Abstract. Some power inequalities for the numerical radius of a product of two operators in Hilbert spaces with applications for commutators and self-commutators are given.

1. Introduction

Let \((H; \langle \cdot, \cdot \rangle)\) be a complex Hilbert space. The numerical range of an operator \(T\) is the subset of the complex numbers \(\mathbb{C}\) given by [11, p. 1]:

\[ W(T) = \{ \langle Tx, x \rangle, \ x \in H, \|x\| = 1 \} . \]

The numerical radius \(w(T)\) of an operator \(T\) on \(H\) is given by [11, p. 8]:

\[ w(T) = \sup \{ |\lambda|, \lambda \in W(T) \} = \sup \{ |\langle Tx, x \rangle|, \|x\| = 1 \} . \quad (1.1) \]

It is well known that \(w(\cdot)\) is a norm on the Banach algebra \(B(H)\) of all bounded linear operators \(T : H \to H\). This norm is equivalent to the operator norm. In fact, the following more precise result holds [11, p. 9]:

\[ w(T) \leq \|T\| \leq 2w(T) , \quad (1.2) \]

for any \(T \in B(H)\)

For other results on numerical radii, see [12], Chapter 11.

If \(A, B\) are two bounded linear operators on the Hilbert space \((H, \langle \cdot, \cdot \rangle)\), then

\[ w(AB) \leq 4w(A)w(B) . \quad (1.3) \]

In the case that \(AB = BA\), then

\[ w(AB) \leq 2w(A)w(B) . \quad (1.4) \]

The following results are also well known [11, p. 38]:

If \(A\) is a unitary operator that commutes with another operator \(B\), then

\[ w(AB) \leq w(B) . \quad (1.5) \]
If $A$ is an isometry and $AB = BA$, then (1.5) also holds true. We say that $A$ and $B$ double commute if $AB = BA$ and $AB^* = B^* A$. If the operators $A$ and $B$ double commute, then [11, p. 38]

$$w(AB) \leq w(B) \|A\|. \quad (1.6)$$

As a consequence of the above, we have [11, p. 39]: Let $A$ be a normal operator commuting with $B$, then

$$w(AB) \leq w(A) w(B). \quad (1.7)$$

For other results and historical comments on the above see [11, p. 39–41]. For recent inequalities involving the numerical radius, see [1] -[9], [13], [14]-[16] and [17].

2. Inequalities for a product of two operators

**Theorem 1.** For any $A, B \in B(H)$ and $r \geq 1$, we have the inequality

$$w^r(B^*A) \leq \frac{1}{2} \|(A^*A)^r + (B^*B)^r\|. \quad (2.1)$$

The constant $\frac{1}{2}$ is best possible.

**Proof.** By the Schwarz inequality in the Hilbert space $(H; \langle ., . \rangle)$ we have

$$|\langle B^*Ax, x \rangle| = |\langle Ax, Bx \rangle| \leq \|Ax\| \cdot \|Bx\| \quad (2.2)$$

$$= \langle A^*Ax, x \rangle^{1/2} \cdot \langle B^*Bx, x \rangle^{1/2}, \quad x \in H.$$ Utilizing the arithmetic mean - geometric mean inequality and then the convexity of the function $f(t) = t^r$, $r \geq 1$, we have successively,

$$\langle A^*Ax, x \rangle^{1/2} \cdot \langle B^*Bx, x \rangle^{1/2} \leq \frac{\langle A^*Ax, x \rangle + \langle B^*Bx, x \rangle}{2} \quad (2.3)$$

$$\leq \left( \frac{\langle A^*Ax, x \rangle^r + \langle B^*Bx, x \rangle^r}{2} \right)^{\frac{1}{r}}$$

for any $x \in H$.

