TRIGONOMETRIC APPROXIMATION OF FUNCTIONS IN WEIGHTED $L^p$ SPACES

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Abstract. The approximation properties of means of trigonometric Fourier series in weighted $L^p$ spaces ($1 < p < \infty$) with Muckenhoupt weights are investigated.

1. Introduction and results

A measurable $2\pi$-periodic function $w : [0, 2\pi] \to [0, \infty]$ is said to be a weight function if the set $w^{-1}(\{0, \infty\})$ has the Lebesgue measure zero. We denote by $L^p_w = L^p_w([0, 2\pi])$, where $1 \leq p < \infty$ and $w$ a weight function, the weighted Lebesgue space of all measurable $2\pi$-periodic functions $f$, that is, the space of all such functions for which

$$\|f\|_{p,w} = \left( \int_0^{2\pi} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.$$  

Let $1 < p < \infty$. A weight function $w$ belongs to the Muckenhoupt class $\mathcal{A}_p$ if

$$\sup_I \left( \frac{1}{|I|} \int_I w(x) \, dx \right) \left( \frac{1}{|I|} \int_I [w(x)]^{-1/p-1} \, dx \right)^{p-1} < \infty,$$

where the supremum is taken over all intervals $I$ with length $|I| \leq 2\pi$.

The weight functions belong to the $\mathcal{A}_p$ class, introduced by Muckenhoupt ([13]), play a very important role in different fields of mathematical analysis.

Let $1 < p < \infty$, $w \in \mathcal{A}_p$ and let $f \in L^p_w$. The modulus of continuity of the function $f$ is defined by

$$\Omega(f, \delta)_{p,w} = \sup_{|h| \leq \delta} \|\Delta_h(f)\|_{p,w}, \quad \delta > 0,$$

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where \[
\Delta_h (f) (x) = \frac{1}{h} \int_0^h |f(x + t) - f(x)| \, dt.
\]
The existence of the modulus \( \Omega (f, \delta)_{p,w} \) follows from the boundedness of the Hardy-Littlewood maximal function in the space \( L^p_w \) (see [13]). The modulus of continuity \( \Omega (f, \cdot)_{p,w} \), defined by N. X. Ky [10], is nondecreasing, nonnegative, continuous function such that
\[
\lim_{\delta \to 0} \Omega (f, \delta)_{p,w} = 0,
\]
\[
\Omega (f_1 + f_2, \cdot)_{p,w} \leq \Omega (f_1, \cdot)_{p,w} + \Omega (f_2, \cdot)_{p,w}.
\]
The modulus of continuity \( \Omega (f, \cdot)_{p,w} \) is defined in this way, since the space \( L^p_w \) is noninvariant, in general, under the usual shift \( f(x) \to f(x + h) \). Note that, in the case \( w \equiv 1 \) the modulus \( \Omega (f, \cdot)_{p,w} \) and the classical integral modulus of continuity \( \omega_p (f, \cdot) \) are equivalent (see [10]).

We define the Lipschitz class \( Lip (\alpha, p, w) \) for \( 0 < \alpha \leq 1 \) by
\[
Lip (\alpha, p, w) = \left\{ f \in L^p_w : \Omega (f, \cdot)_{p,w} = O (\delta^\alpha), \quad \delta > 0 \right\}.
\]
Let \( f \in L^1 \) has the Fourier series
\[
f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).
\]
(1)

Let \( S_n (f) (x), (n = 0, 1, \ldots) \) be the \( n \)th partial sums of the series (1) at the point \( x \), that is,
\[
S_n (f) (x) = \sum_{k=0}^{n} A_k (f) (x),
\]
where
\[
A_0 (f) (x) = \frac{a_0}{2}, \quad A_k (f) (x) = a_k \cos kx + b_k \sin kx, \quad k = 1, 2, \ldots.
\]
Let \( (p_n)_{n=0}^{\infty} \) be a sequence of positive numbers. We consider two means of the series (1) defined by
\[
N_n (f) (x) = \frac{1}{p_n} \sum_{m=0}^{n} p_{n-m} S_m (f) (x)
\]
and
\[
R_n (f) (x) = \frac{1}{p_n} \sum_{m=0}^{n} p_m S_m (f) (x),
\]
where $P_n = \sum_{m=0}^{n} p_m$, $p_{-1} = P_{-1} := 0$. In the case $p_n = 1, n \geq 0$, both of $N_n (f) (x)$ and $R_n (f) (x)$ are equal to the Cesàro mean

