RIEMANN-LIOUVILLE AND CAPUTO FRACTIONAL APPROXIMATION OF CSISZAR’S $f$–DIVERGENCE

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Abstract. Here are established various tight probabilistic inequalities that give nearly best estimates for the Csiszar’s $f$-divergence. These involve Riemann-Liouville and Caputo fractional derivatives of the directing function $f$. Also a lower bound is given for the Csiszar’s distance. The Csiszar’s discrimination is the most essential and general measure for the comparison between two probability measures. This is continuation of [4].

1. Preliminaries

Throughout this paper we use the following.

1) Let $f$ be a convex function from $(0, +\infty)$ into $\mathbb{R}$ which is strictly convex at 1 with $f(1) = 0$. Let $(X, \mathcal{A}, \lambda)$ be a measure space, where $\lambda$ is a finite or a $\sigma$-finite measure on $(X, \mathcal{A})$. And let $\mu_1, \mu_2$ be two probability measures on $(X, \mathcal{A})$ such that $\mu_1 \ll \lambda, \mu_2 \ll \lambda$ (absolutely continuous), e.g. $\lambda = \mu_1 + \mu_2$. Denote by $p = \frac{d\mu_1}{d\lambda}, q = \frac{d\mu_2}{d\lambda}$ the (densities) Radon-Nikodym derivatives of $\mu_1, \mu_2$ with respect to $\lambda$. Here we assume that $0 < a \leq \frac{p}{q} \leq b$, a.e. on $X$ and $a \leq 1 \leq b$.

The quantity

$$
\Gamma_f (\mu_1, \mu_2) = \int_X q(x) f \left( \frac{p(x)}{q(x)} \right) d\lambda(x),
$$

was introduced by I. Csiszar in 1967, see [7], and is called $f$-divergence of the probability measures $\mu_1$ and $\mu_2$. By Lemma 1.1 of [7], the integral (1) is well-defined and $\Gamma_f (\mu_1, \mu_2) \geq 0$ with equality only when $\mu_1 = \mu_2$. In [7] the author without proof mentions that $\Gamma_f (\mu_1, \mu_2)$ does not depend on the choice of $\lambda$.

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For a proof of the last see [4], Lemma 1.1.

The concept of \( f \)-divergence was introduced first in [6] as a generalization of Kullback’s “information for discrimination” or \( I \)-divergence (generalized entropy) [11], [12] and of Rényi’s “information gain” (\( I \)-divergence of order \( \alpha \)) [13]. In fact the \( I \)-divergence of order 1 equals

\[
\Gamma_{u \log_2 u} (\mu_1, \mu_2) .
\]

The choice \( f(u) = (u - 1)^2 \) produces again a known measure of difference of distributions that is called \( \kappa_2 \)-divergence, of course the total variation distance \( |\mu_1 - \mu_2| = \int_X |p(x) - q(x)| \, d\lambda(x) \) equals \( \Gamma_{|u-1|} (\mu_1, \mu_2) \).

Here by assuming \( f(1) = 0 \) we can consider \( \Gamma_f (\mu_1, \mu_2) \) as a measure of the difference between the probability measures \( \mu_1, \mu_2 \). But since \( f \) is convex and strictly convex at 1 (see Lemma 2, [4]) so is

\[
f^*(u) = uf \left( \frac{1}{u} \right)
\]

and as in [7] we get

\[
\Gamma_f (\mu_2, \mu_1) = \Gamma_{f^*} (\mu_1, \mu_2) .
\]

In Information Theory and Statistics many other concrete divergences are used which are special cases of the above general Csiszar \( f \)-divergence, e.g. Hellinger distance \( D_H \), \( \alpha \)-divergence \( D_\alpha \), Bhattacharyya distance \( D_B \), Harmonic distance \( D_{H_\alpha} \), Jeffrey’s distance \( D_J \), triangular discrimination \( D_\Delta \), for all these see, e.g. [5], [9]. The problem of finding and estimating the proper distance (or difference or discrimination) of two probability distributions is one of the major ones in Probability Theory.

The above \( f \)-divergence measures in their various forms have been also applied to Anthropology, Genetics, Finance, Economics, Political Science, Biology, Approximation of Probability distributions, Signal Processing and Pattern Recognition. A great inspiration for this article has been the very important monograph on the topic by S. Dragomir [9].