It is known that if $P$ is a positive operator then for any $r \geq 1$ and $x \in H$ with $\|x\| = 1$ we have the inequality (see for instance [15])

$$\langle Px, x \rangle^r \leq \langle P^r x, x \rangle. \quad (2.4)$$

Applying this property to the positive operator $A^*A$ and $B^*B$, we deduce that

$$\left( \frac{\langle A^*Ax, x \rangle^r + \langle B^*Bx, x \rangle^r}{2} \right)^{\frac{1}{r}} \leq \left( \frac{\langle (A^*A)^r x, x \rangle + \langle (B^*B)^r x, x \rangle}{2} \right)^{\frac{1}{r}} \quad (2.5)$$

$$= \left( \frac{\langle [A^*A]^r + (B^*B)^r \rangle x, x \rangle}{2} \right)^{\frac{1}{r}}.$$
for any $x \in H$, $\|x\| = 1$.

Now, on making use of the inequalities (2.2), (2.3) and (2.5), we get the inequality
\[ |\langle (B^*A)^r x, x \rangle^r| \leq \frac{1}{2} |\langle (A^*A)^r + (B^*B)^r \rangle x, x \rangle \]
for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.6) and since the operator $[(A^*A)^r + (B^*B)^r]$ is self-adjoint, we deduce the desired inequality (2.1).

For $r = 1$ and $B = A$, we get in both sides of (2.1) the same quantity $\|A\|^2$ which shows that the constant $\frac{1}{2}$ is best possible in general in the inequality (2.1).

\[ \Box \]

**Corollary 1.** For any $A \in B(H)$ and $r \geq 1$ we have the inequalities
\[ w^r(A) \leq \frac{1}{2} \| (A^*A)^r + I \| \]
and
\[ w^r(A^2) \leq \frac{1}{2} \| (A^*A)^r + (AA^*)^r \|, \]
respectively.

A different approach is considered in the following result:

**Theorem 2.** For any $A, B \in B(H)$ and any $\alpha \in (0, 1)$ and $r \geq 1$, we have the inequality
\[ w^{2r}(B^*A) \leq \left| \alpha (A^*A)^{\frac{\alpha}{r}} + (1 - \alpha) (B^*B)^{\frac{1}{1-r}} \right|. \]

**Proof.** By Schwarz’s inequality, we have
\[ |\langle (B^*A) x, x \rangle| \leq \langle (A^*A) x, x \rangle \cdot \langle (B^*B) x, x \rangle \]
\[ = \left\langle \left[ (A^*A)^{\frac{\alpha}{r}} \right] x, x \right\rangle \cdot \left\langle \left[ (B^*B)^{\frac{1}{1-r}} \right]^{1-\alpha} x, x \right\rangle, \]
for any $x \in H$.

It is well known that (see for instance [15]) if $P$ is a positive operator and $q \in (0, 1]$ then for any $u \in H$, $\|u\| = 1$, we have
\[ \langle P^q u, u \rangle \leq \langle Pu, u \rangle^q. \]

Applying this property to the positive operators $(A^*A)^{\frac{\alpha}{r}}$ and $(B^*B)^{\frac{1}{1-r}}$ $(\alpha \in (0, 1))$, we have
\[ \left\langle \left[ (A^*A)^{\frac{\alpha}{r}} \right] x, x \right\rangle \cdot \left\langle \left[ (B^*B)^{\frac{1}{1-r}} \right]^{1-\alpha} x, x \right\rangle \]
\[ \leq \left\langle (A^*A)^{\frac{1}{r}} x, x \right\rangle^\alpha \cdot \left\langle (B^*B)^{\frac{1}{1-r}} x, x \right\rangle^{1-\alpha}, \]
for any \( x \in H, \|x\| = 1 \).

Now, utilizing the weighted arithmetic mean - geometric mean inequality, i.e., \( a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha) b, \alpha \in (0, 1), a, b \geq 0 \), we get

\[
\langle (A^*A)^{1/\alpha} x, x \rangle^\alpha \cdot \langle (B^*B)^{1/\alpha} x, x \rangle^{1-\alpha} \\
\leq \alpha \langle (A^*A)^{1/\alpha} x, x \rangle + (1-\alpha) \langle (B^*B)^{1/\alpha} x, x \rangle
\]

(2.13)

for any \( x \in H, \|x\| = 1 \).

