SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.19.02.06
Vol.19 (32), No.2 (2023), 207–214

GAUSSIAN QUATERNION INVOLVING LEONARDO NUMBERS

HASAN GÖKBAŞ

ABSTRACT. In this study, using the Leonardo numbers, we define a new type of quaternion that is called a Leonard Gaussian quaternion. We also give a negative-Leonardo Gaussian quaternion. These numbers are introduced from the set of complex numbers and quaternions. Moreover, we obtain the Binet's formula, generating function formula, d'Ocagne's identity, Catalan's identity, Cassini's identity, Honsberger's identity, like-Vajda's identity and some formulas for these new types of numbers. Morever, we give the matrix representation of the Leonardo Gaussian quaternion.

1. INTRODUCTION AND PRELIMINARIES

The Fibonacci sequence has delighted mathematicians and scientists alike for centuries with its beauty and its propensity to pop up in quite unexpected places. Leonardo de Pisa did not even guess that the number sequences would be so related to the rabbit problem. However, the Fibonacci numbers are found in the Pascal's triangle, Pythagorean triples, computer algorithms, graph theory and many other areas of mathematics. They also ocur in a variety of other fields such as physics, finance, architecture, computer sciences, color image processing, geostatics, music and art. There have been many studies in literature about this special number sequence because of its numerous applications. There are many generalizations on this sequence some of which can be seen in [1], [2], [3], [4], [6], [7], [8], [9], [10], [11], [13], [14], [15], [16], [18], [19], [20], [21], [22], [23], [24], [25], [26].

Quaternions were investigated by Hamilton [12]. A quaternion is a hypercomplex number and is defined by the following equation:

$$q = q_0 + iq_1 + jq_2 + kq_3$$

where q_0, q_1, q_2 and q_3 are real numbers. i, j and k are the standard ortohonormal basis in $\mathbb{R}^3$.

The q_0, q_1, q_2 and q_3 are called the components of the quaternion q. The quaternion multiplication is defined by the following rules:

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik.$$

2010 *Mathematics Subject Classification.* 05A15, 11B39, 11B37, 11R52.

Key words and phrases. Gaussian quaternion, quaternion, Fibonacci numbers, Leonardo numbers.

Leonardo's recurring non-homogeneous sequence, which we shall denote by Le_n is a linear recurrent sequence, having its characteristic recurrence formula defined as:

$$Le_n = Le_{n-1} + Le_{n-2} + 1 \quad or \quad Le_n = 2Le_{n-1} - Le_{n-3}, n \geq 2$$

with Le_n the nth term of the Leonardo sequence and the initial terms indicated by $Le_0 = Le_1 = 1$. Leonardo's numbers also have a relation to the emblematic sequence of Fibonacci, thus there exists a recurrence relation with characteristics of these two sequences. Remembering the formula for obtaining Fibonacci numbers and Lucas numbers, respectively.$F_n = F_{n-1} + F_{n-2}$, with the initial values defined by $F_0 = F_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$, with the initial values defined by $L_0 = 2, L_1 = 1$, respectively. The Leonardo numbers Binet's formula, $Le_n = 2\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right) - 1$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. The relation between Leonardo and Fibonacci numbers is expressed in the following proposition $Le_n = 2F_{n+1} - 1$ [5].

Jordan [17], considered two of the complex Fibonacci sequences and extended some relationships which are known about the common Fibonacci sequences. The author gave many identities related to them $GF_n = F_n + iF_{n-1}$.

2. THE LEONARDO GAUSSIAN QUATERNION

In the following sections, the Leonardo Gaussian quaternion will be defined. In this work, a variety of algebraic properties of both the bicomplex quaternions and the Leonardo Gaussian quaternions are presented in a unified manner. Some identities will be given for the Leonardo Gaussian quaternion such as Binet's formula, generating function formula, d'Ocagne's identity, Catalan's identity, Cassini's identity, Honsberger's identity, like-Vajda's identity and some formulas. The Leonardo Gaussian quaternion' properties will also be obtained using matrix representation.

Definition 2.1. *For $n \geq 3$, the Leonardo Gaussian quaternion $LGSe_n$ is defined by the recurrence relation*

$$LGSe_n = Le_n + iLe_{n-1} + jLe_{n-2} + kLe_{n-3}$$

where Le_n is the nth Leonardo number.