$$\sigma_n (f) (x) = \frac{1}{n+1} \sum_{m=0}^{n} S_m (f) (x).$$

The approximation properties of the means $\sigma_n$ in Lipschitz classes $\text{Lip} (\alpha, p)$, $1 \leq p < \infty$, $0 < \alpha \leq 1$ were investigated by Quade in [14]. The generalizations of Quade’s results were studied by Mohapatra and Russell [12], Chandra ([1], [2], [3], [4]) and Leindler [11]. In [1], Chandra obtained estimates for $\|f - N_n (f)\|_p$, where $f \in \text{Lip} (\alpha, p)$, $1 < p < \infty$, $0 < \alpha \leq 1$ (see [2]). In the paper [4], Chandra gave some conditions on the sequence $(p_n)_{n=0}^{\infty}$ and obtained very satisfactory results about approximation by the means $N_n (f)$ and $R_n (f)$ in $\text{Lip} (\alpha, p)$, $1 \leq p < \infty$, $0 < \alpha \leq 1$.

In the present paper, we give the weighted versions of the results obtained by Chandra in [4] in the case $1 < p < \infty$. Our main results are the following.

**Theorem 1.** Let $1 < p < \infty$, $w \in A_p$, $0 < \alpha \leq 1$, and let $(p_n)_{n=0}^{\infty}$ be a monotonic sequence of positive real numbers such that

$$(n+1) p_n = O (P_n). \quad (2)$$

Then, for every $f \in \text{Lip} (\alpha, p, w)$ the estimate

$$\|f - N_n (f)\|_{p,w} = O (n^{-\alpha}), \quad n = 1, 2, \ldots$$

holds.

**Theorem 2.** Let $1 < p < \infty$, $w \in A_p$, $0 < \alpha \leq 1$, and let $(p_n)$ be a sequence of positive real numbers satisfying the relation

$$\sum_{m=0}^{n-1} \left| \frac{P_m}{m+1} - \frac{P_{m+1}}{m+2} \right| = O \left( \frac{P_n}{n+1} \right). \quad (3)$$

Then, for $f \in \text{Lip} (\alpha, p, w)$ the estimate

$$\|f - R_n (f)\|_{p,w} = O (n^{-\alpha}), \quad n = 1, 2, \ldots$$

satisfied.

If we take $p_n = A_n^{\beta-1}$ ($\beta > 0$), where

$$A_0^\beta = 1, \quad A_k^\beta = \frac{\beta (\beta + 1) \ldots (\beta + k)}{k!}, \quad k \geq 1,$$

we get

$$N_n (f) (x) = \sigma_n^{\beta} (f) (x) = \frac{1}{A_n^\beta} \sum_{m=0}^{n} A_{n-m}^{\beta-1} S_m (f) (x).$$
Hence we can estimate the deviation of \( f \in \text{Lip}(\alpha, p, w) \) from the Cesàro means \( \sigma_n^\beta(f) \):

**Corollary 3.** Let \( 1 < p < \infty, w \in \mathcal{A}_p, 0 < \alpha \leq 1 \) and \( \beta > 0 \). Then, for \( f \in \text{Lip}(\alpha, p, w) \),

\[
\| f - \sigma_n^\beta(f) \|_{p,w} = O\left( n^{-\alpha} \right), \quad n = 1, 2, \ldots.
\]

The trigonometric approximation problems in weighted \( L^p \) spaces with Muckenhoupt weights where \( 1 < p < \infty \) were studied by several authors. Gadjieva [5] obtained the direct and inverse theorems of trigonometric approximation in the spaces \( L^p_w \). Later, Ky investigated the same problems and obtained similar results by using a different modulus of continuity, which in special case coincides with the modulus \( \Omega (f, \cdot)_{p,w} \) ([9], [10]). The improvement of the inverse theorem of Gadjieva was obtained in [6]. Later, in the more general spaces, namely weighted Orlicz spaces, the direct and inverse theorems of trigonometric approximation and the complete characterization of the generalized Lipschitz classes were obtained [8].