Moreover, by the elementary inequality following from the convexity of the function \( f(t) = t^r, r \geq 1 \), namely

\[
\alpha a + (1-\alpha) b \leq (\alpha a^r + (1-\alpha) b^r)^{1/r}, \quad \alpha \in (0, 1), a, b \geq 0,
\]

we deduce that

\[
\alpha \langle (A^*A)^{1/\alpha} x, x \rangle + (1-\alpha) \langle (B^*B)^{1/\alpha} x, x \rangle \\
\leq \left[ \alpha \langle (A^*A)^{1/\alpha} x, x \rangle^r + (1-\alpha) \langle (B^*B)^{1/\alpha} x, x \rangle^r \right]^{1/r} \\
\leq \left[ \alpha \langle (A^*A)^{1/\alpha} x, x \rangle + (1-\alpha) \langle (B^*B)^{1/\alpha} x, x \rangle \right]^{1/r}, \quad (2.14)
\]

for any \( x \in H, \|x\| = 1 \), where, for the last inequality we used the inequality (2.4) for the positive operators \((A^*A)^{1/\alpha}\) and \((B^*B)^{1/\alpha}\).

Now, on making use of the inequalities (2.10), (2.12), (2.13) and (2.14), we get

\[
|\langle (B^*A) x, x \rangle|^{2r} \leq \left[ \alpha \langle (A^*A)^{1/\alpha} x, x \rangle + (1-\alpha) \langle (B^*B)^{1/\alpha} x, x \rangle \right] x, x
\]

(2.15)

for any \( x \in H, \|x\| = 1 \). Taking the supremum over \( x \in H, \|x\| = 1 \) in (2.15) produces the desired inequality (2.9). \( \square \)

**Remark 1.** The particular case \( \alpha = \frac{1}{2} \) produces the inequality

\[
w^{2r} (B^*A) \leq \frac{1}{2} \left\| (A^*A)^{2r} + (B^*B)^{2r} \right\|
\]

(2.16)

for \( r \geq 1 \). Notice that \( \frac{1}{2} \) is best possible in (2.16) since for \( r = 1 \) and \( B = A \) we get in both sides of (2.16) the same quantity \( \|A\|^4 \).

**Corollary 2.** For any \( A \in B(H) \) and \( \alpha \in (0, 1), r \geq 1 \), we have the inequalities

\[
w^{2r} (A) \leq \left\| \alpha (A^*A)^{\frac{1}{\alpha}} + (1-\alpha) I \right\|
\]

(2.17)

and

\[
w^{2r} (A^2) \leq \left\| \alpha (A^*A)^{\frac{1}{\alpha}} + (1-\alpha) (AA^*)^{\frac{1}{\alpha}} \right\|
\]

(2.18)

respectively.
Moreover, we have
\[ \| A \|_{4r} \leq \| \alpha (A^*A)^{\frac{r}{2}} + (1 - \alpha) (A^*A)^{\frac{r}{2}} \| . \] (2.19)

3. Inequalities for the sum of two products

The following result may be stated:

**Theorem 3.** For any \( A, B, C, D \in B(H) \) and \( r, s \geq 1 \) we have
\[ \| \frac{B^*A + D^*C}{2} \|_2 \leq \frac{\| (A^*A)^r + (C^*C)^r \|^{\frac{1}{r}} \cdot \| (B^*B)^s + (D^*D)^s \|^{\frac{1}{r}}}{2} . \] (3.1)

**Proof.** By the Schwarz inequality in the Hilbert space \((H; \langle ., . \rangle)\) we have
\[ \| (B^*A + D^*C) x, y \|_2^2 = \| (B^*Ax, y) + (D^*Cx, y) \|_2^2 \leq \| (B^*Ax, y) \| + \| (D^*Cx, y) \|_2^2 \leq \left( \langle A^*Ax, x \rangle^{\frac{1}{2}} \cdot \langle B^*By, y \rangle^{\frac{1}{2}} + \langle C^*Cx, x \rangle^{\frac{1}{2}} \cdot \langle D^*Dy, y \rangle^{\frac{1}{2}} \right)^2 , \] (3.2)
for any \( x, y \in H \).