The Leonardo Gaussian quaternion starting from $n = 0$ can be written as

$$LGSe_0 = 1 - i + j - 3k, LGSe_1 = 1 + i - j + k, LGSe_2 = 3 + i + j - k,$$
$$LGSe_3 = 5 + 3i + j + k, ...$$

and

$$LGSe_n = LGSe_{n-1} + LGSe_{n-2} + M$$

or

$$LGSe_n = 2LGSe_{n-1} - LGSe_{n-3}$$

where $M = (1 + i + j + k)$ is a recurrence relationship in the Leonardo Gaussian quaternion.

Definition 2.2. *For $n \geq 3$, the negative-Leonardo Gaussian quaternion $LGSe_{-n}$ is defined by the recurrence relation*

$$LGSe_{-n} = (-1)^n \left[Le_{n-2} - iLe_{n-1} + jLe_n - kLe_{n+1}\right] - M$$

where Le_n is the nth Leonardo number and M is $(1+i+j+k)$.

Theorem 2.1. *(Generating Function Formula) Let $LGSe_n$ be the Leonardo Gaussian quaternion. The generating function formula for this number is as follows*

$$H(t) = \frac{(1-i+j-3k)+(-1+3i-3j+7k)t+(1-i+3j-3k)t^2}{(1-2t+t^3)}.$$

Proof. Let $H(t)$ be the generating function formula for the Leonardo Gaussian quaternion as $H(t) = \sum_{n=0}^{\infty} LGSe_n t^n$. Using $H(t)$, $2tH(t)$ and $t^3H(t)$, we get the following equations $2tH(t) = \sum_{n=0}^{\infty} 2LGSe_n t^{n+1}$, $t^3H(t) = \sum_{n=0}^{\infty} LGSe_n t^{n+3}$. After the needed calculations, the generating function formula for the Leonardo Gaussian quaternion is obtained as

$$\begin{aligned}(1-2t+t^3)H(t) &= LGSe_0+(LGSe_1-2LGSe_0)t+(LGSe_2-2LGSe_1)t^2\\ &\quad+\sum_{n=2}^{\infty}(LGSe_{n+1}-2LGSe_n+LGSe_{n-2})\,t^{n+1}.\\ H(t) &= \frac{LGSe_0+(LGSe_1-2LGSe_0)t+(LGSe_2-2LGSe_1)t^2}{(1-2t+t^3)}\\ H(t) &= \frac{(1-i+j-3k)+(-1+3i-3j+7k)t+(1-i+3j-3k)t^2}{(1-2t+t^3)}.\end{aligned}$$ □

Theorem 2.2. *(Binet's Formula) The Binet's formula for the Leonardo Gaussian quaternion $LGSe_n$ is*

$$LGSe_n = \frac{2\alpha^{n-2}\bar{\alpha} - 2\beta^{n-2}\bar{\beta}}{\alpha - \beta} - M$$

where $\bar{\alpha} = \alpha^3 + i\alpha^2 + j\alpha + k, \bar{\beta} = \beta^3 + i\beta^2 + j\beta + k$ and $M = (1+i+j+k)$.

Proof. By using the definition of the Leonardo Gaussian quaternion and the Leonardo numbers of Binet's formula, we obtain the desired result. □

We can give the next theorem that will form the basis of the following equations.

Theorem 2.3. *For nonnegative integer numbers n and m, with $m \geq n$, $LGSe_n$is the Leonardo Gaussian quaternion. We have*

$$\begin{aligned}LGSe_m LGSe_{n+r} - LGSe_{m+r} LGSe_n &= \frac{4\bar{\alpha}\bar{\beta}\left[L_{m-n+r} - (-1)^r L_{m-n-r}\right]}{(\alpha-\beta)^2} +\\ &\quad + M\frac{2\bar{\alpha}(\alpha^r - 1)\left[\alpha^{m-2} - \alpha^{n-2}\right] + 2\bar{\beta}(\beta^r - 1)\left[\beta^{n-2} - \beta^{m-2}\right]}{(\alpha-\beta)}\end{aligned}$$

where L_n is the nth Lucas number and M is $(1+i+j+k)$.

