NON-ADDITIVE MEASURES, ENVELOPES AND EXTENSIONS TO QUASI-MEASURES

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Abstract. In the present paper, we introduce the notions of lower envelope and upper envelope for a \([0, \infty]\)-valued function \(\mu\) defined on a proper sublattice \(M\) of a locally complete \(\sigma\)-continuous lattice \(L\), and we extend a finite-stable, supermodular usc-measure \(\mu\) on a proper sublattice \(M\) of \(L\) to a quasi-*measure (i.e., a supermodular usc-measure) on \(L\), which is \(M\)\(_{\delta}\)-inner regular. Analogously, we extend a submodular lsc-measure on \(M\) to a quasi,-*measure (i.e., a submodular lsc-measure) on \(L\), which is \(M\)\(_{\sigma}\)-outer regular. Furthermore, we have studied notions of measuring envelopes in \(D\)-lattices in the context of null-additive, converse null-additive, superadditive and weak converse null-additive functions.

1. Introduction

In measure theory, a basic procedure is that of extending the notion of a measure on a given class of sets to a larger class of sets. The possibility of extension in measure theory on logics (orthomodular lattices or posets) was presented as an open problem in [12]. Volauf in [28] proved an extension theorem for orthocomplemented lattices and probability measures using Carathéodory measurability. In [25], Riečan studied an extension theorem for subadditive probability measures defined on a suborthomodular lattice of a \(\sigma\)-continuous, \(\sigma\)-complete orthomodular lattice. Later on, in 2001, Avallon and De Simone gave an extension theorem for modular functions that contains Riečan’s result as a particular case. They together with Vitolo further extended the theory in context of lattice ordered effect algebras in [5]. An extension theorem has been proved for measures on MV-algebras in fuzzy measure theory ([6, 26]; see also [18, 21, 27]). Adamski [2] obtained that every non-negative, semifinite, continuous at \(\phi\) and tight function defined on a lattice of sets can be extended to an inner regular measure. The
concept of an effect algebra has been introduced by Bennett and Foulis [7], as a generalization of Hilbert space effects interpreted as “unsharp” quantum events. Different from the “sharp” events the effects do not satisfy the noncontradiction principle, i.e. the conjunction of a and non a may be different from zero. These new logical structures generalize orthomodular lattices (including Boolean algebras) as well as MV-algebras employed by Chang in the analysis of many valued logics [8]. The categorical equivalence of $D$-posets and effect algebras is discussed in [10].

Non-additive set functions, as for example outer measures, semivariations of vector measures, naturally appeared earlier in classical measure theory concerning countable additive set functions or more general finite additive set functions [10]. Non-additive measures appear today in many branches of pure mathematics with many important applications ([23, 29]; see also [15, 16, 19, 20]). Wang [29] gave the concept of null-additive set functions. Wang and Klir [30] introduced the concept of converse null-additive fuzzy measures. The notion of weak converse null-additive function is studied in context of fuzzy measures in [21].

The aim of the present paper is to study an extension problem for non-additive measures defined on a proper sublattice $M$ of a locally complete $\sigma$-continuous lattice $L$. Some basic definitions are collected in Section 2. In Section 3, we introduce notions of lower envelope $\mu_0$ and upper envelope $\mu^*$ of a $[0, \infty]$-valued function $\mu$ defined on $M$, and we extend a finite-stable, supermodular $usc$-measure $\mu$ on a proper sublattice $M$ of $L$ to a quasi*-measure (i.e., a supermodular $usc$-measure) on $L$, which is $M_\delta$-inner regular. Analogously, we extend a submodular $lsc$-measure to a quasi*-measure (i.e., a submodular $lsc$-measure) on $L$, which is $M_\sigma$-outer regular. In Section 4, we have studied the notions of null-additive, converse null-additive, superadditive and weak converse null-additive functions on a proper $D$-sublattice $M$ of $L$, where $L$ is a $\sigma$-complete $\sigma$-continuous $D$-lattice. We have observed that every superadditive function is converse null-additive, and also weak converse null-additive; the converse need not be true, which is established by a counterexample. We proved that if $\mu$ is superadditive, then $\mu_0$ is also superadditive. It is also proved that for a monotone null-additive function defined on a proper $D$-sublattice $M$, if $a$ is $\mu$-measurable, then $\mu_0(a) = \mu^*(a)$ and, if $d \in L$ such that $\mu_0(d) = \mu^*(d) < \infty$, $d$ is $\mu$-measurable, then $\mu$ is weak converse null-additive.

2. Preliminaries and basic facts

Let $L = (L, \leq)$ be a lattice. Let $\{a_n\}$ be a sequence in $L$. We call $a_n \uparrow a, (a \in L)$ if and only if $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots, \bigvee a_n$ exists and $\bigvee a_n = a$. Similarly, call $a_n \downarrow a, (a \in L)$ if and only if $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots, \bigwedge a_n$ exists and $\bigwedge a_n = a$. If $a_n \uparrow a$ and $a_n \downarrow a$, then $a = a_n$. If $L = (L, \leq)$ is a lattice, then $L$ is a $\sigma$-complete $\sigma$-continuous lattice if and only if every increasing sequence $\{a_n\}$ in $L$ has a least upper bound and every decreasing sequence $\{a_n\}$ in $L$ has a greatest lower bound.
exists and \( \bigwedge a_n = a \). In these cases we also write \( a = \lim_{n \to \infty} a_n \). If \( a_n \uparrow a \), \( b_n \uparrow b \) and \( a_n \leq b_n \), for all \( n \), then we may deduce that \( a \leq b \). The symbol 1 denotes the top element (or supremum) of \( L \) and 0 denotes the bottom element (or infimum) of \( L \) (see [22]). A lattice \( L \) is said to be \( \sigma \)-continuous (see [25]) if \( a_n \uparrow a \) implies \( a_n \wedge b \uparrow a \wedge b \) (or equivalently, \( a_n \downarrow a \) implies \( a_n \vee b \downarrow a \vee b \)) for every \( b \in L \). If \( L \) is \( \sigma \)-continuous then, for sequences \( \{a_n\} \) and \( \{b_n\} \) in \( L \) such that \( a_n \uparrow a \) and \( b_n \uparrow b \), we have \( a_n \wedge b_n \uparrow a \wedge b \) (or equivalently, \( a_n \downarrow a, b_n \downarrow b \) implies \( a_n \vee b_n \downarrow a \vee b \)). Every infinitely distributive lattice [22] is \( \sigma \)-continuous.

2.1 [1]. A lattice \( L \) is locally complete if it satisfies one of the following equivalent conditions:

(i) Every non-empty lower bounded subset of \( L \) admits an infimum.

