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DERIVATIVES OF SOLUTIONS OF nTH ORDER DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. (Delta) derivatives of the solutions to an nth order parameter dependent dynamic equation on an arbitrary time scale are shown to exist with respect to the boundary data. This result is achieved by standard uniqueness and continuity assumptions. Moreover, these (delta) derivatives are shown to solve an associated homogeneous and nonhomogeneous dynamic equation on the same time scale.

1. INTRODUCTION

Let $\mathbb{T}$ be a time scale and consider the nth order parameter dependent boundary value problem

$$y^{\Delta^n} = f\big(t, y, y^{\Delta}, y^{\Delta\Delta}, \dots, y^{\Delta^{n-1}}, \lambda\big),\ t \in \mathbb{T}, \tag{1.1}$$

satisfying, for $i = 1, 2, \dots, n$, the boundary conditions

$$y(t_i) = y_i, \tag{1.2}$$

where $t_i \in \mathbb{T}^{\kappa^{n-1}}$ with $\sigma(t_i) < t_{i+1}$ and $y_i,\ \lambda \in \mathbb{R}$.

Definition 1.1. *Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by*

$$\sigma(t) := \inf\{s \in \mathbb{T} : s < t\}.$$

Under suitable hypotheses for f, we will differentiate the solution of (1.1), (1.2) with respect to the boundary values and delta differentiate the solution with respect to the boundary points.

The following are conditions routinely imposed throughout this work.

(H1) $f(t, d_0, d_1, \dots, d_{n-1}, \lambda) : \mathbb{T} \times \mathbb{R}^{n+1} \to \mathbb{R}$ is continuous;
(H2) $\partial f/\partial d_i : \mathbb{T} \times \mathbb{R}^{n+1} \to \mathbb{R}$, $i = 0, 1, \dots, n-1$ are continuous;
(H3) $\partial f/\partial \lambda : \mathbb{T} \times \mathbb{R}^{n+1} \to \mathbb{R}$ is continuous;
(H4) solutions of (1.1) extend to all of $\mathbb{T}$.

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We will show that under the hypotheses above and a few more listed later in the paper that the (delta) derivative of the solution to (1.1), (1.2) solves one of the following two dynamic equations.

Definition 1.2. *The variational equation along a solution $y(t)$ to* (1.1) *is*

$$z^{\Delta^n} = \sum_{i=0}^{n-1} \frac{\partial f}{\partial d_i}\Big(t, y(t), y^{\Delta}(t), \dots, y^{\Delta^{n-1}}(t), \lambda\Big) z^{\Delta^i}. \tag{1.3}$$

Definition 1.3. *An associated nonhomogeneous equation related to the variational equation along a solution $y(t)$ of* (1.1) *is*

$$z^{\Delta^n} = \sum_{i=0}^{n-1} \frac{\partial f}{\partial d_i}\Big(t, y(t), y^{\Delta}(t), \dots, y^{\Delta^{n-1}}(t), \lambda\Big) z^{\Delta^i} + \frac{\partial f}{\partial \lambda}\Big(t, y(t), y^{\Delta}(t), \dots, y^{\Delta^{n-1}}(t), \lambda\Big). \tag{1.4}$$

This work is part of a long line of research into the relationship between derivatives of solutions of differential equations and associated variational or variational-like equations. According to Hartman [9], Peano was the first to investigate the derivative of a solution to a differential equation. In this foundational work by Hartman, the focus was on initial value problems with derivatives taken with respect to the initial data. Building on this work, Spencer [24] was one of the first to shift to boundary value problems, followed by Peterson [23] who considered derivatives with respect to boundary values. These results were then extended by Henderson [10, 11] to include derivatives with respect to boundary points.

More recent results [7, 8, 16] include work on different types of boundary conditions, including multipoint and integral, with the multipoint case generalized to an n-th order case in [12, 19]. Relatedly, research has also been done for difference equations [2, 6, 13, 15, 20] including [21] Lyons' results on the time scale $\mathbb{T} = h\mathbb{Z}$. Also of influence to this work is the addition of a parameter to the differential equation and differentiation thereof as seen in [14, 22]. Finally, this work is most directly related to and the culmination of the following publications. Baxter et al. in [1] considered delta derivatives to a second order dynamic equation on a general time scale. Subsequently, Jensen et al [17] extended this result to a third-order problem and introduced a parameter to the function. Here, we show that under suitable hypotheses the solution of an nth order parameter dependant dynamic boundary value problem (1.1), (1.2) may be delta differentiated with respect to each t_i, y_i, and λ. This work utilizes similar techniques to many papers listed above; employing a dense point argument while making use of continuous dependence and a particular modification of Peano's theorem.

The remainder of the paper is arranged as follows. In Section 2, we present a continuous dependence result for initial value problems and a time scales analogue of Peano's theorem. Section 3 introduces a uniqueness property and establishes continuous dependence for boundary value problems. Finally, in Section 4, we will present the main results. For an exposition on time scales, the author recommends the books by Bohner and Peterson, [3,4].