II) Here we follow [8].

We start with

**Definition 1.** Let \( \nu \geq 0 \), the operator \( J^\nu_a \), defined on \( L_1 (a, b) \) by

\[
J^\nu_a f (x) := \frac{1}{\Gamma (\nu)} \int_a^x (x - t)^{\nu - 1} f(t) \, dt
\]

for \( a \leq x \leq b \), is called the Riemann-Liouville fractional integral operator of order \( \nu \).

For \( \nu = 0 \), we set \( J^0_a := I \), the identity operator. Here \( \Gamma \) stands for the gamma function.
Let $\alpha > 0$, $f \in L_1(a,b)$, $a,b \in \mathbb{R}$, see [8]. Here $\lfloor \cdot \rfloor$ stands for the integral part of the number.

We define the generalized Riemann-Liouville fractional derivative of $f$ of order $\alpha$ by

$$D^\alpha_a f(s) := \frac{1}{\Gamma(m-\alpha)} \left( \frac{d}{ds} \right)^m \int_a^s (s-t)^{m-\alpha-1} f(t) \, dt,$$

where $m := \lfloor \alpha \rfloor + 1$, $s \in [a,b]$, see also [1], Remark 46 there.

In addition, we set

$$D^0_a f := f,$$

$$J^{-\alpha}_a f := D^{-\alpha}_a f,$$  
if $\alpha > 0$,

$$D_{-\alpha}^a f := J^\alpha_a f,$$  
if $0 < \alpha \leq 1$,

$$D^n_a f = f^{(n)},$$  
for $n \in \mathbb{N}$.  \hfill (5)

We need

**Definition 2.** ([3]) We say that $f \in L_1(a,b)$ has an $L_\infty$ fractional derivative $D^\alpha_a f$ ($\alpha > 0$) in $[a,b]$, $a,b \in \mathbb{R}$, iff $D^\alpha_a f \in C([a,b])$, $k = 1, \ldots, m := \lfloor \alpha \rfloor + 1$, and $D^{-\alpha}_a f \in AC([a,b])$ (absolutely continuous functions) and $D^\alpha_a f \in L_\infty(a,b)$.

**Lemma 3.** ([3]) Let $\beta > \alpha \geq 0$, $f \in L_1(a,b)$, $a,b \in \mathbb{R}$, have $L_\infty$ fractional derivative $D^\alpha_a f$ in $[a,b]$, let $D^{\alpha-k}_a f(a) = 0$ for $k = 1, \ldots, \lfloor \beta \rfloor + 1$. Then

$$D^\alpha_a f(s) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^s (s-t)^{\beta-\alpha-1} D^\beta_a f(t) \, dt, \forall s \in [a,b].$$  \hfill (6)

Here $D^\alpha_a f \in AC([a,b])$ for $\beta - \alpha \geq 1$, and $D^\alpha_a f \in C([a,b])$ for $\beta - \alpha \in (0,1)$.

Here $AC^n([a,b])$ is the space of functions with absolutely continuous $(n-1)$-st derivative.

We need to mention

**Definition 4.** ([8]) Let $\nu \geq 0$, $n := \lceil \nu \rceil$, $\lceil \cdot \rceil$ is ceiling of the number, $f \in AC^n([a,b])$. We call Caputo fractional derivative

$$D^\nu_{*a} f(x) := \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t) \, dt, \forall x \in [a,b].$$  \hfill (7)

The above function $D^\nu_{*a} f(x)$ exists almost everywhere for $x \in [a,b]$.

We need

**Proposition 5.** ([8]) Let $\nu \geq 0$, $n := \lceil \nu \rceil$, $f \in AC^n([a,b])$. Then $D^\nu_{*a} f$ exists iff the generalized Riemann-Liouville fractional derivative $D^\alpha_a f$ exists.
Proposition 6. ([8]) Let \( \nu \geq 0 \), \( n := \lceil \nu \rceil \). Assume that \( f \) is such that both \( D_{\nu a}^\nu f \) and \( D_{a}^\nu f \) exist. Suppose that \( f^{(k)}(a) = 0 \) for \( k = 0, 1, \ldots, n-1 \). Then
\[
D_{\nu a}^\nu f = D_{a}^\nu f.
\]