Now, on utilizing the elementary inequality
\[ (ab + cd)^2 \leq (a^2 + c^2)(b^2 + d^2) , \quad a, b, c, d \in \mathbb{R} , \]
we then conclude that
\[ \left( \langle A^*Ax, x \rangle^{\frac{1}{2}} \cdot \langle B^*By, y \rangle^{\frac{1}{2}} + \langle C^*Cx, x \rangle^{\frac{1}{2}} \cdot \langle D^*Dy, y \rangle^{\frac{1}{2}} \right)^2 \leq \langle (A^*Ax, x) + (C^*Cx, x) \rangle \cdot \langle (B^*By, y) + (D^*Dy, y) \rangle , \] (3.3)
for any \( x, y \in H \).

Now, on making use of a similar argument to the one in the proof of Theorem 1, we have for \( r, s \geq 1 \) that
\[ \langle (A^*Ax, x) + (C^*Cx, x) \rangle \cdot \langle (B^*By, y) + (D^*Dy, y) \rangle \leq 4 \left( \frac{\| (A^*A)^r + (C^*C)^r \|}{2} \cdot \left( \frac{\| (B^*B)^s + (D^*D)^s \|}{2} \right) \right)^{\frac{1}{2}} , \] (3.4)
for any \( x, y \in H, \| x \| = \| y \| = 1 \).

Consequently, by (3.2) – (3.4) we have
\[ \left( \left\| \frac{B^*A + D^*C}{2} \right\| \right)^2 \leq \left( \frac{\| (A^*A)^r + (C^*C)^r \|^{\frac{1}{r}} \cdot \| (B^*B)^s + (D^*D)^s \|^{\frac{1}{s}}}{2} \right)^2 . \] (3.5)
for any \( x, y \in H, \|x\| = \|y\| = 1 \).

Taking the supremum over \( x, y \in H, \|x\| = \|y\| = 1 \) we deduce the desired inequality (3.1). \( \square \)

Remark 2. If \( s = r \), then the inequality (3.1) is equivalent with

\[
\left\| \frac{B^*A + D^*C}{2} \right\|^{2r} \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\|^\frac{1}{r} \cdot \left\| \frac{(B^*B)^r + (D^*D)^r}{2} \right\|^\frac{1}{r}. \tag{3.6}
\]

Corollary 3. For any \( A, C \in B(H) \) we have

\[
\left\| \frac{A + C}{2} \right\|^{2r} \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\|^\frac{1}{r}, \tag{3.7}
\]

where \( r \geq 1 \). Also, we have

\[
\left\| \frac{A^2 + C^2}{2} \right\|^{2r} \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\|^\frac{1}{r} \cdot \left\| \frac{(AA^*)^r + (CC^*)^r}{2} \right\|^\frac{1}{r}\tag{3.8}
\]

for all \( r, s \geq 1 \), and in particular

\[
\left\| \frac{A^2 + C^2}{2} \right\|^{2r} \leq \left\| \frac{(A^*A)^r + (C^*C)^r}{2} \right\|^\frac{1}{r} \cdot \left\| \frac{(AA^*)^r + (CC^*)^r}{2} \right\|^\frac{1}{r} \tag{3.9}
\]

for \( r \geq 1 \).

The inequality (3.7) follows from (3.1) for \( B = D = I \), while the inequality (3.8) is obtained from the same inequality (3.1) for \( B = A^* \) and \( D = C^* \).