2. RESULTS FOR INITIAL VALUE PROBLEMS

We begin by building the necessary tools and background for initial value problems. For $i = 0, 1, \dots, n-1$, consider (1.1) satisfying the initial conditions

$$y^{\Delta^i}(t_0) = c_i, \tag{2.1}$$

where $t_0 \in \mathbb{T}^{\kappa^n}$ and $c_i \in \mathbb{R}$.

An additional hypothesis for initial value problems is required:

(H5) solutions to (1.1), (2.1) are unique on all of $\mathbb{T}$.

We will denote the unique solution of (1.1), (2.1) by $u(t,t_0,c_0,c_1,\dots,c_{n-1},\lambda)$. Throughout this paper, we will refer to solutions of BVPs using y and solutions of IVPs using u to help with notation even if referring to the same function.

The following continuous dependence result of IVPs will be employed. See [5] for the proof for the first order IVP. This proof can be easily modified for higher order problems.

Theorem 2.1. *Assume conditions (H1) and (H5) hold. Given an interval $[a,b]_{\mathbb{T}}$, a point $t_0 \in \mathbb{T}^{\kappa^{n-1}}$, $c_0, c_1, \dots, c_{n-1}, \lambda \in \mathbb{R}$, and $\varepsilon > 0$, there exists a*

$$\delta(\varepsilon, [a,b]_{\mathbb{T}}, t_0, c_0, c_1, \dots, c_{n-1}, \lambda) > 0$$

such that if $|c_i - e_i| < \delta$ and $|\lambda - L| < \delta$, then

$$|u(t,t_0,c_0,c_1,\dots,c_{n-1},\lambda) - u(t,t_0,e_0,e_1,\dots,e_{n-1},L)| < \varepsilon$$

for $t \in [a,b]_{\mathbb{T}}$ and $e_0, e_1, \dots, e_{n-1}, L \in \mathbb{R}$.

The next two theorems are analogues of Peano's result for differential equations and may be found in the book by Lakshmikantham et al. [18]. The first involves differentiation of solutions of (1.1), (2.1) with respect to initial values and the parameter λ.

Theorem 2.2. *Assume (H1)-(H2) and (H4)-(H5) hold. Let $c_0, c_1, \dots, c_{n-1}, \lambda \in \mathbb{R}$ and $t_0 \in \mathbb{T}^{\kappa^n}$. Suppose $u(t,t_0,c_0,c_1,\dots,c_{n-1},\lambda)$ solves* (1.1), (2.1). *Then,*

(a) *for $i = 0, 1, \dots, n-1$, $\beta_i(t) := \partial u/\partial c_i$ exists and is the solution of* (1.3) *along $u(t)$ satisfying, for $j = 0, 1, \dots, n-1$, the respective initial conditions $\beta_i^{\Delta^j}(t_0) = \delta_{ij}$.*

(b) *if additionally (H3) holds, $L(t) := \partial u/\partial\lambda$ exists and is the solution of* (1.4) *along $u(t)$ satisfying, for $j = 0, 1, \dots, n-1$, the initial conditions $L^{\Delta^j}(t_0) = 0$.*

The following theorem involves delta differentiation of solutions of (1.1), (2.1) with respect to initial points.

Theorem 2.3. *Assume (H1)-(H2) and (H4)-(H5) hold. Let* $c_0, c_1, \ldots, c_{n-1}, \lambda \in \mathbb{R}$ *and* $t_0 \in \mathbb{T}^{\kappa^n}$. *Then,*

$$\begin{aligned}\gamma(t) :&= u^{\Delta_{t_0}}(t, t_0, c_0, c_1, \ldots, c_{n-1}, \lambda)\\ &= \frac{1}{\mu(t_0)}\left[u(t, \sigma(t_0), c_0, c_1, \ldots, c_{n-1}, \lambda) - u(t, t_0, c_0, c_1, \ldots, c_{n-1}, \lambda)\right]\end{aligned}$$

is the solution of the nth order linear dynamic equation

$$\gamma^{\Delta^n} = \sum_{i=0}^{n-1} A_i(t)\gamma^{\Delta^i},$$

satisfying, for $i = 0, 1, \ldots, n-1$, *the initial conditions*

$$\gamma^{\Delta^i}(t_0) = -u^{\Delta^{i+1}}(t_0, \sigma(t_0), c_0, c_1, \ldots, c_{n-1}, \lambda),$$

where

$$\begin{aligned}A_i(t) = \int_0^1 \frac{\partial f}{\partial d_i}\Big(&t, u(t, \sigma(t_0), c_0, c_1, \ldots, c_{n-1}, \lambda), \ldots, s u^{\Delta^i}(t, t_0, \sigma(t_0), c_0, c_1, \ldots, c_{n-1}, \lambda)\\ &+ (1-s) u^{\Delta^i}(t, t_0, c_0, c_1, \ldots, c_{n-1}, \lambda), \ldots, u^{\Delta^{n-1}}(t, t_0, c_0, c_1, \ldots, c_{n-1}, \lambda)\Big) ds.\end{aligned}$$

Note that if t_0 is right-dense, i.e. $\sigma(t_0) = t_0$, then $\gamma^{\Delta^n} = \sum_{i=0}^{n-1} A_i(t)\gamma^{\Delta^i}$, is the variational equation, (1.3), for (1.1) along $u(t)$.