In conclusion

Corollary 7. ([2]) Let \( \nu \geq 0 \), \( n := \lceil \nu \rceil \), \( f \in AC^n ([a, b]) \), \( D_{\nu a}^\nu f \) exists or \( D_{a}^\nu f \) exists, and \( f^{(k)}(a) = 0 \), \( k = 0, 1, \ldots, n-1 \). Then
\[
D_{\nu a}^\nu f = D_{a}^\nu f.
\]

We need

Theorem 8. ([2]) Let \( \nu \geq 0 \), \( n := \lceil \nu \rceil \), \( f \in AC^n ([a, b]) \) and \( f^{(k)}(a) = 0 \), \( k = 0, 1, \ldots, n-1 \). Then
\[
f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} D_{\nu a}^\nu f(t) \, dt.
\]

We also need

Theorem 9. ([2]) Let \( \nu \geq \gamma + 1 \), \( \gamma \geq 0 \). Call \( n := \lceil \nu \rceil \). Assume \( f \in AC^n ([a, b]) \) such that \( f^{(k)}(a) = 0 \), \( k = 0, 1, \ldots, n-1 \), and \( D_{\nu a}^\nu f \in L_\infty (a, b) \). Then \( D_{\nu a}^\nu f \in AC ([a, b]) \), and
\[
D_{\nu a}^\nu f(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_a^x (x-t)^{\nu-\gamma-1} D_{\nu a}^\nu f(t) \, dt, \quad \forall x \in [a, b].
\]

Theorem 10. ([2]) Let \( \nu \geq \gamma + 1 \), \( \gamma \geq 0 \), \( n := \lceil \nu \rceil \). Let \( f \in AC^n ([a, b]) \) such that \( f^{(k)}(a) = 0 \), \( k = 0, 1, \ldots, n-1 \). Assume \( \exists D_{a}^\nu f(x) \in \mathbb{R}, \forall x \in [a, b] \), and \( D_{\nu a}^\nu f \in L_\infty (a, b) \). Then \( D_{a}^\nu f \in AC ([a, b]) \), and
\[
D_{a}^\nu f(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_a^x (x-t)^{\nu-\gamma-1} D_{a}^\nu f(t) \, dt, \quad \forall x \in [a, b].
\]

2. Results

Here \( f \) and the whole setting is as in 1. Preliminaries (I). We present first results regarding the Riemann-Liouville fractional derivative.

Theorem 11. Let \( \beta > 0 \), \( f \in L_1 (a, b) \), have \( L_\infty \) fractional derivative \( D_{a}^\beta f \) in \([a, b]\), let \( D_{a}^{\beta-k} f (a) = 0 \) for \( k = 1, \ldots, \lceil \beta \rceil + 1 \). Also assume \( 0 < a \leq \frac{p(x)}{q(x)} \leq b \), a.e. on \( X, a < b \). Then
\[
\Gamma_f (\mu_1, \mu_2) \leq \frac{\| D_{a}^\beta f \|_{L_\infty([a, b])}}{\Gamma(\beta + 1)} \int_X q(x)^{1-\beta} (p(x) - aq(x))^{\beta} d\lambda(x).
\]
Proof. By (6), \( \alpha = 0 \), we get

\[
f (s) = \frac{1}{\Gamma (\beta)} \int_a^s (s - t)^{\beta - 1} D_\alpha^\beta f (t) \, dt, \quad \text{all } a \leq s \leq b.
\]  

(14)

Then

\[
|f (s)| \leq \frac{1}{\Gamma (\beta)} \int_a^s (s - t)^{\beta - 1} \left| D_\alpha^\beta f (t) \right| \, dt
\]

\[
\leq \left\| D_\alpha^\beta f \right\|_{[a,b]} \int_a^s (s - t)^{\beta - 1} \, dt
\]

\[
= \frac{\left\| D_\alpha^\beta f \right\|_{[a,b]} (s - a)^\beta}{\Gamma (\beta)}
\]

(15)

I.e. we have that

\[
|f (s)| \leq \frac{\left\| D_\alpha^\beta f \right\|_{[a,b]} (s - a)^\beta}{\Gamma (\beta + 1)}
\]

(16)

Consequently we obtain

\[
\Gamma_f (\mu_1, \mu_2) = \int_X q(x) f \left( \frac{p(x)}{q(x)} \right) d\lambda (x)
\]

\[
\leq \left\| D_\alpha^\beta f \right\|_{[a,b]} \int_X q(x) \left( \frac{p(x)}{q(x)} - a \right)^\beta d\lambda (x)
\]

\[
= \frac{\left\| D_\alpha^\beta f \right\|_{[a,b]} (s - a)^\beta}{\Gamma (\beta + 1)}
\]

(17)

proving the claim. \( \square \)

Next we give an \( L_\delta \) result.

