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CLASS NUMBER ONE PROBLEM FOR A NON-RICHAUD-DEGERT TYPE FAMILY OF REAL QUADRATIC FIELDS

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ABSTRACT. We study the class number one problem for a one parameter family of real quadratic fields $\mathbb{Q}(\sqrt{d})$, where $d = 100m^2 + 28m + 2$ and m is a positive integer. We show that there is no such field with class number one.

1. INTRODUCTION

The class number of a number field K, is an important object of study in algebraic number theory. The ring O_K, of algebraic integers in K, is a unique factorization domain if and only if the class number of K is one. It is known that there are only nine imaginary quadratic fields with class number one (see [2, 11]). On the other hand, it was conjectured by Gauss that there are infinitely many real quadratic fields with class number one (see [7]). This conjecture is still open. However, there are several partial attempts (see [3–6, 10]). For example, class number one criterion is known for real quadratic fields in some families [6, 10].

In [3], Biro proved that, for any positive integer m such that $m^2 + 4$ is square-free, the real quadratic field $\mathbb{Q}(\sqrt{m^2+4})$ has class number one precisely when $m \in \{1, 3, 5, 7, 13, 17\}$. Along similar lines, in [4] Biro again proved a conjecture of Chowla; stating that, if m is a positive integer such that $4m^2 + 1$ is square free then the real quadratic field $\mathbb{Q}(\sqrt{4m^2+1})$ has class number one if and only if $m \in \{1, 2, 3, 5, 7, 13\}$.

Definition 1.1. *[9] A real quadratic field is of the form $\mathbb{Q}(\sqrt{d})$, where d is a square-free integer of the form $n^2 + r$ with $r|4n$ and $-n < r \leq n$ or $r = \pm 4n/3$ is called a real quadratic number field of Richaud-Degert type.*

In [6] the authors discussed a class number one criterion for real quadratic fields of Richaud-Degert type (R–D type). Their approach involves computation of special values of Dedekind zeta function attached to the number fields. We note that this method works only for those number fields whose fundamental unit is explicitly known together with some other restrictions on generalized Dedekind sums.

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In [8] the authors studied the class number one problem for a two parameters family of real quadratic fields of the form $\mathbb{Q}(\sqrt{m^2+4r})$ for certain integers m and r. In this article we use similar techniques as in [8] and deal with a new single parameter family of real quadratic fields which is not in the form of Richaud-Degert type. Certainly the methods in [6] can not be applied to such number fields and therefore we use a method introduced in [1] and later modified in [8] to show some Diophantine equations have no solutions in integers. Throughout this paper K denotes the number field $\mathbb{Q}(\sqrt{d})$ and by h_K we mean the class number of K. We prove the following theorem in this article.

Theorem 1.1. *Let m be any positive integer and suppose $d = 100m^2 + 28m + 2$ is square-free. Then the Diophantine equation $x^2 - dy^2 = \pm 2$ has no solution among integers and hence $h_K > 1$.*

2. Proof of Theorem 1.1

We consider the one parameter family of real quadratic fields $\mathbb{Q}(\sqrt{d})$, where $d = 100m^2 + 28m + 2$ is square-free and m is a positive integer. The following lemma ensures that the number fields in this family are not of R–D type.

Lemma 2.1. *Let $K = \mathbb{Q}(\sqrt{d})$ be a family of real quadratic fields where $d = 100m^2 + 28m + 2$ is square-free and $m \in \mathbb{Z}^+$. Then K is not of R–D type.*

Proof. Given that, $d = 100m^2 + 28m + 2$ this implies that $d = (10m+1)^2 + 8m + 1$. Since $(10m+1)^2 < d < (10m+2)^2$ and $-(10m+1) < 8m+1 \leq 10m+1$, for all positive integer m this shows that $d = (10m+1)^2 + 8m + 1$ is the absolute reduced form of d. But $4(10m+1) = 40m+4$, which is not divisible by $8m+1$ for any $m \in \mathbb{Z}^+$. Therefore, the given family is not a R–D type family of real quadratic fields. $\square$

Now, we are ready to prove our main result.