**Remark.** Theorem 1, Theorem 2 and Corollary 3 also hold in reflexive weighted Orlicz spaces \( L^M_w \).

The general information on weighted Orlicz spaces and approximation results in these spaces can be found in [8].

2. **Some auxiliary results**

**Lemma 4.** Let \( 1 < p < \infty, w \in \mathcal{A}_p \) and \( 0 < \alpha \leq 1 \). Then, the estimate

\[
\| f - S_n(f) \|_{p,w} = O\left( n^{-\alpha} \right) \tag{4}
\]

holds for every \( f \in \text{Lip}(\alpha, p, w) \) and \( n = 1, 2, \ldots \).

**Proof.** Let \( t_n^*(n = 0, 1, \ldots) \) be the trigonometric polynomial of best approximation to \( f \), that is,

\[
\| f - t_n^* \|_{p,w} = \inf \| f - t_n \|_{p,w},
\]

where the infimum is taken over all trigonometric polynomials \( t_n \) of degree at most \( n \). From Theorem 2 of [10], we have

\[
\| f - t_n^* \|_{p,w} = O\left( \Omega(f, 1/n)_{p,w} \right)
\]

and hence

\[
\| f - t_n^* \|_{p,w} = O\left( n^{-\alpha} \right).
\]

By the uniform boundedness of the partial sums \( S_n(f) \) in the space \( L^p_w \) (see [7]), we get

\[
\| f - S_n(f) \|_{p,w} \leq \| f - t_n^* \|_{p,w} + \| t_n^* - S_n(f) \|_{p,w}.
\]
\[
\|f - t_n^*\|_{p,w} + \|S_n (t_n^* - f)\|_{p,w} = O \left(\|f - t_n^*\|_{p,w}\right) = O (n^{-\alpha}).
\]

**Lemma 5.** Let \(1 < p < \infty\) and \(w \in A_p\). Then, for \(f \in \text{Lip}(1, p, w)\) the estimate

\[
\|S_n (f) - \sigma_n (f)\|_{p,w} = O (n^{-1}) , \quad n = 1, 2, \ldots
\]

holds.

**Proof.** If \(f \in \text{Lip}(1, p, w)\), from Theorem 3 of [10] it can be deduced that \(f\) is absolutely continuous and \(f' \in L^p_w\). If \(f\) has the Fourier series

\[
f (x) \sim \sum_{k=0}^{\infty} A_k (f) (x),
\]

then the Fourier series of the conjugate function \(\tilde{f}'\) is

\[
\tilde{f}' (x) \sim \sum_{k=1}^{\infty} k A_k (f) (x).
\]

On the other hand,

\[
S_n (f) (x) - \sigma_n (f) (x) = \sum_{k=1}^{n} \frac{k}{n+1} A_k (f) (x)
\]

\[
= \frac{1}{n+1} S_n \left( \tilde{f}' \right) (x).
\]

Hence, by considering the uniform boundedness of the partial sums and the conjugation operator in the space \(L^p_w\) (see [7]), we obtain

\[
\|S_n (f) - \sigma_n (f)\|_{p,w} = O (n^{-1})
\]

for \(n = 1, 2, \ldots\). □

**Lemma 6.** ([4]). Let \((p_n)\) be a non-decreasing sequence of positive numbers. Then,

\[
\sum_{m=1}^{n} m^{-\alpha} p_{n-m} = O (n^{-\alpha} P_n)
\]

for \(0 < \alpha < 1\).
3. Proof of the new results

Proof of Theorem 1. Let \(0 < \alpha < 1\). Since

\[
f(x) = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} f(x),
\]

we have

\[
f(x) - N_n(f)(x) = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} \{ f(x) - S_m(f)(x) \}.
\]