**Theorem 12.** Same assumptions as in Theorem 11. Let \( \gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1 \) and \( \gamma (\beta - 1) + 1 > 0 \). Then

\[
\Gamma_f (\mu_1, \mu_2) \leq \frac{\left\| D_\alpha^\beta f \right\|_{[a,b]}}{\Gamma (\beta) (\gamma (\beta - 1) + 1)^{1/\gamma}} \int_X q(x)^{2 - \beta - \frac{1}{\gamma}} (p(x) - aq(x))^{\beta - 1 + \frac{1}{\gamma}} d\lambda (x).
\]  

(18)
Proof. By (6), $\alpha = 0$, we get again
\[
f(s) = \frac{1}{\Gamma(\beta)} \int_a^s (s-t)^{\beta-1} D^\beta_a f(t) \, dt, \text{ all } a \leq s \leq b.
\] (19)
Hence
\[
|f(s)| \leq \frac{1}{\Gamma(\beta)} \int_a^s (s-t)^{\beta-1} \left| D^\beta_a f(t) \right| \, dt
\]
\[
\leq \frac{1}{\Gamma(\beta)} \left( \int_a^s (s-t)^{\gamma(\beta-1)} \, dt \right)^{1/\gamma} \left( \int_a^s \left| D^\beta_a f(t) \right|^\delta \, dt \right)^{1/\delta}
\]
\[
\leq \frac{\|D^\beta_a f\|_{\delta, [a,b]}}{\Gamma(\beta)} \frac{(s-a)^{\beta-1+\frac{1}{\gamma}}}{(\gamma (\beta - 1) + 1)^{1/\gamma}}, \text{ all } a \leq s \leq b.
\] (20)
That is
\[
|f(s)| \leq \frac{\|D^\beta_a f\|_{\delta, [a,b]}}{\Gamma(\beta)} \frac{(s-a)^{\beta-1+\frac{1}{\gamma}}}{(\gamma (\beta - 1) + 1)^{1/\gamma}}, \text{ all } a \leq s \leq b.
\] (21)
Consequently we obtain
\[
\Gamma_f(\mu_1, \mu_2) \leq \int_X q \left| f \left( \frac{p}{q} \right) \right| \, d\lambda
\]
\[
\leq \frac{\|D^\beta_a f\|_{\delta, [a,b]}}{\Gamma(\beta)} \frac{(s-a)^{\beta-1+\frac{1}{\gamma}}}{(\gamma (\beta - 1) + 1)^{1/\gamma}} \int_X q \left( \frac{p}{q} - a \right)^{\beta-1+\frac{1}{\gamma}} \, d\lambda
\]
\[
= \frac{\|D^\beta_a f\|_{\delta, [a,b]}}{\Gamma(\beta)} \frac{(s-a)^{\beta-1+\frac{1}{\gamma}}}{(\gamma (\beta - 1) + 1)^{1/\gamma}} \int_X q^{2-\beta+\frac{1}{\gamma}} (p-aq)^{\beta-1+\frac{1}{\gamma}} \, d\lambda,
\] (22)
proving the claim. \hfill \Box

An $L_1$ estimate follows.