Another particular result of interest is the following one:

Corollary 4. For any \( A, B \in B(H) \) we have

\[
\left\| \frac{B^*A + A^*B}{2} \right\|^{2r} \leq \left\| \frac{(A^*A)^r + (B^*B)^r}{2} \right\|^\frac{1}{r} \cdot \left\| \frac{(A^*A)^s + (B^*B)^s}{2} \right\|^\frac{1}{s}\tag{3.10}
\]

for \( r, s \geq 1 \) and, in particular,

\[
\left\| \frac{B^*A + A^*B}{2} \right\|^r \leq \left\| \frac{(A^*A)^r + (B^*B)^r}{2} \right\|\tag{3.11}
\]

for any \( r \geq 1 \).

The inequality (3.9) follows from (3.1) for \( D = A \) and \( C = B \).

Another particular case that might be of interest is the following one.

Corollary 5. For any \( A, D \in B(H) \) we have

\[
\left\| \frac{A + D}{2} \right\|^{2r} \leq \left\| \frac{(A^*A)^r + I}{2} \right\|^\frac{1}{r} \cdot \left\| \frac{(DD^*)^s + I}{2} \right\|^\frac{1}{s}, \tag{3.12}
\]
where \( r, s \geq 1 \). In particular

\[
\|A\|^2 \leq \left\| \frac{(A^*A)^r + I}{2} \right\|^\frac{1}{r} \cdot \left\| \frac{(AA^*)^s + I}{2} \right\|^\frac{1}{s}.
\]  

(3.13)

Moreover, for any \( r \geq 1 \) we have

\[
\|A\|^{2r} \leq \left\| \frac{(A^*A)^r + I}{2} \right\| \cdot \left\| \frac{(AA^*)^r + I}{2} \right\|.
\]

The proof is obvious by the inequality (3.1) on choosing \( B = I, C = I \) and writing the inequality for \( D^* \) instead of \( D \).

**Remark 3.** If \( T \in B(H) \) and \( T = A + iC \), i.e., \( A \) and \( C \) are its Cartesian decomposition, then we get from (3.7) that

\[
\|T\|^{2r} \leq 2^{2r-1} \|A^{2r} + C^{2r}\|,
\]

for any \( r \geq 1 \).

Also, since \( A = \text{Re}(T) = \frac{T + T^*}{2} \) and \( C = \text{Im}(T) = \frac{T - T^*}{2i} \), then from (3.7) we get the following inequalities as well

\[
\|\text{Re}(T)\|^{2r} \leq \left\| \frac{(T^*T)^r + (TT^*)^{r}}{2} \right\|
\]

and

\[
\|\text{Im}(T)\|^{2r} \leq \left\| \frac{(T^*T)^r + (TT^*)^{r}}{2} \right\|
\]

for any \( r \geq 1 \).

In terms of the Euclidean radius of two operators \( w_e(\cdot, \cdot) \), where, as in [1],

\[
w_e(T, U) := \sup_{\|x\|=1} \left( \|\langle Tx, x \rangle\|^2 + \|\langle Ux, x \rangle\|^2 \right)^{\frac{1}{2}},
\]

we have the following result as well.

**Theorem 4.** For any \( A, B, C, D \in B(H) \) and \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), we have the inequality

\[
w_e^2(B^*A, D^*C) \leq \|(A^*A)^p + (C^*C)^q\|^{1/p} \cdot \|(B^*B)^q + (D^*D)^q\|^{1/q}.
\]  

(3.14)

**Proof.** For any \( x \in H, \|x\| = 1 \) we have the inequalities

\[
\begin{align*}
\|\langle B^*A x, x \rangle\|^2 + \|\langle D^*C x, x \rangle\|^2 &
\leq \langle A^*A x, x \rangle \cdot \langle B^*B x, x \rangle + \langle C^*C x, x \rangle \cdot \langle D^*D x, x \rangle \\
&
\leq ((A^*A x, x)^p + (C^*C x, x)^q)^{1/p} \cdot ((B^*B x, x)^q + (D^*D x, x)^q)^{1/q} \\
&
\leq (((A^*A)^p x, x) + ((C^*C)^q x, x))^{1/p} \cdot (((B^*B)^q x, x) + ((D^*D)^q x, x))^{1/q} \\
&
\leq ((A^*A)^p x, x)^{1/p} \cdot ((B^*B)^q x, x)^{1/q}.
\end{align*}
\]
Taking the supremum over \( x \in H, \|x\| = 1 \) and noticing that the operators \((A^*A)^p + (C^*C)^p\) and \((B^*B)^q + (D^*D)^q\) are self-adjoint, we deduce the desired inequality (3.14).