By Lemma 4, Lemma 6 and condition (2) we obtain

\[
\|f - N_n(f)\|_{p,w} \leq \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} \|f - S_m(f)\|_{p,w}
\]

\[
= \frac{1}{P_n} \sum_{m=1}^{n} p_{n-m} O(m^{-\alpha}) + \frac{P_n}{P_n} \|f - S_0(f)\|_{p,w}
\]

\[
= \frac{1}{P_n} O(n^{-\alpha} P_n) + O \left( \frac{1}{n + 1} \right)
\]

\[
= O \left( n^{-\alpha} \right).
\]

Now let \(\alpha = 1\). It is clear that

\[
N_n(f)(x) = \frac{1}{P_n} \sum_{m=0}^{n} P_{n-m} A_m(f)(x).
\]

By Abel transform,

\[
S_n(f)(x) - N_n(f)(x) = \frac{1}{P_n} \sum_{m=1}^{n} \left( P_n - P_{n-m} \right) A_m(f)(x)
\]

\[
= \frac{1}{P_n} \sum_{m=1}^{n} \left( \frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1} \right) \left( \sum_{k=1}^{m} k A_k(f)(x) \right)
\]

\[
+ \frac{1}{n + 1} \sum_{k=1}^{n} k A_k(f)(x),
\]

and hence

\[
\|S_n(f) - N_n(f)\|_{p,w} \leq \frac{1}{P_n} \sum_{m=1}^{n} \left| \frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1} \right|
\]

\[
\times \left\| \sum_{k=1}^{m} k A_k(f) \right\|_{p,w} + \frac{1}{n + 1} \left\| \sum_{k=1}^{n} k A_k(f) \right\|_{p,w}.
\]
Since
\[ S_n (f) (x) - \sigma_n (f) (x) = \frac{1}{n + 1} \sum_{k=1}^{n} kA_k (f) (x) , \]
by Lemma 5 we get
\[ \left\| \sum_{k=1}^{n} kA_k (f) \right\|_{p,w} = (n + 1) \left\| S_n (f) - \sigma_n (f) \right\|_{p,w} = O (1) . \]
Hence,
\[ \left\| S_n (f) - N_n (f) \right\|_{p,w} \leq \frac{1}{P_n} \sum_{m=1}^{n} \left| \frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1} \right| O (1) + O \left( n^{-1} \right) \]
\[ = O \left( \frac{1}{P_n} \right) \sum_{m=1}^{n} \left| \frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1} \right| + O \left( n^{-1} \right) . \quad (6) \]
By a simple computation, one can see that
\[ \frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1} = \frac{1}{m (m+1)} \left( \sum_{k=n-m+1}^{n} p_k - mp_{n-m} \right), \]
which shows that
\[ \left( \frac{P_n - P_{n-m}}{m} \right)_{m=1}^{n+1} \]
is non-increasing whenever \((p_n)\) is non-decreasing and non-decreasing whenever \((p_n)\) is non-increasing. This implies that
\[ \sum_{m=1}^{n} \left| \frac{P_n - P_{n-m}}{m} - \frac{P_n - P_{n-(m+1)}}{m+1} \right| = \left| p_n - \frac{P_n}{n+1} \right| = \frac{1}{n+1} O \left( P_n \right) . \]
This and the inequality (6) yields
\[ \left\| S_n (f) - N_n (f) \right\|_{p,w} = O \left( n^{-1} \right) . \]
Combining the last estimate with (4) we obtain
\[ \left\| f - N_n (f) \right\|_{p,w} = O \left( n^{-1} \right) . \]

**Proof of Theorem 2.** Let \(0 < \alpha < 1\). By definition of \(R_n (f) (x) , \)
\[ f (x) - R_n (f) (x) = \frac{1}{P_n} \sum_{m=0}^{n} p_m \{ f (x) - S_m (f) (x) \} . \]
From Lemma 4, we get
\[
\| f - R_n (f) \|_{p,w} \leq \frac{1}{P_n} \sum_{m=0}^{n} p_m \| f - S_m (f) \|_{p,w}
\]
\[
= O \left( \frac{1}{P_n} \right) \sum_{m=1}^{n} p_m m^{-\alpha} + \frac{P_0}{P_n} \| f - S_0 (f) \|_{p,w}
\]
\[
= O \left( \frac{1}{P_n} \right) \sum_{m=1}^{n} p_m m^{-\alpha}.
\]
(7)