(ii) Every non-empty upper bounded subset of \( L \) admits a supremum.

(iii) There exists a complete lattice, denoted by \( L \), with bottom (or smallest) element 0 and top (or largest) element 1, such that \( L \) is a sublattice of \( L \), \( L = L \cup \{0, 1\} \), \( \inf L = 0 \) and \( \sup L = 1 \).

It can be observed that every complete lattice is locally complete. For any set \( X \), \( (\mathcal{P}(X), \subseteq) \), \( (L^X, \leq) \) (where \( L \) is locally complete \( \sigma \)-continuous lattice) and \( (I, \leq) \) (where \( I \) is the closed unit interval \([0, 1]\) of the real line \( \mathbb{R} \)) are locally complete \( \sigma \)-continuous lattices. For more examples of locally complete lattices, we refer to [1].

2.2 ([10]; see also [17]). An orthomodular poset (OMP), in short, is a bounded poset \( (P, \leq, \vee, 0, 1) \) with a unary operation \( ^{'}: P \to P \) (an orthocomplementation) such that the following conditions are satisfied for all \( a, b, c \in P \):

(i) If \( a \leq b \) then \( b^{'} \leq a^{'} \),

(ii) \( (a^{'} )^{'} = a \),

(iii) \( a \vee a^{'} = 1 \),

(iv) If \( a \leq b^{'} \) then \( a \vee b \) exists in \( P \),

(v) (orthomodular law) If \( a \leq b \), then there is a \( c \in P \) such that \( c \leq a^{'} \) and \( a \vee c = b \). Two elements \( a, b \in P \) are called orthogonal (written as \( a \perp b \)) if \( a \leq b^{'} \) or equivalently \( b \leq a^{'} \). An orthomodular lattice (OML) \( L \) can be defined as a lattice ordered OMP. A subset \( M \) is called a suborthomodular lattice (or sub OML) of an OML \( L \) if it contains 0 and 1, and is closed under the operations \( ^{'} \), \( \wedge \) and \( \vee \), i.e. it is an orthomodular lattice with respect to induced operations. An OMP is called a quantum logic if it is a \( \sigma \)-complete lattice.

2.3 ([7, 9, 10, 13, 24]; see also [4, 5, 14]). An effect algebra \( (L; \oplus, 0, 1) \) is a structure consisting of a set \( L \), two special elements 0 and 1, and a partially defined binary operation \( \oplus \) on \( L \times L \) satisfying the following conditions for every \( a, b, c \in L \):

1. If \( a \oplus b \) is defined, then \( b \oplus a \) is defined and \( a \oplus b = b \oplus a \).

2. If \( b \oplus c \) and \( a \oplus (b \oplus c) \) are defined, then \( a \oplus b \) and \( (a \oplus b) \oplus c \) are defined and \( a \oplus (b \oplus c) = (a \oplus b) \oplus c \).
(3) For every $a \in L$, there exists a unique $a^\perp \in L$ such that $a \oplus a^\perp$ is defined and $a \oplus a^\perp = 1$.

(4) If $a \oplus 1$ is defined, then $a = 0$.

A subset $F$ is called a subeffect algebra of an effect algebra $L$ if (i) $0, 1 \in F$, (ii) for $a \in F$, $a^\perp \in F$, (iii) for $a, b \in F$ with $a \perp b$, $a \oplus b \in F$.

In every effect algebra $L$, a dual operation $\ominus$ to $\oplus$ can be defined as follows: $a \ominus c$ exists and equals $b$ if and only if $b \oplus c$ exists and equals $a$. We say that two elements $a, b \in L$ are orthogonal and we write $a \perp b$, if $a \oplus b$ exists. Also for $a, b \in L$, define $a \leq b$ if there exists $c \in L$ such that $a \perp c$ and $a \oplus c = b$. If $(L, \leq)$ is a lattice, we say that the effect algebra $L$ is a lattice effect algebra, or a $D$-lattice. Any OMP may be regarded as a $D$-poset (which is equivalent to effect algebra) by defining $b \ominus a = b \land a'$ precisely when $a \leq b$.

For $a_1, \ldots, a_n \in L$, we inductively define $a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$, provided that the right hand side exists. The definition is independent of permutation of the elements. A finite subset $\{a_1, \ldots, a_n\}$ of $L$ is said to be orthogonal if $a_1 \oplus \cdots \oplus a_n$ exists. A sequence $\{a_n\}$ in $L$ is called orthogonal if, for every $n$, $\bigoplus_{i \leq n} a_i$ exists. If, moreover $\sup_n \bigoplus_{i \leq n} a_i$ exists, the $\sum \bigoplus_{n \in \mathbb{N}} a_n$ of an orthogonal sequence $\{a_n\}$ in $L$ is defined as $\sup_n \bigoplus_{i \leq n} a_i$.

An effect algebra $L$ is called a $\sigma$-complete effect algebra if every orthogonal sequence in $L$ has its sum.

2.4. Let $a, b, c \in L$ (where $L$ is an OML) such that $a \perp b$ and $b \leq c$. Then $a \lor b \geq c$ if and only if $a \geq c \land b'$.

2.5. A function $\mu : L \to [0, \infty]$ (where $L$ is a lattice) is said to be modular if, for every $a, b \in L$, $\mu(a \lor b) + \mu(a \land b) = \mu(a) + \mu(b)$. We say that $\mu$ is submodular if for every $a, b \in L$, we have $\mu(a) + \mu(b) \geq \mu(a \lor b) + \mu(a \land b)$; $\mu$ is called supermodular if for every $a, b \in L$, we have $\mu(a) + \mu(b) \leq \mu(a \lor b) + \mu(a \land b)$. A function $\mu$ is called subadditive if for every $a, b \in L$, we have $\mu(a \lor b) \leq \mu(a) + \mu(b)$; $\mu$ is called superadditive if for every $a, b \in L$, we have $\mu(a \lor b) \geq \mu(a) + \mu(b)$.

2.6 ([9, 10, 13]). Assume that $a, b$ are elements of an effect algebra $L$.

(i) If $a \leq b$, then $b = a \oplus (b \ominus a)$.
(ii) If $a \leq b \leq c$, then $b \ominus a \leq c \ominus a$.
(iii) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$.
(iv) If $c \leq a, d \leq b$ and $a \perp d$ and $c \perp d \leq a \perp b$.