3. Results for Boundary Value Problems

We require one more hypothesis for (1.1) that will guarantee uniqueness of solutions to boundary value problems of (1.1). To that end, we need the following definition.

Definition 3.1. *The function* $v : \mathbb{T} \to \mathbb{R}$ *is said to have a generalized zero at* $a \in \mathbb{T}$ *if* $v(a) = 0$ *or* $v(\rho(a))v(a) < 0$.

We make two disconjugate-type hypotheses for dynamic equations. The first provides uniqueness for solutions of (1.1), (1.2) and the second provides uniqueness for solutions of nth order linear dynamic equations:

(H6) suppose $y_1(t)$ and $y_2(t)$ are solutions of (1.1). If for $i = 1, 2, \ldots, n$, $y_1(t) - y_2(t)$ has a generalized zero at $t_i \in \mathbb{T}^{\kappa^{n-1}}$ with $\sigma(t_i) < t_{i+1}$, then $y_1(t) - y_2(t) \equiv 0$ on $\mathbb{T}$.

(H7) if for $i = 1, 2, \ldots, n-1$, $s(t)$ is a solution to the linear dynamic equation

$$s^{\Delta^n} = \sum_{i=0}^{n-1} M_i(t) s^{\Delta^i}$$

such that $s(t)$ has a generalized zero at $t_i \in \mathbb{T}^{\kappa^{n-1}}$ with $\sigma(t_i) < t_{i+1}$, then $s(t) \equiv 0$ on $\mathbb{T}$.

Finally, we provide a continuous dependence result with respect to boundary values. The proof involves an application of the Brouwer invariance of domain theorem. See [5] for the proof mechanics.

Theorem 3.1. *Assume conditions (H1)-(H6). Let $y(t)$ be a solution of* (1.1). *For $i = 1,2,\dots,n$, let $t_i \in \mathbb{T}^{\kappa^{n-1}}$ with $\sigma(t_i) < t_{i+1}$ and $y_i, \lambda \in \mathbb{R}$. Then, there exists a $\delta > 0$ such that for $i = 1,2,\dots,n$, $|t_i - s_i| < \delta$ where $s_i \in \mathbb{T}^{\kappa^{n-1}}$ with $\sigma(s_i) < s_{i+1}$, $|y_i - x_i| < \delta$ where $x_i \in \mathbb{R}$, and $|\lambda - L| < \delta$ where $L \in \mathbb{R}$, the boundary value problem for* (1.1) *satisfying*

$$w(s_i) = x_i$$

has a unique solution $w(t, s_1, s_2, \dots, s_n, x_1, x_2, \dots, x_n, L)$. Moreover, as $\delta \to 0$, $w(t)$ converges uniformly to $y(t)$ on $\mathbb{T}$.

4. Derivatives of Solutions to Boundary Value Problems

The first two theorems are BVP analogues of Theorem 2.2 which consider boundary values and the parameter respectively. The proofs of these theorems are similar to the dense case that will be proven later on, and, therefore, we only present the statements of the theorems.

Theorem 4.1. *Assume conditions (H1)-(H7). Suppose $y(t, t_1, t_2, \dots, t_n, y_1, y_2, \dots, y_n, \lambda)$ is the solution of* (1.1), (1.2) *on $\mathbb{T}$ where for $i = 1,2,\dots,n$, $t_i \in \mathbb{T}^{\kappa^{n-1}}$ with $\sigma(t_i) < t_{i+1}$ and $y_i, \lambda \in \mathbb{R}$. Then, for $i = 1,2,\dots,n$, $z_i := \partial y/\partial y_i$ exists on $\mathbb{T}$ and is the solution of* (1.3) *along $y(t)$ that satisfies, for $j = 1,2,\dots,n$, the respective boundary conditions $z_i(t_j) = \delta_{ij}$.*

Theorem 4.2. *Assume conditions (H1)-(H7). Suppose $y(t, t_1, t_2, \dots, t_n, y_1, y_2, \dots, y_n, \lambda)$ is the solution of* (1.1), (1.2) *on $\mathbb{T}$ where for $i = 1,2,\dots,n$, $t_i \in \mathbb{T}^{\kappa^{n-1}}$ with $\sigma(t_i) < t_{i+1}$ and $y_i, \lambda \in \mathbb{R}$. Then, $\Lambda := \partial y/\partial \lambda$ exists on $\mathbb{T}$ and is the solution of* (1.4) *along $y(t)$ that satisfies, for $j = 1,2,\dots,n$, the boundary conditions $\Lambda(t_j) = 0$.*

The third result deals with delta differentiation of the solution $y(t)$ of (1.1), (1.2) with respect to the boundary points. Since the boundary points could be dense or scattered, we consider both cases separately in the proof.