By Abel transform,
\[
\sum_{m=1}^{n} p_m m^{-\alpha} = \sum_{m=1}^{n-1} p_m \left\{ m^{-\alpha} - (m + 1)^{-\alpha} \right\} + n^{-\alpha} P_n
\]
\[
\leq \sum_{m=1}^{n-1} m^{-\alpha} \frac{P_m}{m + 1} + n^{-\alpha} P_n,
\]
and
\[
\sum_{m=1}^{n-1} m^{-\alpha} \frac{P_m}{m + 1} = \sum_{m=1}^{n-1} \left( \frac{P_m}{m + 1} - \frac{P_{m+1}}{m + 2} \right) \left( \sum_{k=1}^{m} k^{-\alpha} \right) + \frac{P_n}{n + 1} \sum_{m=1}^{n-1} m^{-\alpha}
\]
\[
= O \left( n^{-\alpha} P_n \right)
\]
by condition (3). This yields
\[
\sum_{m=1}^{n} p_m m^{-\alpha} = O \left( n^{-\alpha} P_n \right)
\]
and from this and (7) we get
\[
\| f - R_n (f) \|_{p,w} = O \left( n^{-\alpha} \right).
\]

Let’s consider the case \( \alpha = 1 \). By Abel transform,
\[
R_n (f) (x) = \frac{1}{P_n} \sum_{m=0}^{n-1} \left\{ P_m (S_m (f) (x) - S_{m+1} (f) (x)) + P_n S_n (f) (x) \right\}
\]
\[
= \frac{1}{P_n} \sum_{m=0}^{n-1} P_m (-A_{m+1} (f) (x)) + S_n (f) (x),
\]
and hence
\[
R_n (f) (x) - S_n (f) (x) = - \frac{1}{P_n} \sum_{m=0}^{n-1} P_m A_{m+1} (f) (x).
\]
Using Abel transform again yields

\[
\sum_{m=0}^{n-1} P_m A_{m+1} (f) (x) = \sum_{m=0}^{n-1} \left( \frac{P_m}{m+1} - \frac{P_{m+1}}{m+2} \right) (k+1) A_{k+1} (f) (x)
\]

\[
+ \frac{P_n}{n+1} \sum_{k=0}^{n-1} (k+1) A_{k+1} (f) (x).
\]

Thus, by considering (5) and (3) we obtain

\[
\left\| \sum_{m=0}^{n-1} P_m A_{m+1} (f) \right\|_{p,w} \leq \sum_{m=0}^{n-1} \left| \frac{P_m}{m+1} - \frac{P_{m+1}}{m+2} \right| \left\| \sum_{k=0}^{m} (k+1) A_{k+1} (f) \right\|_{p,w}
\]

\[
+ \frac{P_n}{n+1} \left\| \sum_{k=0}^{n-1} (k+1) A_{k+1} (f) \right\|_{p,w}
\]

\[
= \sum_{m=0}^{n-1} \left| \frac{P_m}{m+1} - \frac{P_{m+1}}{m+2} \right| (m+2) \| S_{m+1} (f) - \sigma_{m+1} (f) \|_{p,w}
\]

\[
+ P_n \| S_n (f) - \sigma_n (f) \|_{p,w}
\]

\[
= O \left( 1 \right) \sum_{m=0}^{n-1} \left| \frac{P_m}{m+1} - \frac{P_{m+1}}{m+2} \right| + O \left( \frac{P_n}{n} \right).
\]

This gives

\[
\| R_n (f) - S_n (f) \|_{p,w} = \frac{1}{P_n} \left\| \sum_{m=0}^{n-1} P_m A_{m+1} (f) \right\|_{p,w}
\]

\[
= \frac{1}{P_n} O \left( \frac{P_n}{n} \right) = O \left( \frac{1}{n} \right).
\]

Combining this estimate with (4) yields

\[
\| f - R_n (f) \|_{p,w} = O \left( n^{-1} \right).
\]

\[\square\]

**References**