Proof.

$$
\begin{aligned}
&LGSe_m LGSe_{n+r} - LGSe_{m+r} LGSe_n = \\
&\qquad = \left(\frac{2\bar{\alpha}\alpha^{m-2} - 2\bar{\beta}\beta^{m-2}}{\alpha-\beta} - M\right)\left(\frac{2\bar{\alpha}\alpha^{n+r-2} - 2\bar{\beta}\beta^{n+r-2}}{\alpha-\beta} - M\right) - \\
&\qquad - \left(\frac{2\bar{\alpha}\alpha^{m+r-2} - 2\bar{\beta}\beta^{m+r-2}}{\alpha-\beta} - M\right)\left(\frac{2\bar{\alpha}\alpha^{n-2} - 2\bar{\beta}\beta^{n-2}}{\alpha-\beta} - M\right) \\
&\qquad = \frac{4\bar{\alpha}\bar{\beta}(-1)^n\left[L_{m-n+r} - (-1)^r L_{m-n-r}\right]}{(\alpha-\beta)^2} + \\
&\qquad + M\frac{2\bar{\alpha}\left(\alpha^r - 1\right)\left[\alpha^{m-2} - \alpha^{n-2}\right] + 2\bar{\beta}\left(\beta^r - 1\right)\left[\beta^{n-2} - \beta^{m-2}\right]}{(\alpha-\beta)}
\end{aligned}
$$

where L_n is the nth Lucas number and M is $(1+i+j+k)$. □

Theorem 2.4. *(Catalan's Identity) For nonnegative integer numbers n and r, with $n \geq r$, $LGSe_n$ is the Leonardo Gaussian quaternion. We have*

$$
\begin{aligned}
LGSe_{n-r} LGSe_{n+r} - LGSe_n^2 &= \frac{4\bar{\alpha}\bar{\beta}\left[2(-1)^r - L_{2r}\right]}{(\alpha-\beta)^2(-1)^r} + \\
&+ M\frac{\bar{\alpha}\alpha^{n-r-2}\left[4\alpha^r - 2\left(1+\alpha^{2r}\right)\right] + \bar{\beta}\beta^{n-r-2}\left[2\left(1+\beta^{2r}\right) - 4\beta^r\right]}{(\alpha-\beta)}
\end{aligned}
$$

where L_n is the nth Lucas number and M is $(1+i+j+k)$.

Proof. Since Catalan's identity is a special case of the Theorem 2.3, the proof is seen by taking $m = n-r$. □

Theorem 2.5. *(Cassini's Identity) For $n \geq 1$, $LGSe_n$ is the Leonardo Gaussian quaternion. We have*

$$
LGSe_{n-1} LGSe_{n+1} - LGSe_n^2 = 4\bar{\alpha}\bar{\beta} + M\frac{\bar{\alpha}\alpha^{n-3}\left[\sqrt{5}-3\right] + \bar{\beta}\beta^{n-3}\left[\sqrt{5}+3\right]}{\alpha-\beta}
$$

where $M = (1+i+j+k)$.

Proof. Since Cassini's identity is a special case of Catalan's identity, the proof is seen by taking $r = 1$. □

Theorem 2.6. *(d'Ocagne's Identity) For nonnegative integer numbers n and m, with $m \geq n$, $LGSe_n$is the Leonardo Gaussian quaternion. We have*

$$
LGSe_m LGSe_{n+1} - LGSe_{m+1} LGSe_n = \frac{4\bar{\alpha}\bar{\beta}(-1)^n\left[L_{m-n+1} + L_{m-n-1}\right]}{(\alpha-\beta)^2} -
$$

$$-M\frac{2\bar{\alpha}\beta\left[\alpha^{m-2}-\alpha^{n-2}\right]+2\bar{\beta}\alpha\left[\beta^{n-2}-\beta^{m-2}\right]}{(\alpha-\beta)}$$

where L_n is the nth Lucas number and M is $(1+i+j+k)$.