**Theorem 13.** Same assumptions as in Theorem 11. Let $\beta \geq 1$. Then
\[
\Gamma_f(\mu_1, \mu_2) \leq \frac{\|D^\beta_a f\|_{1, [a,b]}}{\Gamma(\beta)} \left( \int_X q(x)^{2-\beta} (p(x) - aq(x))^{\beta-1} \, d\lambda(x) \right)
\] (23)

Proof. By (19) we have
\[
|f(s)| \leq \frac{1}{\Gamma(\beta)} \int_a^s (s-t)^{\beta-1} \left| D^\beta_a f(t) \right| \, dt
\]
\[
\leq \frac{(s-a)^{\beta-1}}{\Gamma(\beta)} \int_a^b \left| D^\beta_a f(t) \right| \, dt = \frac{(s-a)^{\beta-1}}{\Gamma(\beta)} \|D^\beta_a f\|_{1, [a,b]}.
\] (24)
I.e.
\[ |f(s)| \leq \frac{(s-a)^{\beta-1}}{\Gamma(\beta)} \left\| D_\alpha^\beta f \right\|_{1,[a,b]}, \tag{25} \]
for all \( s \) in \([a, b]\). Therefore
\begin{align*}
\Gamma_f(\mu_1, \mu_2) &\leq \int_X q \left| f \left( \frac{p}{q} \right) \right| d\lambda \\
&\leq \frac{\left\| D_\alpha^\beta f \right\|_{1,[a,b]}}{\Gamma(\beta)} \int_X q \left( \frac{p}{q} - a \right)^{\beta-1} d\lambda \\
&= \frac{\left\| D_\alpha^\beta f \right\|_{1,[a,b]}}{\Gamma(\beta)} \left( \int_X q^{2-\beta} (p - a q)^{\beta-1} d\lambda \right), \tag{26}
\end{align*}
proving the claim. □

We continue with results regarding the Caputo fractional derivative.

**Theorem 14.** Let \( \nu > 0 \), \( n := \lceil \nu \rceil \), \( f \in AC^n([a, b]) \) and \( f^{(k)}(a) = 0 \), \( k = 0, 1, \ldots, n-1 \). Assume \( D_\alpha^\nu f \in L_\infty(a, b) \), \( 0 < a \leq \frac{p(x)}{q(x)} \leq b \), a.e. on \( X \), \( a < b \). Then
\[ \Gamma_f(\mu_1, \mu_2) \leq \left\| D_\alpha^\nu f \right\|_{\infty,[a,b]} \left( \int_X q^{2-\nu} (p(x) - a q(x))^\nu d\lambda(x) \right). \tag{27} \]

**Proof.** Similar to Theorem 11, using Theorem 8. □

Next we give an \( L_\delta \) result.

**Theorem 15.** Assume all as in Theorem 14. Let \( \gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1 \) and \( \gamma (\nu - 1) + 1 > 0 \). Then
\[ \Gamma_f(\mu_1, \mu_2) \leq \frac{\left\| D_\alpha^\nu f \right\|_{\delta,[a,b]}}{\Gamma(\nu) (\gamma (\nu - 1) + 1)^{1/\gamma}} \int_X q(x)^{2-\nu - \frac{1}{\gamma}} (p(x) - a q(x))^{\nu - 1 + \frac{1}{\gamma}} d\lambda(x). \tag{28} \]

**Proof.** Similar to Theorem 12, using Theorem 8. □

It follows an \( L_1 \) estimate.

**Theorem 16.** Assume all as in Theorem 14. Let \( \nu \geq 1 \). Then
\[ \Gamma_f(\mu_1, \mu_2) \leq \frac{\left\| D_\alpha^\nu f \right\|_{1,[a,b]}}{\Gamma(\nu)} \left( \int_X (q(x))^{2-\nu} (p(x) - a q(x))^{\nu - 1} d\lambda(x) \right). \tag{29} \]

**Proof.** Similar to Theorem 13, using Theorem 8. □

Regarding again the Riemann-Liouville fractional derivative we need:
Corollary 17. Let $\nu \geq 0$, $n := \lceil \nu \rceil$, $f \in AC^n ([a, b])$, $\exists D^\nu_a f (x) \in \mathbb{R}$, $\forall x \in [a, b]$, $f^{(k)} (a) = 0$, $k = 0, 1, \ldots, n - 1$. Then
\[
f(x) = \frac{1}{\Gamma (\nu + 1)} \int_a^x (x-t)^{\nu-1} D^\nu_a f (t) \, dt. \tag{30}\]

Proof. By Corollary 7 and Theorem 8. \hfill \Box

We continue with results again regarding the Riemann-Liouville fractional derivative.