Proof. Suppose that the Diophantine equation

$$x^2 - dy^2 = \pm 2 \tag{2.1}$$

has a solution among integers, then without loss of generality, we can assume that (u, v) is a solution with the minimum possible $u \geq 0$ and $v > 0$. Then

$$u^2 - dv^2 = \pm 2. \tag{2.2}$$

Consider the algebraic integer $\sigma := u - v\sqrt{d}$. Then clearly by (2.2), $N(\sigma) = \pm 2$. Similarly define another algebraic integer as $\tau := (50m+7) + 5\sqrt{d}$. Then $N(\tau) = -1$. Now consider the product

$$\sigma\tau = \big(u(50m+7) - 5vd\big) + \big(5u - v(50m+7)\big)\sqrt{d}. \tag{2.3}$$

We take norm on both sides to get

$$\big(u(50m+7)-5vd\big)^2 - d\big(5u-v(50m+7)\big)^2 = \pm 2. \tag{2.4}$$

Clearly both $\big(u(50m+7)-5vd\big)$ and $\big(5u-v(50m+7)\big)$ are integers. Thus, using the minimality condition of v, we get from (2.4)

$$\big|5u-v(50m+7)\big| \geq v. \tag{2.5}$$

First we consider the case when $5u-v(50m+7)\geq v$. In this case we have

$$\begin{aligned} & 5u \geq v(50m+8) \\ \Longrightarrow\ & 25u^2 \geq v^2(50m+8)^2. \end{aligned}$$

Replacing the value of u^2 from (2.2) one obtains

$$\begin{aligned} & 25(\pm 2+dv^2) \geq v^2(50m+8)^2 \\ \Longrightarrow\ & \pm 50 \geq v^2\big((50m+8)^2-25(100m^2+28m+2)\big) \\ \Longrightarrow\ & \pm 50 \geq v^2(100m+14) \end{aligned}$$

which is not possible.
Now we consider the remaining case when $v(50m+7)-5u\geq v$. Here we have

$$\begin{aligned} & 5u \leq v(50m+6) \\ \Longrightarrow\ & 25u^2 \leq v^2(50m+6)^2. \end{aligned}$$

Again replacing the value of u^2 from (2.2) we get

$$\begin{aligned} & 25(\pm 2+dv^2) \leq v^2(50m+6)^2 \\ \Longrightarrow\ & \pm 50 \leq v^2\big((50m+6)^2-25(100m^2+28m+2)\big) \\ \Longrightarrow\ & \pm 50 \leq -v^2(100m+14) \end{aligned}$$

which is again not possible.

We have $d=100m^2+28m+2 \implies d\equiv 2 \pmod 4$ and the discriminant d_K of the real quadratic field $K=\mathbb{Q}(\sqrt{d})$ is $4d$. From the basic algebraic number theory we know that the ring of integers $O_K=\mathbb{Z}[\sqrt{d}]$ and the rational prime 2 ramifies in O_K. Suppose $2O_K=\mathfrak{p}^2$ for some prime ideal $\mathfrak{p}$ with $N(\mathfrak{p})=2$. If $h_K=1$, then $\mathfrak{p}$ is a principal ideal and hence we can write $\mathfrak{p}=(\alpha+\beta\sqrt{d})$ for some $\alpha,\beta\in\mathbb{Z}$. This implies that $\alpha^2-d\beta^2=\pm 2$, which is a contrary to the above shown arguments. □

Remark 2.1. We can deduce a divisibility criteria for the class numbers of the above discussed family of real quadratic fields. In the proof of Theorem 1.1 we have shown that the rational prime 2 ramifies in O_K and the prime ideal $\mathfrak{p}$ above 2 is non-principal. This implies that the ideal class in the ideal class group of K containing $\mathfrak{p}$ is of order 2. This proves that $2\mid h_K$.

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