The following particular case is of interest.

**Corollary 6.** For any \( A, C \in B(H) \) and \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), we have

\[
\varpi^2(A, C) \leq 2^{1/q} \|(A^*A)^p + (C^*C)^p\|^{1/p}.
\]

The proof follows from (3.14) for \( B = D = I \).

**Corollary 7.** For any \( A, D \in B(H) \) and \( p, q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), we have

\[
\varpi^2(A, D) \leq \|(A^*A)^p + I\|^{1/p} \cdot \|(D^*D)^q + I\|^{1/q}.
\]

4. **Norm inequalities for the commutator**

The commutator of two bounded linear operators \( T \) and \( U \) is the operator \( TU - UT \). For the usual norm \( \|\cdot\| \) and for any two operators \( T \) and \( U \), by using the triangle inequality and the submultiplicity of the norm, we can state the following inequality

\[
\|TU - UT\| \leq 2 \|U\| \|T\|. \tag{4.1}
\]

In [10], the following result has been obtained as well

\[
\|TU - UT\| \leq 2 \min \{\|T\|, \|U\|\} \min \{\|T - U\|, \|T + U\|\}. \tag{4.2}
\]

By utilizing Theorem 3 we can state the following result for the numerical radius of the commutator.

**Proposition 1.** For any \( T, U \in B(H) \) and \( r, s \geq 1 \) we have

\[
\|TU - UT\|^{2r} \leq 2^{2r-1} \|(T^*T)^r + (U^*U)^r\| \cdot \|(TT^*)^s + (UU^*)^s\|^{1/2}. \tag{4.3}
\]

**Proof.** Follows by Theorem 3 on choosing \( B = T^*, A = U, D = -U^* \) and \( C = T \). \( \square \)

**Remark 4.** In particular, for \( r = s \) we get from (4.3) that

\[
\|TU - UT\|^{2r} \leq 2^{2r-2} \|(T^*T)^r + (U^*U)^r\| \cdot \|(TT^*)^r + (UU^*)^r\| \tag{4.4}
\]

and for \( r = 1 \) we get

\[
\|TU - UT\|^2 \leq \|T^*T + U^*U\| \cdot \|(TT^*) + (UU^*)\|. \tag{4.5}
\]

For a bounded linear operator \( T \in B(H) \), the self-commutator is the operator \( T^*T - TT^* \). Observe that the operator \( V := -i(T^*T - TT^*) \) is self-adjoint and \( w(V) = \|V\| \), i.e.,

\[
w(T^*T - TT^*) = \|T^*T - TT^*\|.
\]

Now, utilizing (4.3) for \( U = T^* \) we can state the following corollary.
Corollary 8. For any $T \in B(H)$ we have the inequality
\[
\|T^*T - TT^*\| \leq 2^{2 - \frac{1}{r} - \frac{1}{s}} \|(T^*T)^r + (TT^*)^r\|^{\frac{1}{r}} \|(T^*T)^s + (TT^*)^s\|^{\frac{1}{s}}. \tag{4.6}
\]
In particular, we have
\[
\|T^*T - TT^*\|^r \leq 2^{r-1} \|(T^*T)^r + (TT^*)^r\|, \tag{4.7}
\]
for any $r \geq 1$.
Moreover, for $r = 1$ we have
\[
\|T^*T - TT^*\| \leq \|T^*T + TT^*\|. \tag{4.8}
\]

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References