Proof. Since d'Ocagne's identity is a special case of the Theorem 2.3, the proof is seen by taking $r=1$. □

Theorem 2.7. *(Honsberger's Identity) For nonnegative integer numbers n and m, with $m\geq n$, $LGSe_n$is the Leonardo Gaussian quaternion. We have*

$$LGSe_mLGSe_n+LGSe_{m+1}LGSe_{n+1}=-\frac{4\bar{\alpha}^2\alpha^{n+m-3}+4\bar{\beta}^2\beta^{n+m-3}+16(-1)^n\bar{\alpha}\bar{\beta}\beta^{m-n}}{(\alpha-\beta)^2}$$
$$+M\frac{2\bar{\beta}\alpha\left[\beta^{n-2}+\beta^{m-2}\right]-2\bar{\alpha}\beta\left[\alpha^{m-2}+\alpha^{n-2}\right]}{(\alpha-\beta)}$$

where $M=(1+i+j+k)$.

Proof.

$$\begin{aligned}
&LGSe_mLGSe_n+LGSe_{m+1}LGSe_{n+1}=\\
&=\left(\frac{2\bar{\alpha}\alpha^{m-2}-2\bar{\beta}\beta^{m-2}}{\alpha-\beta}-M\right)\left(\frac{2\bar{\alpha}\alpha^{n-2}-2\bar{\beta}\beta^{n-2}}{\alpha-\beta}-M\right)+\\
&+\left(\frac{2\bar{\alpha}\alpha^{m-1}-2\bar{\beta}\beta^{m-1}}{\alpha-\beta}-M\right)\left(\frac{2\bar{\alpha}\alpha^{n-1}-2\bar{\beta}\beta^{n-1}}{\alpha-\beta}-M\right)\\
&=-\frac{4\bar{\alpha}^2\alpha^{n+m-3}+4\bar{\beta}^2\beta^{n+m-3}+16(-1)^n\bar{\alpha}\bar{\beta}\beta^{m-n}}{(\alpha-\beta)^2}+\\
&+M\frac{2\bar{\beta}\alpha\left[\beta^{n-2}+\beta^{m-2}\right]-2\bar{\alpha}\beta\left[\alpha^{m-2}+\alpha^{n-2}\right]}{(\alpha-\beta)}
\end{aligned}$$

where $M=(1+i+j+k)$. □

Theorem 2.8. *(Like-Vajda's Identity) For nonnegative integer numbers n and m, with $m\geq n$, $LGSe_n$is the Leonardo Gaussian quaternion. We have*

$$LGSe_{n+1}^2-LGSe_nLGSe_{n+2}=4(-1)^n\bar{\alpha}\bar{\beta}+M\frac{2\bar{\alpha}\alpha^{n-2}(\alpha-1)^2-2\bar{\beta}\beta^{n-2}(\beta-1)^2}{(\alpha-\beta)}$$

where $M=(1+i+j+k)$.

Proof.

$$LGSe_{n+1}^2-LGSe_nLGSe_{n+2}=\left(\frac{2\bar{\alpha}\alpha^{n-1}-2\bar{\beta}\beta^{n-1}}{\alpha-\beta}-M\right)\left(\frac{2\bar{\alpha}\alpha^{n-1}-2\bar{\beta}\beta^{n-1}}{\alpha-\beta}-M\right)-$$

$$
\begin{aligned}
&-\left(\frac{2\bar{\alpha}\alpha^{n-2}-2\bar{\beta}\beta^{n-2}}{\alpha-\beta}-M\right)\left(\frac{2\bar{\alpha}\alpha^{n}-2\bar{\beta}\beta^{n}}{\alpha-\beta}-M\right)\\
&=4(-1)^{n}\bar{\alpha}\bar{\beta}+M\frac{2\bar{\alpha}\alpha^{n-2}(\alpha-1)^{2}-2\bar{\beta}\beta^{n-2}(\beta-1)^{2}}{(\alpha-\beta)}
\end{aligned}
$$

where $M=(1+i+j+k)$. □

Lemma 2.1. *Let F_n be the Fibonacci number. In this case*

$$
\begin{aligned}
\sum_{i=1}^{n}F_i&=F_{n+2}-1\\
\sum_{i=0}^{n}F_{2i+1}&=F_{2n+2}\\
\sum_{i=1}^{n}F_{2i}&=F_{2n+1}-1
\end{aligned}
$$

Proof. The proofs are seen by induction on n. □

In the next theorem, we can give the sum of the finite, finite odd and finite even terms of the Leonardo Gaussian quaternion numbers.