Theorem 4.3. *Assume conditions (H1)-(H7). Suppose $y(t, t_1, t_2, \dots, t_n, y_1, y_2, \dots, y_n, \lambda)$ is the solution of* (1.1), (1.2) *on $\mathbb{T}$, where for $i = 1,2,\dots,n$, $t_i \in \mathbb{T}^{\kappa^{n-1}}$ with $\sigma(t_i) < t_{i+1}$ and $y_i, \lambda \in \mathbb{R}$. Then, for $j = 1,2,\dots,n$, $\nu_j := y^{\Delta_{t_j}}(t, t_1, t_2, \dots, t_n, y_1, y_2, \dots, y_n, \lambda)$ is a solution of the linear dynamic equation*

$$\nu_j^{\Delta^n} = \sum_{i=0}^{n-1} A_{ij}(t) \nu_j^{\Delta^i}$$

where

$$\begin{aligned}A_{ij}(t) = \int_0^1 \frac{\partial f}{\partial d_i}\Big(&t, y(t,t_1,\dots,t_j,\dots,t_n,y_1,y_2,\dots,y_n,\lambda),\dots,\\ &sy^{\Delta^i}(t,t_1,\dots,\sigma(t_j),\dots,t_n,y_1,y_2,\dots,y_n,\lambda)\\ &+(1-s)y^{\Delta^i}(t,t_1,\dots,t_j,\dots,t_n,y_1,y_2,\dots,y_n,\lambda),\dots,\\ &y^{\Delta^{n-1}}(t,t_1,\dots,\sigma(t_j),\dots,t_n,y_1,y_2,\dots,y_n,\lambda)\Big)ds,\end{aligned}$$

with respective boundary conditions, for $k = 1,2,\dots,n$,

$$\nu_j(t_k) = -y^{\Delta}(t_k,t_1,\dots,\sigma(t_j),\dots,t_n,y_1,y_2,\dots,y_n,\lambda)\delta_{jk}.$$

Proof. Set $1 \le j \le n$ an integer. For notational ease since only t and t_j are not fixed, we denote $y(t,t_1,t_2,\dots,t_n,y_1,y_2,\dots,y_n,\lambda)$ by $y(t,t_j)$ and consider two cases; t_j is right-scattered and t_j is right-dense.
Case 1: Assume $t_j < \sigma(t_j)$, i.e. t_j is right-scattered.

First, we show that

$$\nu_j(t) = y^{\Delta_{t_j}}(t,t_j) = \frac{1}{\mu(t_j)}\left[y(t,\sigma(t_j)) - y(t,t_j)\right]$$

is a solution of the linear dynamic equation

$$\nu_j^{\Delta^n} = \sum_{i=0}^{n-1} A_{ij}(t)\nu_j^{\Delta^i}$$

with the stated boundary conditions.

Checking the boundary conditions and using a telescoping sum, we see that

$$\begin{aligned}\nu_j(t_j) &= y^{\Delta_{t_j}}(t_j,t_j)\\ &= \frac{1}{\mu(t_j)}[y(t_j,\sigma(t_j)) - y(t_j,t_j)]\\ &= \frac{1}{\mu(t_j)}[y(t_j,\sigma(t_j)) - y(\sigma(t_j),\sigma(t_j)) + y(\sigma(t_j),\sigma(t_j)) - y_j]\\ &= -y^{\Delta}(t_j,\sigma(t_j)) + \frac{1}{\mu(t_j)}[y_j - y_j]\\ &= -y^{\Delta}(t_j,\sigma(t_j)),\end{aligned}$$

and for $i = 1,2,\dots,n$ with $i \neq j$,

$$\nu_j(t_i) = y^{\Delta_{t_j}}(t_i,t_j) = \frac{1}{\mu(t_j)}[y(t_i,\sigma(t_j)) - y(t_i,t_j)] = \frac{1}{\mu(t_j)}[y_i - y_i] = 0.$$

Now, we show ν_j solves the dynamic equation. Notice

$$\nu_j^{\Delta^n} = \left[y^{\Delta_{t_j}}(t,t_j)\right]^{\Delta^n}$$

$$= \frac{1}{\mu(t_j)}\left[y^{\Delta^n}(t,\sigma(t_j)) - y^{\Delta^n}(t,t_j)\right]$$
$$= \frac{1}{\mu(t_j)}\left[f\Big(t, y(t,\sigma(t_j)), y^{\Delta}(t,\sigma(t_j)), \dots, y^{\Delta^{n-1}}(t,\sigma(t_j))\Big) - f\Big(t, y(t,t_j), y^{\Delta}(t,t_j), \dots, y^{\Delta^{n-1}}(t,t_j)\Big)\right].$$

We apply $n-1$ telescoping sums

$$\begin{aligned}\nu_j^{\Delta^n} = \frac{1}{\mu(t_j)}\Big[& f\Big(t, y(t,\sigma(t_j)), y^{\Delta}(t,\sigma(t_j)), \dots, y^{\Delta^{n-1}}(t,\sigma(t_j))\Big) \\ & - f\Big(t, y(t,t_j), y^{\Delta}(t,\sigma(t_j)), \dots, y^{\Delta^{n-1}}(t,\sigma(t_j))\Big) \\ & + f\Big(t, y(t,t_j), y^{\Delta}(t,\sigma(t_j)), \dots, y^{\Delta^{n-1}}(t,\sigma(t_j))\Big) \\ & - \cdots \\ & - f\Big(t, y(t,t_j), y^{\Delta}(t,t_j), \dots, y^{\Delta^{n-1}}(t,\sigma(t_j))\Big) \\ & + f\Big(t, y(t,t_j), y^{\Delta}(t,t_j), \dots, y^{\Delta^{n-1}}(t,\sigma(t_j))\Big) \\ & - f\Big(t, y(t,t_j), y^{\Delta}(t,t_j), \dots, y^{\Delta^{n-1}}(t,t_j)\Big)\Big].\end{aligned}$$