3. Extension of semi-continuous measures to quasi-measures

Throughout in this section, let $L$ be a locally complete $\sigma$-continuous lattice, let $C$ be a non-empty subset of $L$, and $M$ be a proper sublattice of $L$ containing 0. We denote $M_\sigma = \{ b \in L : \text{there exists a sequence } \{a_n\} \text{ in } M \text{ such that } a_n \uparrow b \}$, and $M_\delta = \{ b \in L : \text{there exists a sequence } \{a_n\} \text{ in }$
\(M\) such that \(a_n \downarrow b\). We may also describe \(M_\mu\) as the family of all countable joins of elements from \(M\) and \(M_\delta\) as the family of all countable meets of elements from \(M\).

**Definition 3.1.** A function \(\mu : C \to [0, \infty)\) is called a semi-continuous measure (or non-additive measure) on \(C\), if it satisfies the following conditions:

(i) \(\mu(0) = 0\), whenever \(0 \in C\),
(ii) (monotone) if \(a \leq b\), \(a, b \in C\), then \(\mu(a) \leq \mu(b)\),
(iii) (semi-continuous from below) if \(a_n \uparrow a\), \(a, a_n \in C\) \((n \in \mathbb{N})\), then
\[
\lim_{n \to \infty} \mu(a_n) = \mu(a),
\]
(iv) (semi-continuous from above) if \(a_n \downarrow a\), \(a, a_n \in C\) \((n \in \mathbb{N})\),
\[
\mu(a_1) < \infty, \text{ then } \lim_{n \to \infty} \mu(a_n) = \mu(a).
\]

The function \(\mu\) is said to be a lower semi-continuous measure (or lsc-measure) if it satisfies (i), (ii) and (iii), while \(\mu\) is said to be an upper semi-continuous measure (or usc-measure) if it satisfies (i), (ii) and (iv).

**Definition 3.2.** A function \(\mu : C \to [0, \infty)\) is called finite-stable if, for all \(c_1, c_2 \in C\), \(\max(\mu(c_1), \mu(c_2)) < \infty\) implies \(\mu(c_1 \lor c_2) < \infty\).

Define \(\nu(c) = \sup\{\mu(d) : d \leq c, d \in C\}; \mu(d) < \infty\}, c \in C\). If \(\mu = \nu\), then \(\mu\) is called semifinite.

If, in addition, \(L\) is an orthomodular lattice (OML), \(M\) is a proper sub-orthomodular lattice of \(L\) and \(\rho : L \to [0, \infty)\) with \(\rho(0) = 0\), we write \(\mathcal{M}(\rho; M) = \{a \in L : \rho(b) = \rho(b \land a) + \rho(b \land (b \land a)^\perp) \text{ for all } b \in M\}\).

**Definition 3.3.** Let \(\mu : M \to [0, \infty)\) with \(\mu(0) = 0\). Define \(\mu_* : L \to [0, \infty]\) and \(\mu^* : L \to [0, \infty]\) by
\[
\mu_*(a) = \sup\{\mu(b) : b \leq a, b \in M\}, a \in L
\]
and
\[
\mu^*(a) = \inf\{\mu(b) : a \leq b, b \in M\}, a \in L,
\]
\(\mu_*\) is called the lower envelope and \(\mu^*\) the upper envelope of \(\mu\).

We get: (i) \(\mu^*(0) = 0, \mu_*(0) = 0\), (ii) both \(\mu^*\) and \(\mu_*\) are monotone, (iii) \(\mu^*|M \leq \mu \leq \mu_*|M\), (iv) \(\mu\) is semifinite if and only if \(\mu\) is monotone if and only if \(\mu^*|M = \mu = \mu_*|M\).

**Proposition 3.1.**

(i) If \(\mu\) is finite-stable, then \(\mu^*\) is finite-stable.
(ii) If \(\mu\) is supermodular, then \(\mu_*\) is supermodular (see [3]).
(iii) If \(\mu\) is submodular, then \(\mu^*\) is submodular (see [3]).
(iv) If \(\mu\) is superadditive, then \(\mu_*\) is superadditive.
(v) If \(\mu\) is subadditive, then \(\mu^*\) is subadditive.
(vi) Let \(M\) be a proper suborthomodular lattice of an orthomodular lattice \(L\). Then we have:
We shall prove only (i), (ii) and (v).

(a) If $\mu$ is superadditive $\Rightarrow M(\mu^*; M) = \{ a \in L : \mu(b) \leq \mu_a(b \land a) + \mu_b(b \land b \lor a) \}$ for all $b \in M$.

(b) If $\mu$ is subadditive $\Rightarrow M(\mu^*; M) = \{ a \in L : \mu(b) \geq \mu^*(b \land a) + \mu^*(b \land b \lor a) \}$ for all $b \in M$.

Proof. We shall prove only (i), (ii) and (v).

(i) Let $a_1, a_2 \in L$ with $\mu^*(a_1) < \infty$, $\mu^*(a_2) < \infty$. Let $\epsilon > 0$. Then there exist $b_1, b_2 \in M$, $a_1 \leq b_1$, $a_2 \leq b_2$, $\mu(b_1) < \mu^*(a_1) + \epsilon/2$ and $\mu(b_2) < \mu^*(a_2) + \epsilon/2$. Since $a_1 \lor a_2 \leq b_1 \lor b_2$ and $\mu$ is finite-stable, we obtain $\mu^*(a_1 \lor a_2) < \infty$. Hence $\mu^*$ is finite-stable.

(ii) Let $a, b \in L$. Let $\epsilon > 0$. Then there exist $c, d \in M$, $c \leq a$, $d \leq b$, $\mu^*(a) - \epsilon/2 < \mu(c)$ and $\mu^*(b) - \epsilon/2 < \mu(d)$. It follows that $\mu^*(a) + \mu^*(b) - \epsilon < \mu(c) + \mu(d)$. Since $\mu$ is supermodular, we obtain $\mu^*(a) + \mu^*(b) - \epsilon < \mu(c \lor d) + \mu(c \land d)$. Now, we have $\mu^*(a) + \mu^*(b) - \epsilon < \mu^*(a \lor b) + \mu^*(a \land b)$. Thus $\mu^*$ is supermodular.

(v) Let $a, b \in L$. Let $\epsilon > 0$. Then there exist $c, d \in M$, $a \leq c$, $b \leq d$, $\mu(c) < \mu^*(a) + \epsilon/2$ and $\mu(d) < \mu^*(b) + \epsilon/2$. Since $a \lor b \leq c \lor d$ and $\mu^*$ is monotone, we get $\mu^*(a \lor b) \leq \mu^*(c \lor d) < \mu^*(a) + \epsilon/2 + \mu^*(b) + \epsilon/2 = \mu^*(a) + \mu^*(b) + \epsilon$. Hence $\mu^*$ is subadditive.