Theorem 18. Let $\nu > 0$, $n := \lceil \nu \rceil$, $f \in AC^n ([a, b])$, $\exists D^\nu_a f (x) \in \mathbb{R}$, $\forall x \in [a, b]$, $f^{(k)} (a) = 0$, $k = 0, 1, \ldots, n - 1$. Assume $D^\nu_a f \in L^\infty (a, b)$, $0 < a \leq p(x) \leq b$, a.e. on $X$, $a < b$. Then
\[
\Gamma (\mu_1, \mu_2) \leq \frac{\| D^\nu_a f \| \infty_{[a,b]}}{\Gamma (\nu + 1)} \int_X q(x)^{1-\nu} (p(x) - q(x))^{\nu} d\lambda (x). \tag{31}\]

Proof. Similar to Theorem 11, using Corollary 17. \hfill \Box

Next we give the corresponding $L_\delta$ result.

Theorem 19. Assume all as in Theorem 18. Let $\gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$ and $\gamma (\nu - 1) + 1 > 0$. Then
\[
\Gamma (\mu_1, \mu_2) \leq \frac{\| D^\nu_a f \| L^\delta_{[a,b]}}{\Gamma (\nu + 1)} (\int_X q(x)^{2-\nu} (p(x) - q(x))^{\nu-1} d\lambda (x)). \tag{32}\]

Proof. Similar to Theorem 12, using Corollary 17. \hfill \Box

It follows the $L_1$ estimate.

Theorem 20. Assume all as in Theorem 18. Let $\nu \geq 1$. Then
\[
\Gamma (\mu_1, \mu_2) \leq \frac{\| D^\nu_a f \| L^1_{[a,b]}}{\Gamma (\nu)} (\int_X (q(x))^{2-\nu} (p(x) - q(x))^{\nu-1} d\lambda (x)). \tag{33}\]

Proof. Similar to Theorem 13, using Corollary 17. \hfill \Box

We need

Theorem 21. (Taylor expansion for Caputo derivatives, [8], p. 40) Assume $\nu \geq 0$, $n = \lceil \nu \rceil$, and $f \in AC^n ([a, b])$. Then
\[
f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)} (a)}{k!} (x-a)^k + \frac{1}{\Gamma (\nu)} \int_a^x (x-t)^{\nu-1} D^\nu_s f (t) \, dt, \forall x \in [a, b]. \tag{34}\]
We make

**Remark 22.** Let $\nu > 0$, $n = \lceil \nu \rceil$, and $f \in AC^n ([a, b])$.
If $D^\nu_{sa} f \geq 0$ over $[a, b]$, then

$$\int_a^x (x - t)^{\nu - 1} D^\nu_{sa} f (t) \, dt \geq 0 \quad \text{on } [a, b].$$

By (34) then we obtain

$$f (x) \geq (\leq) \sum_{k=0}^{n-1} \frac{f^{(k)} (a)}{k!} (x - a)^k,$$

(35)

$\forall x \in [a, b]$. Hence

$$q f \left( \frac{p}{q} \right) \geq (\leq) \sum_{k=0}^{n-1} \frac{f^{(k)} (a)}{k!} q \left( \frac{p}{q} - a \right)^k,$$

(36)

Consequently we get

$$\Gamma f (\mu_1, \mu_2) \geq (\leq) \sum_{k=0}^{n-1} \frac{f^{(k)} (a)}{k!} \int_X q^{1-k} (p - aq)^k \, d\lambda.$$  

(37)

We have established

**Theorem 23.** Let $\nu > 0$, $n = \lceil \nu \rceil$, and $f \in AC^n ([a, b])$. If $D^\nu_{sa} f \geq 0$ on $[a, b]$, then

$$\Gamma f (\mu_1, \mu_2) \geq (\leq) \sum_{k=0}^{n-1} \frac{f^{(k)} (a)}{k!} \left( \int_X (q (x))^{1-k} (p (x) - aq (x))^k \, d\lambda (x) \right).$$

(38)

We finish with

**Remark 24.** Using Lemma 3, Theorem 9 and Theorem 10 and in their settings, for $g$ any of $D^n_{sa} f$, $D^\nu_{sa} f$, $D^\nu_{sa} f$, which fulfill the conditions and assumptions of 1. Preliminaries (I), we can find as above similar estimates for $\Gamma g (\mu_1, \mu_2)$.

**References**


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