Then, using the fundamental theorem of calculus n times, we write,

$$\begin{aligned}\nu_j^{\Delta^n} = & \\ = & \frac{1}{\mu(t_j)}\int_0^1 \frac{df}{ds}\Big(t, sy(t,\sigma(t_j)) + (1-s)y(t,t_j), y^{\Delta}(t,\sigma(t_j)), \dots, y^{\Delta^{n-1}}(t,\sigma(t_j))\Big)ds \\ & + \frac{1}{\mu(t_j)}\int_0^1 \frac{df}{ds}\Big((t, y(t,t_j), sy^{\Delta}(t,\sigma(t_j)) + (1-s)y^{\Delta}(t,t_j), \dots, y^{\Delta^{n-1}}(t,\sigma(t_j))\Big)ds \\ & + \cdots \\ & + \frac{1}{\mu(t_j)}\int_0^1 \frac{df}{ds}\Big(t, y(t,t_j), y^{\Delta}(t,t_j), \dots, sy^{\Delta^{n-1}}(t,\sigma(t_j)) + (1-s)y^{\Delta^{n-1}}(t,t_j)\Big)ds.\end{aligned}$$

Finally, applying the mean value theorem n times yields

$$\begin{aligned}\nu_j^{\Delta^n} = & \\ = & \int_0^1 \frac{\partial f}{\partial d_0}\Big(t, sy(t,\sigma(t_j)) + (1-s)y(t,t_j), y^{\Delta}(t,\sigma(t_j)), \dots, y^{\Delta^{n-1}}(t,\sigma(t_j))\Big)ds \\ & \times \left(\frac{y(t,\sigma(t_j)) - y(t,t_j)}{\mu(t_j)}\right)\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 \frac{\partial f}{\partial d_1}\Big(t, y(t,t_j), sy^{\Delta}(t,\sigma(t_j)) + (1-s)y^{\Delta}(t,t_j), \ldots, y^{\Delta^{n-1}}(t,\sigma(t_j))\Big)ds \\
&\times\left(\frac{y^{\Delta}(t,\sigma(t_j)) - y^{\Delta}(t,t_j)}{\mu(t_j)}\right) \\
&+\cdots \\
&+\int_0^1 \frac{\partial f}{\partial d_{n-1}}\Big(t, y(t,t_j), y^{\Delta}(t,t_j), \ldots, sy^{\Delta^{n-1}}(t,\sigma(t_j)) + (1-s)y^{\Delta^{n-1}}(t,t_j)\Big)ds \\
&\times\left(\frac{y^{\Delta^{n-1}}(t,\sigma(t_j)) - y^{\Delta^{n-1}}(t,t_j)}{\mu(t_j)}\right) \\
&= A_{0j}(t)\nu_j + A_{1j}(t)\nu_j^{\Delta} + \cdots + A_{(n-1)j}(t)\nu_j^{\Delta^{n-1}}.
\end{aligned}
$$

Case 2: Assume $t_j = \sigma(t_j)$ i.e. t_j is right-dense.

First, notice that in this case,

$$\nu_j^{\Delta^n} = A_{0j}(t)\nu_j + A_{1j}(t)\nu_j^{\Delta} + \cdots + A_{(n-1)j}(t)\nu_j^{\Delta^{n-1}}$$

is the variational equation (1.3) along $y(t)$. Because $t_j = \sigma(t_j)$, t_j is right-dense in $\mathbb{T}$, and so, for any $\delta > 0$, $\mathrm{card}(t_j - \delta, t_j + \delta) = \infty$. Choose δ as in Theorem 3.1 and for each $t_j + h \in (t_j - \delta, t_j + \delta)_{\mathbb{T}}$, define

$$\nu_{jh}(t) = \frac{1}{h}[y(t, t_j + h) - y(t, t_j)].$$

Now, we investigate the boundary conditions. Note that by using a telescoping sum

$$
\begin{aligned}
\nu_{jh}(t_j) &= \frac{1}{h}[y(t_j, t_j + h) - y(t_j, t_j)] \\
&= \frac{1}{h}[y(t_j, t_j + h) - y(t_j + h, t_j + h) + y(t_j + h, t_j + h) - y(t_j, t_j)] \\
&= \frac{1}{h}[y(t_j, t_j + h) - y(t_j + h, t_j + h) + y_j - y_j] \\
&= \frac{1}{h}[y(t_j, t_j + h) - y(t_j + h, t_j + h)],
\end{aligned}
$$

and for $i = 1, 2, \ldots, n$ with $i \neq j$,

$$\nu_{jh}(t_i) = \frac{1}{h}[y(t_i, t_j + h) - y(t_i, t_j)] = \frac{1}{h}[y_i - y_i] = 0.$$