Lemma 3.1. (see [3])

(a) Let $\mu$ be a $[0, \infty]$-valued finite-stable usc-measure on $M$. Then for any $b \in M_\delta$ with $\mu^*(b) < \infty$, there exists a sequence $\{ b_n \}_{n=1}^\infty$ in $M$ such that $b_n \downarrow b$ and for each such sequence $\mu^*(b) = \inf_n \mu(b_n)$.

(b) Let $\mu$ be a $[0, \infty]$-valued lsc-measure on $M$. Then for any $b \in M_\delta$, there exists a sequence $\{ b_n \}_{n=1}^\infty$ in $M$ such that $b_n \uparrow b$ and for each such sequence $\mu^*(b) = \sup_n \mu(b_n)$.

Proof. (a) Let $b \in M_\delta$ with $\mu^*(b) < \infty$. Then there exists a sequence $\{ b_n \}_{n=1}^\infty$ in $M$ such that $b_n \downarrow b$. Choose $c \in M$ such that $b \leq c$ and $\mu(c) < \infty$. Replace $b_n$ by $b_n \land d$ (where $d \in M$, $b \leq d$ and $\mu(d) < \infty$). Then we have $b_n \lor c \leq c$ (as $L$ is $\sigma$-continuous), and $\inf_n \mu(b_n \lor c) < \infty$. Again since $\mu$ is semi-continuous from above, we get $\lim_{n \to \infty} \mu(b_n \lor c) = \inf_n \mu(b_n \lor c) = \mu(c)$. Also $b_n \leq b_n \lor c$ for each $n$, yields $\inf_n \mu(b_n) \leq \mu^*(b)$. On the other hand, $b \leq b_n$ ($n \in \mathbb{N}$) and $\mu^*$ is monotone, imply $\mu^*(b) \leq \inf_n \mu(b_n)$. Hence $\mu^*(b) = \inf_n \mu(b_n)$.

(b) It follows using similar arguments as in (a).

Theorem 3.1.

(a) If $\mu$ is a $[0, \infty]$-valued finite-stable usc-measure on $M$, then $\mu^*|_{M_\delta}$ is a usc-measure.

(b) If $\mu$ is a $[0, \infty]$-valued lsc-measure on $M$, then $\mu^*|_{M_\delta}$ is a lsc-measure.

Proof. (a) Let $\{ a_n \}_{n=1}^\infty$ be a decreasing sequence in $M_\delta$ with $\mu^*(a_1) < \infty$ and $a_n \downarrow a$ ($n \in \mathbb{N}$), $a \in M_\delta$. Then there exists a sequence $\{ a_{ni} \}_{i=1}^\infty$ in $M$ such
that \(a_{ni} \downarrow a_n\) and \(\mu^*(a_n) = \lim_{i \to \infty} \mu(a_{ni})\). For \(i \in \mathbb{N}\), set \(b_i = a_{1i} \wedge a_{2i} \ldots \wedge a_{ii}\). Then \(b_i \in M\), \(\{b_i\}_{i=1}^\infty\) is a decreasing sequence and \(b_i \geq a_i \geq a\) for all \(i\), which yield that \(b = \lim_{i \to \infty} b_i = \bigwedge b_i \geq \bigwedge a_i = a\). It may be noted that \(b \in M_\delta\).

Also \(a_{ki} \geq b_i\) for \(1 \leq k \leq i\). Therefore \(a_k = \lim_{i \to \infty} a_{ki} \geq \lim_{i \to \infty} b_i = b\). It follows that \(a = \bigwedge a_k \geq b\). Thus \(a = b\). Now, by Lemma 3.1(a), we have \(\mu^*(a) = \lim_{i \to \infty} \mu(b_i) = \lim_{i \to \infty} \mu^*(b_i) \geq \lim_{i \to \infty} \mu^*(a_i)\). Also since \(a \leq a_n\), the result follows.

(b) It follows using similar arguments as in (a).

**Definition 3.4.** Let \(\mu^- : L \to [0, \infty]\) be defined by

\[
\mu^-(a) = \sup \{ \mu^*(b) : b \leq a, b \in M_\delta, \mu^*(b) < \infty \}, \quad a \in L.
\]

We obtain: (i) \(\mu^-(0) = 0\), (ii) \(\mu^-\) is monotone, (iii) \(\mu^- \leq \mu^*\), (iv) \(\mu^-|M_\delta = \mu^*|M_\delta\); particularly \(\mu^-|M = \mu^*|M\), (v) if \(\mu\) is monotone, then \(\mu^-|M = \mu\), i.e. \(\mu^-\) is an extension of \(\mu\), (vi) if \(\mu\) is subadditive, then \(\mu^-\) is subadditive.

**Proposition 3.2.** Let \(M = M_\delta\). Then

(i) \(\mu^- \leq \mu_*\),
(ii) \(\mu^- = \mu_*\) on \(M\), provided \(\mu\) is semifinite.

**Proof.** (i) Let \(a \in L\) and \(b \in M\) with \(b \leq a\). Since \(\mu^*|M \leq \mu_*|M\) and \(\mu_*\) is monotone, we obtain \(\mu^*(b) \leq \mu_*(b) \leq \mu_*(a)\). Hence \(\mu^-(a) \leq \mu_*(a)\).

(ii) Since \(M = M_\delta\) and \(\mu\) is semifinite, so \(\mu^- = \mu^*\). Using Definition 3.3 (iv), we obtain \(\mu^- = \mu_*\) on \(M\).

**Theorem 3.2.** Let \(\mu\) be a finite-stable usc-measure on \(M\). Then (i) \(\Rightarrow\) (ii) \(\Rightarrow\) (iii), where

(i) \(\mu\) is supermodular.
(ii) \(\mu^*|M_\delta\) is supermodular.
(iii) (a) \(\mu^-\) is supermodular. (b) \(\mu^-\) is semi-continuous from above.

In addition, if \(L\) is an orthomodular lattice, \(M\) is a proper suborthomodular lattice of \(L\) and \(\mu\) is subadditive, then (iii) \(\Rightarrow\) (iv) \(\Rightarrow\) (v), where

(iv) \(\mathcal{M}(\mu^-; M) = \{a \in L : \mu(b) \leq \mu^-(b \wedge a) + \mu^-(b \wedge (b \wedge a)^+\}\) for all \(b \in M\) with \(\mu(b) < \infty\).

(v) \(\mathcal{M}(\mu_*; M) \subseteq \mathcal{M}(\mu^-; M)\).

**Proof.** (i)\(\Rightarrow\) (ii) Let \(\mu\) be supermodular. Let \(a, b \in M_\delta\) with \(\mu^*(a) < \infty\), \(\mu^*(b) < \infty\). Then, by Lemma 3.1(a) there exist sequences \(\{a_n\}\) and \(\{b_n\}\) in \(M\) such that \(a_n \downarrow a, b_n \downarrow b, \mu^*(a) = \lim_{n \to \infty} \mu(a_n)\) and \(\mu^*(b) = \lim_{n \to \infty} \mu(b_n)\).