Theorem 2.9. *Let $LGSe_n$ be the Leonardo Gaussian quaternion. In this case*

$$
\begin{aligned}
\sum_{m=0}^{n}LGSe_m&=2F_{n+3}+2iF_{n+2}+2j(F_{n+1}+1)+2kF_n-(n+3)M\\
\sum_{m=0}^{n}LGSe_{2m+1}&=2F_{2n+3}+2i(F_{2n+2}+1)+2jF_{2n+1}+2k(F_{2n}+2)-(n+3)M\\
\sum_{m=1}^{n}LGSe_{2m}&=2F_{2n+2}+2iF_{2n+1}+2j(F_{2n}+1)+2kF_{2n-1}-(n+2)M
\end{aligned}
$$

where $M=(1+i+j+k)$.

Proof.

$$
\begin{aligned}
\sum_{m=0}^{n}LGSe_m&=\sum_{m=0}^{n}(Le_m+iLe_{m-1}+jLe_{m-2}+kLe_{m-3})\\
&=\sum_{m=0}^{n}Le_m+i\sum_{m=0}^{n}Le_{m-1}+j\sum_{m=0}^{n}Le_{m-2}+k\sum_{m=0}^{n}Le_{m-3}\\
&=\sum_{m=0}^{n}(2F_{m+1}-1)+i\sum_{m=0}^{n}(2F_m-1)+j\sum_{m=0}^{n}(2F_{m-1}-1)+k\sum_{m=0}^{n}(2F_{m-2}-1)\\
&=(2F_{n+3}-n-3)+i(2F_{n+2}-n-3)+j(2F_{n+1}-n-1)+k(2F_n-n-3)\\
&=2F_{n+3}+2iF_{n+2}+2j(F_{n+1}+1)+2kF_n-(n+3)M
\end{aligned}
$$

where $M=(1+i+j+k)$.

Other sum formulas are proven using through the same method. □

With the next theorem, we can give the matrix representations of the positive and negative index terms of the Leonardo Gaussian quaternion numbers.

Theorem 2.10. *Let $LGSe_n$ be the Leonardo Gaussian quaternion. Let for $n \geq 1$ be an integer. Then*

$$\begin{bmatrix} LGSe_{n+3} & LGSe_{n+2} & LGSe_{n+1} \\ LGSe_{n+2} & LGSe_{n+1} & LGSe_{n} \\ LGSe_{n+1} & LGSe_{n} & LGSe_{n-1} \end{bmatrix} = \begin{bmatrix} LGSe_{3} & LGSe_{2} & LGSe_{1} \\ LGSe_{2} & LGSe_{1} & LGSe_{0} \\ LGSe_{1} & LGSe_{0} & LGSe_{-1} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^n$$

and

$$\begin{bmatrix} LGSe_{-n+3} & LGSe_{-n+2} & LGSe_{-n+1} \\ LGSe_{-n+2} & LGSe_{-n+1} & LGSe_{-n} \\ LGSe_{-n+1} & LGSe_{-n} & LGSe_{-n-1} \end{bmatrix} = \begin{bmatrix} LGSe_{3} & LGSe_{2} & LGSe_{1} \\ LGSe_{2} & LGSe_{1} & LGSe_{0} \\ LGSe_{1} & LGSe_{0} & LGSe_{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}^n.$$

Proof. The proof is seen by induction on n. □

3. CONCLUSION

This study presents the Leonardo Gaussian quaternion. We obtain this new quaternion not defined in the literature before. We generate Binet's formula, generating function formula and matrix representation. Also we give d'Ocagne's identity, Catalan's identity, Cassini's identity, Honsberger's identity and like-Vajda's identity. Since this study includes some new results, it contributes to literature by providing essential information concerning the bicomplex quaternions. The main contribution of this research that is one can get a great number of distinct quaternion sequences by providing the initial values in the Leonardo sequence. For further studies, we plan to find some properties of these new numbers.

ACKNOWLEDGEMENT

The author expresses his sincere thanks to the anonymous referees and the associate editor for their careful reading, suggestions and comments, which improved the presentation of results.