To show that $\nu_{jh}(t)$ solves the variational equation, we view $y(t)$ as a solution of an initial value problem at the initial point t_j. For $i = 1, 2, \ldots, n-1$, let

$$\mu_i = y^{\Delta^i}(t_j, t_j)$$

and

$$\varepsilon_i = y^{\Delta^i}(t_j, t_j + h) - \mu_i.$$

Also, let

$$\varepsilon_0 = y(t_j, t_j + h) - y_j.$$

Notice that by continuous dependence that for each $i = 0, 1, \dots, n-1$ as $t_j + h \to t_j$, we have $\varepsilon_i \to 0$. Thus, our solution $y(t)$ may be written using initial value problem notation at initial point t_j as $u(t, t_j, y_j, \mu_1, \mu_2, \dots, \mu_{n-1})$. Therefore, in terms of $u(t)$, we have

$$\begin{aligned}\nu_{jh}(t) = \frac{1}{h}\Big[&u(t, t_j, y_j + \varepsilon_0, \mu_1 + \varepsilon_1, \mu_2 + \varepsilon_2, \dots, \mu_{n-1} + \varepsilon_{n-1})\\ &- u(t, t_j, y_j, \mu_1, \mu_2, \dots, \mu_{n-1})\Big]\end{aligned}$$

By employing $n-1$ telescoping sums,

$$\begin{aligned}\nu_{jh}(t) = \frac{1}{h}\Big[&u(t, t_j, y_j + \varepsilon_0, \mu_1 + \varepsilon_1, \mu_2 + \varepsilon_2, \dots, \mu_{n-1} + \varepsilon_{n-1})\\ &- u(t, t_j, y_j, \mu_1 + \varepsilon_1, \mu_2 + \varepsilon_2, \dots, \mu_{n-1} + \varepsilon_{n-1})\\ &+ u(t, t_j, y_j, \mu_1 + \varepsilon_1, \mu_2 + \varepsilon_2, \dots, \mu_{n-1} + \varepsilon_{n-1})\\ &- \cdots\\ &- u(t, t_j, y_j, \mu_1, \mu_2, \dots, \mu_{n-1} + \varepsilon_{n-1})\\ &+ u(t, t_j, y_j, \mu_1, \mu_2, \dots, \mu_{n-1} + \varepsilon_{n-1}) - u(t, t_j, y_j, \mu_1, \mu_2, \dots, \mu_{n-1})\Big].\end{aligned}$$

By the standard mean value theorem for each $i = 1, 2, \dots, n-1$, there exists $\bar{\varepsilon}_i \in (-\varepsilon_i, \varepsilon_i)$ such that

$$\begin{aligned}&u(t, t_j, y_j, \mu_1, \mu_2, \dots \mu_i + \varepsilon_i, \dots, \mu_{n-1} + \varepsilon_{n-1}) - u(t, t_j, y_j, \mu_1, \mu_2, \dots, \mu_i, \dots, \mu_{n-1} + \varepsilon_{n-1})\\ &\qquad = \beta_i(t, u(t, t_j, y_j, \mu_1, \mu_2, \dots, \mu_i + \bar{\varepsilon}_i, \dots, \mu_{n-1} + \varepsilon_{n-1}))(\mu_i + \varepsilon_i - \mu_i),\end{aligned}$$

where $\beta_i(t, u(\cdot))$ is as defined in Theorem 2.2. Similarly, there exists an $\bar{\varepsilon}_0 \in (-\varepsilon_0, \varepsilon_0)$ such that

$$\begin{aligned}&u(t, t_j, y_j + \varepsilon_0, \mu_1 + \varepsilon_1, \dots, \mu_{n-1} + \varepsilon_{n-1}) - u(t, t_j, y_j, \mu_1, \dots, \mu_{n-1} + \varepsilon_{n-1})\\ &\qquad = \beta_0(t, u(t, t_j, y_j + \bar{\varepsilon}_0, \mu_1 + \varepsilon_1, \dots, \mu_{n-1} + \varepsilon_{n-1}))(y_j + \varepsilon_0 - y_j),\end{aligned}$$

Combining the above together and suppressing the respective function components of u, we have

$$\nu_{jh}(t) = \frac{\varepsilon_0}{h}\beta_0(t, u(\cdot)) + \frac{\varepsilon_1}{h}\beta_1(t, u(\cdot)) + \cdots + \frac{\varepsilon_{n-1}}{h}\beta_{n-1}(t, u(\cdot)).$$

Thus, to show $\lim_{t_j+h\to t_j} \nu_{jh}(t)$ exists, we need that for $i = 0, 1, \dots, n-1$, $\lim_{t_j+h\to t_j} \dfrac{\varepsilon_i}{h}$ exist. From above,

$$\lim_{t_j+h\to t_j} \frac{\varepsilon_0}{h} = \lim_{t_j+h\to t_j} \frac{1}{h}[y(t_j, t_j + h) - y_j]$$

$$= \lim_{t_j+h\to t_j} \frac{1}{h}[y(t_j,t_j+h) - y(t_j+h,t_j+h)]$$
$$= -y^{\Delta}(t_j,t_j).$$