Since \(L\) is \(\sigma\)-continuous, so \(a_n \vee b_n \downarrow a \vee b\) and also we have \(a_n \wedge b_n \downarrow a \wedge b\).
Therefore
\[
\mu^*(a) + \mu^*(b) = \lim_{n \to \infty} \mu(a_n) + \lim_{n \to \infty} \mu(b_n)
\]
\[
= \lim_{n \to \infty} (\mu(a_n) + \mu(b_n))
\]
\[
\leq \lim_{n \to \infty} (\mu(a_n \lor b_n) + \mu(a_n \land b_n))
\]
\[
= \lim_{n \to \infty} \mu(a_n \lor b_n) + \lim_{n \to \infty} \mu(a_n \land b_n)
\]
\[
= \mu^*(a \lor b) + \mu^*(a \land b).
\]

(ii)⇒(iii) (a) Let \(a, b \in L\). For \(\varepsilon > 0\), we have \(c, d \in M_\delta\) such that \(c \leq a, d \leq b\), with \(\mu^*(c) < \infty, \mu^*(d) < \infty\). Then we have \(\mu^-(a) - \varepsilon/2 < \mu^*(c)\) and \(\mu^-(b) - \varepsilon/2 < \mu^*(d)\). Thus
\[
\mu^-(a) + \mu^-(b) - \varepsilon < \mu^*(c) + \mu^*(d)
\]
\[
\leq \mu^*(c \lor d) + \mu^*(c \land d)
\]
\[
\leq \mu^-(a \lor b) + \mu^-(a \land b).
\]

Since \(\varepsilon\) is arbitrary, we have \(\mu^-(a) + \mu^-(b) \leq \mu^-(a \lor b) + \mu^-(a \land b)\).

(ii)⇒(iii) (b) Let \(\{a_n\}_{n=1}^\infty\) be a sequence in \(L\) with \(\mu^-(a_1) < \infty\) and \(a_n \downarrow a\) \((n \in \mathbb{N})\), \(a \in L\). Let \(\varepsilon > 0\). For each \(n\), we choose \(b_n \in M_\delta\) such that \(b_n \leq a_n\), \(\mu^*(b_n) < \infty\) and
\[
\mu^*(b_n) > \mu^-(a_n) - \varepsilon/2^n. \tag{3.2.1}
\]
Put \(c_n = b_1 \land b_2 \land \ldots \land b_n\). Then \(c_n \in M_\delta\) and \(c_n \downarrow c, c \in M_\delta\). Now, by
(3.2.1), we get \(\mu^*(c_1) = \mu^*(b_1) > \mu^-(a_1) - \varepsilon/2\). Since \(\mu^*|M_\delta\) is supermodular, it follows that
\[
\mu^*(c_2) = \mu^*(b_1 \land b_2)
\]
\[
\geq \mu^*(b_1) + \mu^*(b_2) - \mu^*(b_1 \lor b_2)
\]
\[
> \mu^-(a_1) - \varepsilon/2 + \mu^-(a_2) - \varepsilon/2 - \mu^-(a_1 \lor a_2)
\]
\[
= \mu^-(a_2) - (\varepsilon/2 + \varepsilon/2^2).
\]
Suppose that \(\mu^*(c_m) \geq \mu^-(a_m) - \sum_{i=1}^m \varepsilon/2^i\). Then
\[
\mu^*(c_{m+1}) = \mu^*(c_m \land b_{m+1})
\]
\[
\geq \mu^*(c_m) + \mu^*(b_{m+1}) - \mu^*(c_m \lor b_{m+1})
\]
\[
> \mu^-(a_m) - \sum_{i=1}^m \varepsilon/2^i + \mu^-(a_{m+1}) - \varepsilon/2^{m+1} - \mu^-(a_m \lor a_{m+1})
\]
\[
= \mu^-(a_{m+1}) - \sum_{i=1}^{m+1} \varepsilon/2^i.
\]
Thus by induction, we obtain that $\mu^*(c_n) \geq \mu^-(a_n) - \sum_{i=1}^{n} \varepsilon / 2^i$, for all $n$. As $b_n \leq a_n$ ($n \in \mathbb{N}$), and $L$ is locally complete, therefore $c = \bigwedge_{n=1}^{\infty} c_n = \bigwedge_{n=1}^{\infty} b_n \leq \bigwedge_{n=1}^{\infty} a_n = a$, which yield, using Theorem 3.1(a) that $\mu^-(a) \geq \mu^*(c) = \lim_{n \to \infty} \mu^*(c_n) \geq \lim_{n \to \infty} (\mu^-(a_n) - \sum_{i=1}^{n} \varepsilon / 2^i) = \lim_{n \to \infty} \mu^-(a_n) - \varepsilon$. So $\mu^-(a) \geq \lim_{n \to \infty} \mu^-(a_n)$. Now since $a \leq a_n$ for all $n$ and $\mu^-$ is monotone the result follows.

(iii)$\Rightarrow$(iv) Put $\mathcal{D} = \{a \in L : \mu(b) \leq \mu^-(b \wedge a) + \mu^-(b \wedge (b \wedge a)') \}$ for all $b \in M$ with $\mu(b) < \infty$. Since $\mu^-$ is subadditive, so $\mathcal{M}(\mu^-; M) \subseteq \mathcal{D}$. Let $a \in \mathcal{D}$. Let $c \in L$, $b \in M_\sigma$ with $b \leq c$ and $\mu^*(b) < \infty$. Then, by Lemma 3.1(a) there exists a sequence $\{b_n\}$ in $M$ such that $b_n \downarrow b$ and $\mu^*(b_n)$ exists. Now, we have $\mu(b_n) \leq \mu^-(b_n \wedge a) + \mu^-(b_n \wedge (b_n \wedge a))$. Since $L$ is $\sigma$-continuous, and $\mu^-$ is monotone, using 2.6 we obtain,

$$
\mu^*(b) = \lim_{n \to \infty} \mu(b_n) \\
\leq \lim_{n \to \infty} \mu^-(b_n \wedge a) + \lim_{n \to \infty} \mu^-(b_n \wedge (b_n \wedge a)) \\
= \mu^-(b \wedge a) + \mu^-(b \wedge (b \wedge a)) \\
\leq \mu^-(c \wedge a) + \mu^-(c \wedge (c \wedge a')).
$$

Now by definition of $\mu^-$, we have $\mu^-(c) \leq \mu^-(c \wedge a) + \mu^-(c \wedge (c \wedge a'))$. The reverse inequality follows from the supermodularity of $\mu^-$ and the ortho-modular law. Thus $a \in \mathcal{M}(\mu^-; M)$. Hence $\mathcal{D} \subseteq \mathcal{M}(\mu^-; M)$.

