^2 \theta, \forall j = 1, \ldots, k.
\]

**Proof.** Let \( p \in M \) and \( \{ e_1, \ldots, e_n = \xi \} \) be an orthonormal basis of \( T_p M \). Since we use subspaces invariant by \( P \), we may choose \( e_2 = \frac{1}{\cos \theta} Pe_1, \ldots, e_{2k} = \frac{1}{\cos \theta} Pe_{2k-1} \).

Let \( L_1, \ldots, L_k \) be \( k \) mutually orthogonal subspaces of \( T_p M \), \( \dim L_j = n_j \), defined by:

\[
L_1 = \text{Sp} \{ e_1, \ldots, e_n \}, \\
L_2 = \text{Sp} \{ e_{n+1}, \ldots, e_{n_1+n_2} \}, \\
\vdots \\
L_k = \text{Sp} \{ e_{n_{k-1}+1}, \ldots, e_{n_1+\cdots+n_k} \}.
\]

Thus

\[
\Psi(L_j) = \frac{n_j}{2} \cos^2 \theta, \forall j = 1, \ldots, k.
\]

\[
\square
\]

**References**


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