Now, we need the limit to exist for the remaining ε_i/h's. Since for $i = 1,2,\dots,n$ with $i \neq j$, $\nu_{jh}(t_i) = 0$, we have $n-1$ equations with $n-1$ remaining unknown limits

$$0 = \nu_{jh}(t_i) = \frac{\varepsilon_0}{h}\beta_0(t_i,u(\cdot)) + \frac{\varepsilon_1}{h}\beta_1(t_i,u(\cdot)) + \cdots + \frac{\varepsilon_{n-1}}{h}\beta_{n-1}(t_i,u(\cdot))$$

or

$$-\frac{\varepsilon_0}{h}\beta_0(t_i,u(\cdot)) = \frac{\varepsilon_1}{h}\beta_1(t_i,u(\cdot)) + \cdots + \frac{\varepsilon_{n-1}}{h}\beta_{n-1}(t_i,u(\cdot)).$$

We write this in matrix equation form

$$-\frac{\varepsilon_0}{h}\begin{bmatrix} \beta_0(t_1,u(\cdot)) \\ \vdots \\ \beta_0(t_{j-1},u(\cdot)) \\ \beta_0(t_{j+1},u(\cdot)) \\ \vdots \\ \beta_0(t_n,u(\cdot)) \end{bmatrix} = \begin{bmatrix} \beta_1(t_1,u(\cdot)) & \beta_2(t_1,u(\cdot)) & \cdots & \beta_{n-1}(t_1,u(\cdot)) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1(t_{j-1},u(\cdot)) & \beta_2(t_{j-1},u(\cdot)) & \cdots & \beta_{n-1}(t_{j-1},u(\cdot)) \\ \beta_1(t_{j+1},u(\cdot)) & \beta_2(t_{j+1},u(\cdot)) & \cdots & \beta_{n-1}(t_{j+1},u(\cdot)) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1(t_n,u(\cdot)) & \beta_2(t_n,u(\cdot)) & \cdots & \beta_{n-1}(t_n,u(\cdot)) \end{bmatrix} \begin{bmatrix} \frac{\varepsilon_1}{h} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \frac{\varepsilon_{n-1}}{h} \end{bmatrix}$$

More succinctly,

$$-\frac{\varepsilon_0}{h}B_{0h} = B_h E_h.$$

To solve for E_h, we need to show that B_h^{-1} exists. Therefore, we investigate the following matrix along $u(t)$:

$$B = \begin{bmatrix} \beta_1(t_1,u(t)) & \beta_2(t_1,u(t)) & \cdots & \beta_{n-1}(t_1,u(t)) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1(t_{j-1},u(t)) & \beta_2(t_{j-1},u(t)) & \cdots & \beta_{n-1}(t_{j-1},u(t)) \\ \beta_1(t_{j+1},u(t)) & \beta_2(t_{j+1},u(t)) & \cdots & \beta_{n-1}(t_{j+1},u(t)) \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1(t_n,u(t)) & \beta_2(t_n,u(t)) & \cdots & \beta_{n-1}(t_n,u(t)) \end{bmatrix}.$$

By continuous dependence, if B^{-1} exists, then so does B_h^{-1}.

For contradiction, assume that B is not invertible. Then, for $i = 1, 2, \ldots, n-1$, there exist coefficients $c_i \neq 0$ in $\mathbb{R}$ such that

$$\sum_{i=1}^{n-1} c_i \begin{bmatrix} \beta_i(t_1, u(t)) \\ \vdots \\ \beta_i(t_{j-1}, u(t)) \\ \beta_i(t_{j+1}, u(t)) \\ \vdots \\ \beta_i(t_n, u(t)) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Set $p(t) = \sum_{i=1}^{n-1} c_i \beta_i(t, u(t))$. Therefore, $p(t)$ solves (1.3) as it is a linear combination of $\beta_1, \beta_2, \ldots, \beta_{n-1}$ which each solve it. Thus, for $k = 1, 2, \ldots, n-1$ with $k \neq j$,

$$p(t_k) = \sum_{i=1}^{n-1} c_i \beta_i(t_k, u(t)) = 0.$$

Recalling that for $i = 1, 2, \ldots, n-1$, $\beta_i(t_j, u(t)) = 0$, we have that

$$p(t_j) = \sum_{i=1}^{n-1} c_i \beta_i(t_j, u(t)) = \sum_{i=1}^{n-1} c_i(0) = 0.$$

Thus, by (H7), $p(t) \equiv 0$ which means that $c_1 = c_2 = \cdots = c_{n-1} = 0$. This is a contradiction to the choice of the c_i's. Therefore, B is invertible, and so B_h^{-1} exists.

Apply B_h^{-1} to each side the matrix equation to find

$$E_h = -\frac{\varepsilon_0}{h} B_h^{-1} B_{0h}.$$

Thus,

$$E = \lim_{t_j+h \to t_j} E_h = -y^{\Delta}(t_j) B^{-1} B_0.$$

For each $i = 1, 2, \ldots, n-1$, set $e_i = \lim_{t_j+h \to t_j} \varepsilon_i / h$ which exists as the matrix E above exists.

Now, let $\nu_j(t) = \lim_{t_j+j \to t_j} \nu_{jh}(t)$ and note by construction that

$$\nu_j(t) = y^{\Delta_{t_j}}(t).$$

Furthermore,

$$\nu_j(t) = -y^{\Delta}(t_j)\beta_0(t, y(t)) + \sum_{i=1}^{n-1} e_i \beta_i(t, y(t))$$

which solves the variational equation along $y(t)$.