(iv)$\Rightarrow$(v) Let $a \in \mathcal{M}(\mu_\sigma; M)$. Let $b \in M$ with $\mu(b) < \infty$. $\mu(b) = \mu_\sigma(b) \leq \mu_\sigma(b \wedge a) + \mu_\sigma(b \wedge (b \wedge a')) \leq \mu^-(b \wedge a) + \mu^-(b \wedge (b \wedge a'))$. Thus $a \in \mathcal{M}(\mu^--; M)$.

Definition 3.5. Define $\mu_- : L \to [0, \infty]$ by

$$
\mu_-(a) = \inf\{\mu_\sigma(b) : a \leq b, b \in M_\sigma\}, \quad a \in L.
$$

We observe that: (i) $\mu_-(0) = 0$, (ii) $\mu_-$ is monotone, (iii) $\mu_- \geq \mu_\sigma$, (iv) $\mu_-|M_\sigma = \mu_\sigma|M_\sigma$; particularly $\mu_-|M = \mu_\sigma|M$, (v) if $\mu$ is monotone, then $\mu_-|M = \mu$, i.e. $\mu_-$ is an extension of $\mu$, (vi) if $\mu$ is superadditive, then $\mu_-$ is superadditive.

Proposition 3.3. Let $M = M_\sigma$. Then $\mu^* \leq \mu_-$.

Proof. Let $a \in L$ and $b \in M$ with $a \leq b$. Since $\mu^*|M \leq \mu_\sigma|M$ and $\mu^*$ is monotone, we obtain $\mu^*(a) \leq \mu^*(b) \leq \mu_\sigma(b)$. Hence $\mu^*(a) \leq \mu_-(a)$.\qed

Using similar arguments as in Theorem 3.2, we obtain the following:

Theorem 3.3. Let $\mu$ be an lsc-measure on $M$. Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), where (i) $\mu$ is submodular. (ii) $\mu_\sigma|M_\sigma$ is submodular. (iii) (a) $\mu_-$ is submodular. (b) $\mu_-$ is semi-continuous from below.
In addition, if $L$ is an orthomodular lattice, $M$ is a proper suborthomodular lattice of $L$ and $\mu$ is superadditive, then (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v), where (iv) $\mathcal{M}(\mu_\ast; M) = \{ a \in L : \mu(b) \geq \mu_\ast(b \wedge a) + \mu_\ast(b \wedge (b \land a)) \}$ for all $b \in M$. (v) $\mathcal{M}(\mu_\ast; M) \subseteq \mathcal{M}(\mu_\ast; M)$.

**Definition 3.6.**

Let $C \subseteq L$. We call a function $\rho : L \to [0, \infty]$ $C$-inner regular if for any $a \in E$, $\rho(a) = \sup\{\rho(b) : b \leq a, b \in C, \mu(b) < \infty\}$, and is said to be $C$-outer regular if for any $a \in E$, $\rho(a) = \inf\{\rho(b) : a \leq b, b \in C\}$. The function $\rho$ is said to be quasi$^\ast$-measure (respectively, quasi$^\ast_r$-measure) on $L$, if it is supermodular usc-measure (respectively, submodular lsc-measure); the pair $(L, \rho)$ is called quasi$^\ast$-measure space (respectively, quasi$^\ast_r$-measure space). The function $\rho$ is said to be quasi-measure on $L$, if it is either $M_\ast$-inner regular quasi$^\ast$-measure or $M_\ast_r$-outer regular quasi$^\ast_r$-measure.

Since $\mu_\ast|_{M_\ast} = \mu_\ast|_{M_\ast_r}$, the function $\mu_\ast$ is $M_\ast$-outer regular. Also, since $\mu^\ast|_{M_\ast} = \mu^\ast|_{M_\ast_r}$, the function $\mu^\ast$ is $M_\ast^\ast$-inner regular.

**Theorem 3.4.**

(a) Every finite-stable, supermodular, usc-measure defined on a proper sublattice $M$ of $L$, can be extended to an $M_\ast$-inner regular quasi$^\ast$-measure on $L$.

(b) Every submodular, lsc-measure defined on a proper sublattice $M$ of $L$, can be extended to an $M_\ast_r$-outer regular quasi$^\ast_r$-measure on $L$.

**Proof.** Follows from Theorem 3.2 and Theorem 3.3. \qed

**Theorem 3.5.** Let $\mu$ be a semi-continuous measure on $M$. Then we have:

(i) $\mu^\ast$ is semifinite and $M_\ast$-inner regular.

(ii) $\mu^\ast$ is a semirextension of $\mu$ (i.e. $\mu(a) = \mu^\ast(a)$ for $a \in M$, $\mu(a) < \infty$), and $\mu^\ast$ is the smallest semirextension of $\mu$ within the class of all monotone, $[0, \infty]$-valued functions on $L$ which are semi-continuous from above on $M_\ast$.

(iii) $\mu^\ast$ is an extension of $\mu$ if and only if $\mu$ is semifinite.

(iv) $\mu_\ast$ is the largest extension of $\mu$ within the class of all monotone, $[0, \infty]$-valued functions on $L$ which are semi-continuous from below on $M_\ast$.

**Proof.** We shall prove only (ii) and (iii).

Obviously $\mu^\ast$ is a semirextension of $\mu$. Now, let $\eta : L \to [0, \infty]$ be a monotone semirextension of $\mu$, and is semi-continuous from above on $M_\ast$. Let $a \in L, b \in M_\ast$ with $b \leq a$ and $\mu^\ast(b) < \infty$. Then there exists a sequence $\{b_n\}_{n=1}^\infty$ in $M$ such that $b_n \downarrow b$ and $\mu^\ast(b) = \inf_{n} \mu^\ast(b_n)$. Thus $\mu^\ast(b) = \inf_{n} \eta(b_n) = \eta(b) \leq \eta(a)$, and so $\mu^\ast(a) \leq \eta(a)$, which proves (ii).
Define a function \( \eta \) on \( M \) by \( \eta(a) = \sup\{\mu(b) : b \leq a, b \in M, \mu(b) < \infty\} \). Now, \( \mu \) is semifinite if and only if \( \mu = \eta \). Since \( \mu^- \) is an extension of \( \eta \), (iii) follows.