REFERENCES

[1] S. L. Adler, *Quaternionic quantum mechanics and quantum fields*, New York Oxford Univ. Press, 1994.
[2] F. Alves, R. Vieira, *The Newton Fractal's Leonardo Sequence Study with the Google Colab*, International Electronic Journal of Mathematics Education 2020; 15 (2): 1-9.
[3] J. Baez, *The Octonians*, Bull. Amer. Math. Soc., 2001; 145 (39): 145-205.
[4] G. Bilgici, *Two generalizations of Lucas sequence*, Applied Mathematics and Computation 2014; 245: 526-538.
[5] P. Catarino, A. Borges, *On Leonardo Numbers*, Acta Math. Univ. Comenianae 2020; 1: 75-86.
[6] C. B. Çimen, A. Ípek, *On Pell Quaternions and Pell-Lucas Quaternions*, Adv. Appl. Clifford Algebras 2016; 26 (1): 39–51.

[7] M. Edson, O. Yayenie, *A new generalization of Fibonacci sequences and the extended Binet's formula*, Inyegers Electron. J. Comb. Number Theor. 2009; 9: 639-654.
[8] S. Falcon, A. Plaza, *On the Fibonacci k-numbers*, Chaos, Solitions and Fractals 2007; 32 (5): 1615-1624.
[9] A. H. George, *Some formula for the Fibonacci sequence with generalization*, Fibonacci Quart. 1969; 7: 113-130.
[10] H. Gökbaş, *A Note on BiGaussian Pell and Pell-Lucas Numbers*, Jourrnal of Science and Arts 2021; 3 (56): 669-680.
[11] S. Halıcı, *On Fibonacci Quaternions*, Adv. Appl. Clifford Algebras 2012; 22: 321-327.
[12] W. R. Hamilton, *Li on Quaternions; or on a New System of Imaginaries in Algebra*, Philos. Mag. Ser. Taylor and Francis 1844; 25 (163): 10.
[13] C. J. Harman, *Complex Fibonacci Numbers*, The Fibonacci Quart. 1981; 19 (1): 82-86.
[14] A. F. Horadam, *A generalized Fibonacci sequence*, Math. Mag. 1961; 68: 455-459.
[15] A. F. Horadam, *Complex Fibonacci numbers and Fibonacci quaternions*, Amer. Math. Monthly 1963; 70: 289-291.
[16] R. A. Horn, *Johnson CR. Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.
[17] J. H. Jordan, *Gaussian Fibonacci and Lucas Numbers*, Fib. Quart. 1965; 3: 315.
[18] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, A Wiley-Intersience Publication, USA, 2001.
[19] M. Mangueria, F. Alves, P. Catarino, *Os Biquternions Elipticos de Leonardo*, Revista Eletronica Paulista de Matematica 2021; 21: 130-139.
[20] S. P. Pethe, C. N. Phadte, *A generalization of the Fibonacci sequence*, Applications of Fibonacci numbers 1992; 5: 465-472.
[21] J. C. Pond, *Generalized Fibonacci Summations*, Fibonacci Quart. 1968; 6: 97-108.
[22] L. Ramirez, *Some combinatorial properties of the k-Fibonacci and the k-Lucas quaternions*, An. S t. Univ. Ovidius Constanta, Ser. Mat. 2015; 23 (2): 201-212.
[23] A. G. Shannon, *A Note on Generalized Leonardo Numbers*, Notes on Number theory and Discrete Mathematics 2019; 25 (3): 97-101.
[24] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Ellis Horwood Limited Publ., England, 1989.
[25] R. Vieira, F. Alves, P. Catarino, *Relaçoes Bidimension Ais e Identidades da Sequencia de Leonardo*, Revista Sergipana de Matematica e Educaçao Matematica 2019; 2: 156-173.
[26] J. P. Ward, *Quaternions and Cayley Numbers*, Algebra and Applications, Kluwer Academic Publishers, London, 1997.

(Received: August 20, 2022)
(Revised: December 08, 2023)

Hasan Gökbaş
Bitlis Eren University
Science-Arts Faculty
Mathematics Department
13000, Bitlis, Turkey
e-mail: hgokbas@beu.edu.tr