Finally, checking the boundary conditions for $i = 1, 2, \ldots, n$ with $i \neq j$ gives us:

$$\nu_j(t_i) = \lim_{t_j+h \to t_j} \nu_{jh}(t_i) = \lim_{t_j+h \to t_j} 0 = 0,$$

and

$$\nu_j(t_j) = \lim_{t_j+h \to t_j} \nu_{jh}(t) = \lim_{t_j+h \to t_j} \frac{1}{h}[y(t_j, t_j+h) - y(t_j+h, t_j+h)] = -y^{\Delta}(t_j, t_j). \quad \square$$

Remark 4.1. In conclusion, the results presented here consider conjugate boundary conditions for ease of notation. However, the result generalizes to to any type of n-point boundary conditions such as right-focal, etc.

REFERENCES

[1] B. H. Baxter, J. W. Lyons and J. T. Neugebauer, *Differentiating solutions of a boundary value problem on a time scale*, Bull. Aust. Math. Soc. **94**, (2016), 101–109.

[2] M. Benchohra, S. Hamani, J. Henderson, S. K. Ntouyas and A. Ouahab, *Differentiation and differences for solutions of nonlocal boundary value problems for second order difference equations*, Int. J. Difference Equ. **2**, 1 (2007), 37–47.

[3] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[4] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[5] C. J. Chyan, *Uniqueness implies existence on time scales*, J. Math. Anal. Appl. **258**, 1 (2001), 359–365.

[6] A. Datta, *Differences with respect to boundary points for right focal boundary conditions*, J. Differ. Equations Appl. **4**, 6 (1998), 571–578.

[7] J. A. Ehme, *Differentiation of solutions of boundary value problems with respect to nonlinear boundary conditions*, J. Differential Equations **101**, 1 (1993), 139–147.

[8] J. Ehrke, J. Henderson, C. Kunkel and Q. Sheng, *Boundary data smoothness for solutions of nonlocal boundary value problems for second order differential equations*, J. Math. Anal. Appl. **333**, 1 (2007), 191–203.

[9] P. Hartman, *Ordinary Differential Equations*, Wiley, New York, 1964.

[10] J. Henderson, *Right focal point boundary value problems for ordinary differential equations and variational equations*, J. Math. Anal. Appl. **98**, 2 (1984), 363–377.

[11] J. Henderson, *Disconjugacy, disfocality, and differentiation with respect to boundary conditions*, J. Math. Anal. Appl. **121**, 1 (1987), 1–9.

[12] J. Henderson, B. Hopkins, E. Kim and J. W. Lyons, *Boundary data smoothness for solutions of nonlocal boundary value problems for n-th order differential equations*, Involve **1**, 2 (2008), 167–181.

[13] J. Henderson, M. Horn and L. Howard, *Differentiation of solutions of difference equations with respect to boundary values and parameters*, Comm. Appl. Nonlinear Anal. **1**, 2 (1994), 47–60.

[14] J. Henderson and X. Jiang, *Differentiation with respect to parameters of solutions of nonlocal boundary value problems for difference equations*, Involve **8**, 4 (2015), 629–636.

[15] B. Hopkins, E. Kim, J. W. Lyons and K. Speer, *Boundary data smoothness for solutions of nonlocal boundary value problems for second order difference equations*, Comm. Appl. Nonlinear Anal. **16**, 2 (2009), 1–12.

[16] A. F. Janson, B. T. Juman and J. W. Lyons, *The connection between variational equations and solutions of second order nonlocal integral boundary value problems*, Dynam. Systems Appl. **23**, 2-3 (2014), 493–503.

[17] W. M. Jensen, J. W. Lyons, R. Robinson, *Delta derivatives of the solution to a third-order parameter dependent boundary value problem on an arbitrary time scale*, Differ. Equ. Appl. **14** (2022), no. 2, 291-304.

[18] V. Lakshmikantham, S. Sivasundaram and B. Kaymakçalan, *Dynamic Systems on Measure Chains*, Kluwer Academic Publishers, Boston, 1996.
[19] J. W. Lyons, *Differentiation of solutions of nonlocal boundary value problems with respect to boundary data*, Electon. J. Qual. Theory Differ. Equ. **2011**, 51, 1–11.
[20] J. W. Lyons, *Disconjugacy, differences, and differentiation for solutions of nonlocal boundary value problems for nth order difference equations*, J. Difference Equ. Appl. **20**, 2 (2014), 296–311.
[21] J. W. Lyons, *On differentiation of solutions of boundary value problems for second order dynamic equations on a time scale*, Comm. App. Anal. **18**, 2014, 215–224.
[22] J. W. Lyons and J. K. Miller, *The derivative of a solution to a second order parameter dependent boundary value problem with a nonlocal integral boundary condition*,Journal of Mathematics and Statistical Science **2015**, 43–50.
[23] A. Peterson, *Comparison theorems and existence theorems for ordinary differential equations*, J. Math. Anal. Appl. **55**, 3 (1976), 773–784.
[24] J. Spencer, *Relations between boundary value functions for a nonlinear differential equation and its variational equations*, Canad. Math. Bull. **18**, 2 (1975), 269–276.

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