4. Measuring envelopes in \( D \)-lattices

In this Section we shall consider \( L \) to be a \( \sigma \)-complete \( \sigma \)-continuous \( D \)-lattice and study a few properties of the function \( \mu \) defined on \( L \), and its measuring envelopes \( \mu_+ \) and \( \mu^- \) (introduced in Section 3), in the context of some weaker forms of additivity of the function \( \mu \).

**Definition 4.1.** Let \( E \) be a difference poset (i.e. a \( D \)-poset) or effect algebra and \( \mu : E \to [0, \infty] \) be a function. Then \( \mu \) is called:

(i) **null-additive**, if \( \mu(b \oplus c) = \mu(b) \) provided \( b, c \in E, b \perp c \) and \( \mu(c) = 0 \).

(ii) **converse null-additive**, if \( \mu(c \ominus b) = 0 \) provided \( b, c \in E, b \leq c, \mu(b) = \mu(c) < \infty \).

(iii) **superadditive**, if \( \mu(b \oplus c) \geq \mu(b) + \mu(c) \) provided \( b, c \in E, b \perp c \).

Observe that, \( \mu \) is null-additive if and only if \( \mu(b \ominus c) = \mu(b) \) provided \( b, c \in M, c \leq b \) and \( \mu(c) = 0 \). Also, observe that if \( \mu \) is superadditive, then \( \mu \) is converse null-additive. The converse of this statement need not be true as \( \mu_2 \) (given below) is converse null-additive but not superadditive.

**Example 4.1.** Let \( E_1 = \{0, a, b, c, d, e, 1\} \). Let us define: \( a \oplus b = b \oplus a = c \), \( b \oplus c = c \oplus b = a \oplus d = d \oplus a = e \oplus e = 1 \) and let \( x \oplus 0 = 0 \oplus x = x \) for all \( x \in E_1 \). Then \( E_1 \) is an effect algebra. Define functions \( \mu_1 \) and \( \mu_2 \) on \( E_1 \) as follows:

(I) \( \mu_1(x) = 0 \) if \( x \in \{0, b, c, d, 1\} \) and \( \mu_1(x) = 1 \) if \( x \in \{a, e\} \);

(II) \( \mu_2(x) = 0 \) if \( x \in \{0, a, b, c, d, 1\} \) and \( \mu_2(e) = 1 \).

Then we have,

(i) \( \mu_2 \) is null-additive, but \( \mu_1 \) is not;

(ii) \( \mu_1 \) and \( \mu_2 \) are not superadditive;

(iii) \( \mu_2 \) is converse null-additive, but \( \mu_1 \) is not.

**Example 4.2.** Let \( E_2 = \{0, a, b, c, 1\} \). Let us define: \( a \oplus b = b \oplus a = c \), \( b \oplus c = c \oplus b = a \oplus a = 1 \) and let \( x \oplus 0 = 0 \oplus x = x \) for all \( x \in E_2 \). Then \( E_2 \) is an effect algebra. Define functions \( \mu_3, \mu_4 \) and \( \mu_5 \) on \( E_2 \) as follows:

(I) \( \mu_3(x) = 0 \) if \( x \in \{0, b\} \), and \( \mu_3(x) = 1 \) if \( x \in \{a, c, 1\} \).

(II) \( \mu_4(x) = 0 \) if \( x \in \{0, a, b\} \), and \( \mu_4(x) = 1 \) if \( x \in \{c, 1\} \).

(III) \( \mu_5(x) = 0 \) for all \( x \in E_2 \).

Then we have,

(i) \( \mu_3 \) and \( \mu_5 \) are null-additive, but \( \mu_4 \) is not;

(ii) \( \mu_4 \) and \( \mu_5 \) are superadditive, but \( \mu_3 \) is not.
(iii) $\mu_4$ and $\mu_5$ are converse null-additive, but $\mu_3$ is not.

**Proposition 4.1.** Let $\mu_1$ and $\mu_2$ be two $[0, \infty)$-valued functions defined on an effect algebra $E$. Define $(\mu_1 + \mu_2)(a) = \mu_1(a) + \mu_2(a)$, and $(\mu_1 - \mu_2)(a) = \mu_1(a) - \mu_2(a), a \in E$. Then the following hold:

(i) If $\mu_1$ and $\mu_2$ are supermodular (submodular, respectively) then $\mu_1 + \mu_2$ is supermodular (submodular, respectively).

(ii) If $\mu_1$ and $\mu_2$ are modular, then so are $\mu_1 + \mu_2$ and $\mu_1 - \mu_2$.

(iii) If $\mu_1$ and $\mu_2$ are null-additive, then so are $\mu_1 + \mu_2$ and $\mu_1 - \mu_2$.

**Proof.** (i) Let $\mu_1$ and $\mu_2$ be supermodular. Let $a, b \in E$. Then

$$(\mu_1 + \mu_2)(a) + (\mu_1 + \mu_2)(b) = \mu_1(a) + \mu_2(a) + \mu_1(b) + \mu_2(b) \leq \mu_1(a \vee b) + \mu_1(a \wedge b) + \mu_2(a \vee b) + \mu_2(a \wedge b) = (\mu_1 + \mu_2)(a \vee b) + (\mu_1 + \mu_2)(a \wedge b).$$

If $\mu_1$ and $\mu_2$ are submodular, then by similar arguments, $\mu_1 + \mu_2$ is submodular.

(ii) The proof is obvious.

(iii) Let $a, b \in E$ and $a \perp b$. Since $\mu_1$ and $\mu_2$ are null-additive, therefore, we have $\mu_1(a \oplus b) = \mu_1(a)$ and $\mu_2(a \oplus b) = \mu_2(a)$, whenever $\mu_1(b) = 0$ and $\mu_2(b) = 0$. So $(\mu_1 + \mu_2)(b) = 0$ and $(\mu_1 + \mu_2)(a \oplus b) = \mu_1(a \oplus b) + \mu_2(a \oplus b) = (\mu_1 + \mu_2)(a)$.

By similar arguments, we can show that $\mu_1 - \mu_2$ is also null-additive. $\square$

From now onwards, let $L$ be a $\sigma$-complete $\sigma$-continuous $D$-lattice, $M$ be a proper $D$-sublattice of $L$ and $\mu$ be a $[0, \infty]$-valued function defined on $M$.

**Proposition 4.2.** Let $\mu$ be a superadditive function defined on $M$. Then the function $\mu_s$ is superadditive.

**Proof.** Let $a, b \in L$ and $a \perp b$. By definition of $\mu_s$, we have $c, d \in M$, $c \leq a$, $d \leq b$, $\mu_s(a) - \varepsilon/2 < \mu(c)$ and $\mu_s(b) - \varepsilon/2 < \mu(d)$. Now, by 2.8 (iv) we have $c \oplus d \leq a \oplus b$. Since $\mu_s$ is monotone, we have $\mu_s(a \oplus b) \geq \mu_s(c \oplus d) \geq \mu(c \oplus d) \geq \mu(c) + \mu(d) > \mu_s(a) - \varepsilon/2 + \mu_s(b) - \varepsilon/2 = \mu_s(a) + \mu_s(b) - \varepsilon$. Since $\varepsilon$ is arbitrary, the result follows. $\square$

**Definition 4.2.** Let $\mu$ be a monotone null-additive function on $M$. Let $N_\mu = \{a \in L : \exists a_1, a_2 \in M$ such that $a_1 \leq a \leq a_2$ and $\mu(a_2 \ominus a_1) = 0\}$. The function $\overline{\mu} : N_\mu \rightarrow [0, +\infty]$ is defined as $\overline{\mu}(a) = \mu(a_1)$, where $a_1, a_2 \in M$, $a_1 \leq a \leq a_2$ and $\mu(a_2 \ominus a_1) = 0$. Elements of $N_\mu$ are sometimes also referred to as $\mu$-measurable.

Since $a_1 \leq a_2$, so by 2.8(i), we have $a_2 = a_1 \oplus (a_2 \ominus a_1)$. Hence by null-additivity of $\mu$, we get $\mu(a_2) = \mu(a_1)$. So $\overline{\mu}(a)$ may equally be defined as $\mu(a_2)$. Further, if $d \leq a$ and $d \in M$, then $\mu(d) \leq \mu(a_2) = \mu(a_1)$. Hence
\( \mu(a_1) = \sup \{ \mu(d) : d \leq a, d \in M \}. \) Thus \( \overline{\pi} \) is well-defined. Also \( \overline{\pi} \) is an extension of \( \mu \) on \( N_\mu; \) in particular \( \overline{\pi}(0) = \mu(0). \)

Also if \( a_1, a_2 \in N_\mu \) and \( a_1 \leq a_2, \) then \( \pi(a_1) \leq \pi(a_2) \): Since \( a_1, a_2 \in N_\mu, \) then there exist \( b_1, c_1, b_2, c_2 \in M \) such that \( b_1 \leq a_1 \leq c_1, \mu(c_1 \ominus b_1) = 0 \) and \( b_2 \leq a_2 \leq c_2, \mu(c_2 \ominus b_2) = 0. \) Again, since \( b_1 \leq a_1 \leq a_2 \leq c_2, \) so \( \overline{\pi}(a_1) = \mu(b_1) \leq \mu(c_2) = \overline{\pi}(a_2). \)

Let \( \mu \) and \( \nu \) be monotone null-additive functions defined on \( M \). If \( \ker \mu = \ker \nu, \) then \( N_\mu = N_\nu, \) where \( \ker \mu = \{ a \in M : \mu(a) = 0 \}. \)

**Proposition 4.3.** Let \( \mu \) be a monotone null-additive function defined on \( M. \) If \( \mu \) is converse null-additive, then the function \( \overline{\pi} \) is also converse null-additive on \( N_\mu. \)

**Proof.** Let \( b, c \in N_\mu \) with \( b \leq c \) and \( \overline{\pi}(b) = \overline{\pi}(c) < \infty. \) Then there exist \( b_1, c_1, b_2, c_2 \in M \) such that \( b_1 \leq b \leq b_2, \mu(b_2 \ominus b_1) = 0 \) and \( c_1 \leq c \leq c_2, \mu(c_2 \ominus c_1) = 0. \) Since \( \mu \) is converse null-additive on \( M \) and \( \mu(b_1) = \overline{\pi}(b) = \overline{\pi}(c) = \mu(c_2), \) so \( \mu(c_2 \ominus b_1) = 0. \) Since \( b \leq c \leq c_2, \) then by 2.8 (ii) and (iii) we have \( c \ominus b \leq c_2 \ominus b \leq c_2 \ominus b_1. \) Again since \( \overline{\pi} \) is monotone, \( \overline{\pi}(c \ominus b) \leq \mu(c_2 \ominus b_1) = 0, \) so \( \overline{\pi}(c \ominus b) = 0. \) Hence \( \overline{\pi} \) is converse null-additive on \( N_\mu. \)

**Definition 4.3.** Let \( \mu \) be a function defined on \( M. \) Let \( d \in L \) such that \( d \notin M \) and \( \mu^*(d) < \infty. \) A function \( \mu \) is called weak converse null-additive (on \( M \)) with respect to \( d, \) if for all \( a, b \in M \) such that \( a \leq d \leq b \) and \( \mu(a) = \mu(b) < \infty, \) there exist elements \( a_0, b_0 \in M \) with \( a_0 \leq d \leq b_0 \) such that \( \mu(b_0 \ominus a_0) = 0; \) \( \mu \) is called weak converse null-additive (on \( M \)), if it is weak converse null-additive (on \( M \)) with respect to \( d, \) whenever \( d \notin M \) and \( \mu^*(d) < \infty. \)

Observe that if \( \mu \) is converse null-additive, then it is weak converse null-additive, which yields that every superadditive function is weak converse null-additive. The converse of this statement need not be true as \( \mu_2 \) in Example 4.1 is weak converse null-additive but not superadditive.

**Theorem 4.1.** Let \( \mu \) be a monotone null-additive function defined on \( M. \) Then the following hold:

(i) If \( a \in N_\mu, \) then \( \mu_*(a) = \mu^*(a). \)

(ii) If for every \( d \in L \) such that \( \mu_*(d) = \mu^*(d) < \infty, \) \( d \in N_\mu, \) then \( \mu \) is weak converse null-additive.

**Proof.** (i) Let \( a \in N_\mu. \) Then there exist \( b, c \in M \) such that \( b \leq a \leq c \) and \( \mu(c \ominus b) = 0. \) So \( \mu(b) \leq \mu_*(a) \leq \mu^*(a) \leq \mu(c). \) Since \( \mu(b) = \mu(c), \) we have \( \mu_*(a) = \mu^*(a). \)

(ii) Suppose that \( \mu \) is not weak converse null-additive (on \( M \)), then there exists an element \( d \in L \) such that \( \mu^*(d) < \infty \) and for all \( b, c \in M, \)
$b \leq d \leq c$ and $\mu(b) = \mu(c) < \infty$, $\mu(c \ominus b) \neq 0$. Since $\mu_*(d) = \mu^*(d) < \infty$, we have that $d \in N_\mu$. Then there exist $b_0, c_0 \in M$ such that $b_0 \leq d \leq c_0$ and $\mu(c_0 \ominus b_0) = 0$, a contradiction. \hfill \Box

References


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